HODGE OPERATORS OF MAPPING SPACES; LOCAL STUDY

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Abstract

Hodge operator of the mapping space $Map(X,M)$, M is a smooth manifold and X is a compact spin manifold, is defined by defined by using a non-degenerate selfadjoint 1-st order ψDO D on X. Choosing D means to choice a Soboles metric of the Sobolev space over X. The $(\omega - p)$-form on $Map(X,M)$ is defined by using the Sobolev duality. We propse to compute $(-1)^\infty$ and so on replacing ∞ by $\nu = \zeta_{|D|}(0)$. It is shown the integrity of ν is necessary and sufficient to the anticommutativity and associativity of the Grassmann algebra with $(\infty - p)$-degree parts. Similar results for Clifford algebra is also shown. Differential and integral calcul uses on $(\infty - p)$-forms are discussed and necessity of some kinds of complete continuity to the differentiability and integrability of $(\infty - p)$-forms is shown. Global definition of Hodge operators and related problems are also discussed.

AMS Subject Classification: Primary 58D15, Secondary 58A10, 58C99.
Key words: Hodge operator, maping space, Sobulev space

In the talk last workshop, we studied a family of Dirac operators and associated a loop group bundle to this family ([4]). It was shown this bundle is determinated by the information of the changes of sign of proper values of the Dirac operator (+lower order termen). Such information is topological and geometric use of other informations of spectres (they must be metrical, cf. [8]) were remained open.

In this talk, we propose a definition of Hodge operators of mapping spaces by using Dirac operator (added mass term) and its Green operator. To pursue this study, we need to use several informations of spectres of the Dirac operator via aspectre eta and zeta functions. In fact, to define a Hodge operator on an infinite dimensional manifold, we need to answer the following two questions;

(I) What are $(\infty - p)$-forms ?

Editor Gr.Tsagas *Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, 1995,* 11-20

(II) How compute $(-1)^{\infty}$ and so on ?

We propose candidates of answers to these qustions as follows ;

Proposal I . $(\infty - p)$-forms are defined via Sobolev duality, and the duality, is determined by choosing an elliptic operator D.

Proposal II . To compute $(-1)^{\infty}$ and so on, ∞ should be replaced by $\nu = \varsigma_{|D|}(0)$.

To explain precise meaning of proposal I , we need to deal with the Sobolev structure of a mapping space.

1 Sobolev structure of a mapping space

Let X be a d-dimensional compact spin manifold, M an N-dimensional smooth manifold. Then the appropriate regularity of the elements of $Map(X, M)$ is Sobolev k-class, . In fact, in this case, we have

(i) $Map(X, M)$ becomes a Hilbert manifold. So it allows smoth partition of unity subordinate to locally finite open covering.

(ii) Any map in $Map(X, M)$ is continuous.

Precisely saying, $Map(X, M)$ is not connected in general. In the rest, $Map(X, M)$ means the connected component of $Map(X, M)$ consisted by contractible maps.

By (i) , the model of $Map(X, M)$ is $W^k(X) \otimes I\!R^n$, where $W^k(X)$ is the Sobolev k-space over X. So the canonical fibres of the tangent and cotangent bundles of $Map(X, M)$ are $W^k(X) \otimes I\!R^n$ and $W^{-k}(X) \otimes I\!R^n$, respectively .

Note. Let be $\{g_{UV}\}$ the transition function of the tangent bundle of M , then the transition function of the tangent bundle of $Map(X, M)$ is given by $\{g_{UV}^{X}\}$, where

$$\left(g_{UV}^{X}(f)\right)(x) = g_{UV}(f(x)),\ x \in X,\ f \in Map(X, U) \cap Map(X, V).$$

Hence the tangent bundle of $Map(X, M)$ is a $Map(X, G)$-bundle, G is the structure group of the tangent bundle of M, and we may consider the associated bundle of the tangent bundle of $Map(X, M)$ with the fibre $W^k(X, E) \otimes I\!R^n$, etc.. Here $W^k(X, E)$ means the k-Sobolev space of spinor fields over X.

For simple, we write $W^k(X)$ etc, instead of $W^k(X, E) \otimes I\!R^n$, $W^k(X, E)$ $\otimes I\!R^n$, etc ..

Since a p-form on $Map(X, M)$ is a smooth cross-section of the associated $\Lambda^P W^{-k}(X)$-bundle of the cotangent bundle, which is the dual of the associated $\Lambda^P W^k(X)$-bundle of the tangent bundle of $Map(X, M)$. So we restate proposal I as follows ;

Proposal I" An $(\infty - p)$- form on $Map(X, M)$ is a smooth cross-section of the associated $\Lambda^P W^k(X)$-bundle of the tangent bundle of $Map(X, M)$.

But to treat $(\infty - p)$- forms, we need to determine Sobolev duality exactly. This argument will relate to proposal II.

2 Sobolev duality via elliptic operator

We fix a non-degenerate selfadjoint 1-st order elliptic (pseudo) differential operator D on X. On $K^2(X)$ $(= W^0(X))$, D allows spectral decomposition

$$D = \sum (\,, \Phi_\lambda)\, \Phi_\lambda. \tag{1}$$

Then to set

$$e_{\lambda,k} = sgn\lambda \mid \lambda \mid^{-k} \Phi_\lambda, \tag{2}$$

we have

$$W^k(X) = \left\{ \sum c_\lambda e_{\lambda,k} \mid \sum \mid c_\lambda \mid^2 < \infty \right\}. \tag{3}$$

Expression (3) means the Sobolev norm of $f = \sum c_\lambda e_{\lambda,k} \in W^k(X)$ is determined to be

$$\| f \|^2 = \sum \mid c_\lambda \mid^2 .$$

We denote the Green operator of D by G. Then the pairing of $u \in W^{-k}(X)$ and $f \in W^k(X)$ is given by

$$< u, f > = \left(G^k u, D^k f \right), \tag{4}$$

$(\,,)$ is the inner product of $L^2(X)$.

Since $W^k(X)$ is spanned by the proper functions of D, the number of linear independent proper functions of D might be the (virtual) dimension of $W^k(X)$. To compute this number, we use the spectre η- and ς-functions of D and $\mid D \mid$ defined by

$$\eta_D(s) = \sum\nolimits_{\lambda \in SpecD} sgn\lambda \mid \lambda \mid^{-s},$$

$$\varsigma_{|D|}(s) = \sum\nolimits_{\lambda \in SpecD} \mid \lambda \mid^{-s} = \eta_D 2(s/2).$$

We use the following facts [7], [9]:

(i) These functions continued meromorphically on whole complex plane with poles possively at d, $d-1$, $d-2$, ..., at most order 1. (ii) These functions are holomorphic at $s = 0$.

By (ii), $\nu = \varsigma_{|D|}(0)$ exists. We call ν to be the virtual dimension of $W^k(X) = W^k(X, D)$. By definition, virtual dimension does not depend on k but depends on D. For example, adding a mass-term m, we get

$$\varsigma_{|D+m|}(0) = \tag{5}$$

$$= \varsigma_{|D|}(0) - m \underset{s=1}{Res}\, \eta_D(s) + \frac{m^2}{2} \underset{s=2}{Res}\, \varsigma_{|D|}(s) - \cdots - \frac{m^d}{d} \underset{s=d}{Res}\, \eta_D(s),$$

d, is odd,

$$= \varsigma_{|D|}(0) - m \underset{s=1}{Res}\, \eta_D(s) + \cdots + \frac{m^d}{d} \underset{s=d}{Res}\, \eta_{|D|}(s),$$

d, is even.

Here, m is a real number not to be a proper value of D.

Virtual dimension is invariant under the scale change of D. But since we get

$$\varsigma_{t|D|}(s) = t^{-s}\varsigma_{|D|}(s), \tag{6}$$

we have $\varsigma'_{t|D|}(0) = -\varsigma_{|D|}(0)\log t + \varsigma'_{|D|}(0)$. So to define det$|\ D\ |$ by $\exp(-\tau'_{|D|}(0))$ we get

$$\det(t \mid D \mid) = t^{\nu}\det \mid D \mid, \ t > 0. \tag{7}$$

Note. We set

$$\nu_{+} = \frac{1}{2}\left(\nu + \eta_D(0)\right), \ \nu_{-} = \frac{1}{2}\left(\nu - \eta_D(0)\right).$$

They are virtual numbers of positive and negative proper values of D, respectively. By using these numbers, we define $\det D$ by

$$\det D = (-1)^{\nu_-}\exp\left(-\varsigma'_{|D|}(0)\right).$$

When ν_- is not an integer, determination of $(-1)^{\nu_-}$ involves delicate problem related to the integrity of ν. This will be remarked in the next section.

3 Grassman algebra with $(\infty - p)$-degree parts

Elements of $\Lambda^P W^{\pm k}(X)$ are alternative functions of p-product of X . We define the wedge product of a q-form u and $(\infty - p)$-form f $(q \le p)$ by

$$(u \wedge f)(x_1, \ldots, x_{p-q}) =$$

$$< u(x_{p-q+1}, \ldots, x_p), f(x_{p-q+1}, \ldots, x_p; x_1, \ldots, x_{p-q}) >$$

The orthonormal basis of $\Lambda^P W^{-k}(X)$ is given by $\{e_{\lambda_1,-k} \wedge \cdots \wedge e_{\lambda_p,-k}\}$. Here $f_1 \wedge \cdots \wedge f_p$ means $1/p! \sum_{\sigma \in S_p} sgn\sigma f_1(x_{\sigma(1)}) \cdots f_p(x_{\sigma(p)})$. The dual element of $e_{\lambda_{1'},-k} \wedge \cdots \wedge e_{\lambda_{p'},-k}$ is $e_{\lambda_{1'}k} \wedge \cdots \wedge e_{\lambda_{p'}k}$. Regarding this element to be the subspace of $W^k(X)$ orthogonal to the subspace of $W^{-k}(X)$ spanned by $e_{\lambda_{1'},-k}, \ldots, e_{\lambda_{p'},-k}$, we denote

$$e_{\lambda_{1'}k} \wedge \cdots \wedge e_{\lambda_{p'}k} = \Lambda^{\infty - A} e_{\lambda',-k}, \quad A = (\lambda_1, \ldots, \lambda_p).$$

For a q-form u and $(\infty - p)$-form f we must have

$$f \wedge u = (-1)^{q(\infty - p)}\, u \wedge f.$$

Proposal II asserts this ∞ must be replaced by ν. Then, since $u \wedge f$ must be $(-1)^{q(\infty-p)} f \wedge u.$, it needs

$$(-1)^{2q(\infty - p)} = 1, \tag{8}$$

for all integers p, q. Conversely, if (8) is hold, we get

$$\left(f^q\right)\left(u^{\infty-p}\right) \wedge g^r = f^q \wedge \left(u^{\infty-p}\right)\left(g^r\right),$$

$$\left(u^{\infty-p}\right)\left(f^q\right) \wedge g^r = u^{\infty-p} \wedge \left(f^q \wedge g^r\right)\left(= (-1)^{q(\nu-p)}\left(f^q \wedge u^{\infty-p}\right) \wedge g^r\right).$$

Therefore we conclude

Theorem 1 *The Grassmann algebra* $\sum \Lambda^p W^{-k}(X) \oplus \sum \Lambda^p W^k(X)$ *(=* $\sum \Lambda^p W^{-k}(X) \oplus \sum \Lambda^{\infty - p} W^{-k}(X)$*) becomes anticommutative and associative if and only if the virtual dimension of* $W^{-k}(X)$ *is an integer.*

Note This Theorem holds allowing arbitrary expression of -1 as the power of e. But restricting the expression of -1, we relux the conclusion of Theorem as follows;

Theorem 2 $\sum \Lambda^p W^{-k}(X) \oplus \sum \Lambda^{\infty - p} W^{-k}(X)$ *becomes anticommutative and associative if the virtual dimension of* $W^{-k}(X)$ *is* n/m*,* $(n, m) = 1$*, and* m *is an odd number, restricted the expression of* (-1) *to be* $-1 = \exp(ml\pi\sqrt{-1})$*,* l *is an odd number.*

4 Clifford algebra with $(\infty - p)$ degree spinors

We define the Clifford multiplication of the basis of $W^{-k}(X)$ by

$$e_{\lambda\prime, -k} \vee e_{\mu\prime, -k} = -e_{\mu\prime, -k} \vee e_{\lambda\prime, -k}\ ,\ \lambda \neq \mu,$$

$$e_{\lambda\prime, -k} \vee e_{\lambda\prime, -k} = sgn\lambda \mid \lambda \mid^{2k};$$

([11], [12]). For the index set $A = \{\lambda_1, \ldots, \lambda_p\}$, we set

$$e_A = \underset{\lambda \in A}{\vee} e_{\lambda\prime, -k}\ ,\ e^{\infty - \lambda} = \underset{\lambda \notin A}{\vee} e_{\lambda\prime, -k} \left(= \underset{\lambda \in A}{\vee} e_{\lambda\prime, k} \right),$$

$\mid A \mid = p$, $\mid A \mid_-$ is the number of $-$ elements in A,

$$\mid \lambda_A \mid = \underset{\lambda \in A}{\Pi} \mid \lambda \mid,$$

Then we define

$$e^{\infty - A} \vee e^{\infty - A} = (-1)^{(\nu - |A|)(\nu - |A| + 1)/2 + \nu_- - |A|_-} \mid \lambda_A \mid^{-2k} \mid \det D \mid^{2k}.$$

Under these computation rules , if A, B, C, D, E, F, and G are disjoint finite index sets , we get

$$\begin{aligned}
&\left(\left(e_A \vee e_D \vee e_F \vee e^{\infty - G}\right) \vee \left(e_B \vee e_D \vee e_E \vee e^{\infty - G}\right)\right) \vee \\
&\left(e_C \vee e_E \vee e_F \vee e^{\infty - G}\right) \qquad (9) \\
&= (-1)^{(\nu - |G|)(|E| - |C| - |F|)} \left(e_A \vee e_D \vee e_F \vee e^{\infty - G}\right) \vee \\
&\left(\left(e_B \vee e_D \vee e_E \vee e^{\infty - G}\right) \vee \left(e_C \vee e_E \vee e_F \vee e^{\infty - G}\right)\right).
\end{aligned}$$

We set $\mid E \mid + \mid C \mid + \mid F \mid = p$, then, if ν is an integer , we get

$$(\nu - \mid G \mid)(\mid E \mid - \mid C \mid - \mid F \mid) = (\nu - \mid G \mid)(\mid G \mid - p).$$

Therefore we obtain

Theorem 3 *If the virtual dimension of $W^{-k}(X)$ is even, then the module of even spinors and (∞-odd)-degree spinors becomes an associative algebra.*

If the virtual dimension of $W^{-k}(X)$ is odd, then the module of even spinors and (∞-even)-degree spinors becomes an associative algebra.

In Theorem 3, the expression of -1 is arbitrary. If we restrict the expression of -1, following Theorem holds.

Theorem 4 *If the virtual dimension of $W^{-k}(X)$ is a rational number s/r, r is an odd number and $(r, s) = 1$, then restricting $-1 = \exp(rt\pi\sqrt{-1})$, t is an odd number, the module even spinors and (∞-odd)-degree spinors becomes an associative algebra if s is even, and the module of even spinors and (∞-even)-degree spinors becomes an associative algebra if s is odd.*

The restriction of the expression of -1 in Theorems 2 and 4 might reflects to the definition of $\det D$.

5 Calculus of $(\infty - p)$-forms

We denote the Frechét differential of $f : U \to \Lambda^p W^k(X)$, $U \subset W^k(X)$ by $\bar{d}f$. Then $\bar{d}f(x)$ belongs to $W^{-k}(X) \otimes \Lambda^p W^k(X)$. Since $W^{-k}(X) \otimes W^k(X)$ is contained in the algebra of bounded linear operators on $W^k(X)$, we define the exterior differential of f (as an $(\infty - p)$-form) by

$$df\,(x_1, \ldots, x_{p-1}) = (-1)^{p-1}\, tr\left(\bar{d}f\,(x_0, x_p; x_2, \ldots, x_{p-1})\right).$$

Hence an $(\infty - p)$-form f is differentiable if and only if $\bar{d}f(x_0, x_p)(x_1, \ldots, x_{p-1}) = \bar{d}(x_0, x_p; x_1, \ldots, x_{p-1})$ becomes a trace class operator of $W^k(X)$.

Note This differentiability condition is strong. But since formally we get $d(\sum f_i e^{-i})$ $= \sum(-1)^i \partial f/\partial e_i \wedge e_i$, the trace class condition seems necessary . We also note that trace class condition implies some complete continuity of f. That is, if f is differentiable, then f must be continuous in weaker topology of $W^k(X)$.

Next we consider integrals of $(\infty{-}p)$-forms (cf[2], [5], [6]). If Y is a p-codimensional smooth submanifold of$x$, taking the p-form n corresponding to the ortho - normal basis of the normal space of Y at x, the pairing $< n(x), g(x) >$ gives a function on Y. Hence we need only to consider integrals of ∞-forms.

We set $Q_{+m,t} = \{\sum c_\lambda \phi_\lambda \mid 0 \le c_\lambda \le \mid t\lambda \mid^m\}$, $Q_{m,t} = \{\sum c_\lambda \phi_\lambda \mid - \mid t\lambda \mid^m \le c_\lambda \le \mid t\lambda \mid^m\}$. They are subsets of $W^{-m-d/2}(X)$. We define their volumes as

$$vol\,(Q_{+m,t}) = t^{m_\lambda}\,(\det \mid D \mid)^m\,,\ vol\,(Q_{m,t}) = (2t)^{m_\lambda}\,(\det \mid D \mid)^m\,.$$

Since the definitions are similar , hereafter we only consider integrals on $Q_{m,t}$. Let $0.s_1 s_2 \ldots$ be the binary expansion of $s \in [0, 1]$. Then we set

$$Q_s = \{\Sigma c_i \phi_i \mid - \mid t\lambda_i \mid^m \le c_i \le 0,\ if\ s_i = 0,\ 0 \le c_i \le \mid t\lambda_i \mid^m,\ if\ s_i = 1\}\,.$$

By definition, $Q_{m,t} = \underset{s\in[0,1]}{\cup} Q_s$, and $Q_s \wedge Q_s$ is contained in a subset of $W^{-m-d/2}(X)$ of non-zero codimension , if $s \neq s'$. For a real valued function f on , we set

$$\bar{f}_1(s) = \sup_{x\in Q_s} f(x), \quad \underline{f}_1(s) = \inf_{x\in Q_s} f(x).$$

Then $\int_0^1 \bar{f}_1(s) ds vol(Q_{m,t})$ and $\int_0^1 \underline{f}_1(s) ds vol(Q_{m,t})$ are the upper and lower Riemann sums for the partition $\{Q_s \mid s \in [0,1]\}$ of $Q_{m,t}$.

We assume the partition $Q_{s^1,\ldots,s^{n-1}}$ for the vector $(s^1,\ldots,s^{n-1})$ in $[0,1]\times \overset{n-1}{\cdots} \times$ has been determined. Then by using the bynary expansion $s^n = 0.s_1^n s_2^n \ldots$, we define $Q_{s^1,\ldots,s^n}$ by

$$Q_{s^1,\ldots,s^n} = \{\Sigma c_i\phi_i \mid a_i \leq c_i \leq a_i + (b_i - a_i)/2,\ if\ s_i^n = 0,$$

$$\begin{aligned} Q_{s^1,\ldots,s^n} = \{\Sigma c_i\phi_i \mid \quad & a_i \leq c_i \leq a_i + (b_i - a_i)/2,\ if\ s_i^n = 0, \\ & a_i + (b_i - a_i)/2 \leq c_i \leq b_i,\ if\ s_i^n = 1,\ where \\ & Q_{s^1,\ldots,s^{n-1}} = \{\Sigma c_i\phi_i \mid a_i \leq c_i \leq b_i\}\}. \end{aligned}$$

The functions $\overline{f}_n$ and $\underline{f}_n$ are defined by

$$\overline{f}_n\left(s^1,\ldots,s^n\right) = \sup_{x\in Q_{s^1,\ldots,s^n}} f(x),$$

$$\underline{f}_n\left(s^1,\ldots,s^n\right) = \inf_{x\in Q_{s^1,\ldots,s^n}} f(x).$$

The integral of $f(x)$ on $Q_{m,t}$ is defined to be

$$\int_{Q_{m,t}} f(x)dx = \lim_{n\to\infty} \int_{I^n} \overline{f}_n\left(s^1,\ldots,s^n\right) ds^1 \ldots ds^n vol(Q_{m,t}),$$

provided

$$\lim_{n\to\infty} \int_{I^n} \overline{f}_n\left(s^1,\ldots,s^n\right) ds^1 \ldots ds^n = \lim_{n\to\infty} \int_{I^n} \underline{f}_n\left(s^1,\ldots,s^n\right) ds^1 \ldots ds^n.$$

Here I^n means n-th Cartesian product of $I = [0,1]$.

As for the existence of the integral, we can show

Theorem 5 *Let a be larger than $d/2$. Then f is integrable on $Q_{m,t}$ if f is continuous by the topology of $W^{-m-a}(X)$.*

Example Let $f(x)$ be $\| x \|_{-k}^2$ $(=\Sigma x_n^2 \lambda_n^{-2k})$. Then we have for $Q_{m,t}$

$$\overline{f}_n\left(s^1,\ldots,s^n\right) = t^{2(m-k)}\Sigma\left(\left(2^{n-1}s_i^1 + 2^{n-2}s_i^2 + \ldots + s_i^n + 1\right)/2^n\lambda_i\right)^{2(m-k)},$$

$$\underline{f}_n\left(s^1,\ldots,s^n\right) = t^{2(m-k)}\Sigma\left(\left(2^{n-1}s_i^1 + 2^{n-2}s_i^2 + \ldots + s_i^n\right)/2^n\lambda_i\right)^{2(m-k)}.$$

Hence we get

$$\int_{Q_{+m,t}} f(x)dx = \frac{1}{2} t^{(2+\nu)m-2k} \varsigma_{|D|}\left(2\left(k-m\right)\right)\left(\det \mid D \mid\right)^{m} .$$

So the integral exists if $(k-m) > d/2$.

Note In the above integral, $\Lambda\phi_\lambda$ is taken to be the volume form. Since $\det D$ is defined , we get volume form on $W^m(X)$ for arbitrary m. On the other hand, if $g \in Map(X, SU(n))$, we may regard $\Lambda\phi_\lambda = \Lambda g(\phi_\lambda)$. So we need not consider the influence of g.

6 Global definition of the Hodge operator

We assume the Sobolev structure of $W^k(X)$ is given by a non-degenerate selfadjoint 1-st order eliptic operator D. In general, D is not gauge covariant as the operator acting on the sections of the (co)tangent bundle of $Map(X, M)$. But it is possible to introduce selfadjoint connection $\{A_U)$ such that

$$\left(D + A_U\right) g_{UV} = g_{UV}\left(D + A_V\right) \tag{10}$$

where $\{g_{UV}\}$ is the transition function of the tangent bundle of $Map(X, M)$ ([4]). For simple, we write D_A instead of $\{D + A_U\}$ in the rest. The (family of) Green operator of D_A is denoted by G_A if eists.

If D_A is non-degenerate at any point of $Map(X, M)$, we define the Hodge operator $*$ on $Map(X, M)$ by

$$*u^p\left(x_1, \ldots, x_p\right) = \left(G_A^{2k} \times \cdots \times G_A^{2k}\right) u^p\left(x_1, \ldots, x_p\right),$$

$$*f^{\infty-p}\left(x_1, \ldots, x_p\right) = (-1)^{p(\nu_A - p)}\left(G_A^{2k} \times \cdots \times G_A^{2k}\right) f^{\infty-p}\left(x_1, \ldots, x_p\right).$$

Here, ν_A means $\varsigma_{|D_A|}(0)$.

If D is positive, $D + A_U(x)$ is bounded below for any $x \in Map(X, M)$. Hence to take $m(x)$ larger than $\min(Spec(D + A_U(x)))$, $\{D + A_{U+m}\}$ becomes a non-degenerate operator acting on the sections of the tangent bundle of $Map(X, M)$. Hence if M is orientable, $Map(X, M)$ has a positive definite Hodge operator.

On the other hand , if D is the Dirac operator , we can not get a connection $\{A_U\}$ of D with respect to the tangent bundle of $Map(X, M)$ such that D_A is not-degenerate at any point of $Map(X, M)$, unless $Map(X, M)$ is parallelisable ([3] , [4]) . But as a global operator, G_A may allow mild singularity (with respect to the parameter). In this connection , we pose the following conjecture.

Conjecture 1 *Let* $\bar{c}^i \in H^{2i+1}\left(Map(X, M), Z\right)$ *be the i-th string class of the tangent bundle of* $Map(X, M)$ *(regarded as a loop gruop bundle (cf. [3])) . Then by using the Dirac operator on* X *we can define a Hodge operator on* $Map(X, M)$ *if*

$$\bar{c}^1 = \cdots = \bar{c}^d = 0. \tag{11}$$

If $d = 2k - 1$, denoting the i-th Pontrjagin class of M by $p_i(M)$, (11) is replaced by

$$p_1(M) = \ldots = p_k(M) = 0. \tag{12}$$

This conjecture may relate rigoulus definition of the Dirac-Raomond operator (cf. [1] , [13]).

Note As an alternative function of infinite many variables, fixed volume form $\Lambda\phi_\lambda$ only changed to $\Lambda T\phi_\lambda$, T is inversible and takes the form $D + S$, S is a compact operator. This suggests $\Lambda\, H$ may be parametrized by $GL(H)/K(H)$, $H = W^{-k}(X)$. Here $GL(H)$ is the group of inversible bounded linear operators on H and $K(H)$ is the subgroup of $GL(H)$ consisted by the operators of the form $I + S$, S is compact.

We regard a $Map(X, G)$-bundle ξ to be a GL_c-bundle , where $GL_c = \{T \in GL(H) \mid [\varepsilon, T] \textit{ is compact}\}$, $\varepsilon = D^{-1} \mid D \mid$. Then choosing a volume form of ξ should be choosing a cross-section of the associated $GL_c/K(H)$-bundle of ξ which is equivalent to $\xi \oplus \xi^{-1}$ ([3]). Hence such cross-section exists if the structure group of ξ can be lifted to its central extension (as a loop group bundle). Hence vanishing of the 1-st string class of ξ may need to the existence of the volume from on ξ. (cf [1], [11], [13]).

To get anticommutative and associative Grassmann or Clifford algebra on $Map(X, M)$, we need to add a mass-term m to the connection (Hodge operator), to satisfy the condition $\varsigma_{A+m}(0)$ to be an integer (or a rational number with odd denominator). Such selection of m is locally possible by (5). Possibility of the global selection of m relates to the (non-)existence of the branching point of the d-th degree equation of $m$determined by (5) (and the prescribed virtual dimension) (cf. [10], [14]).

Acknowledgments

Supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture, Ziuten 07210235

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