ON VECTORIAL FINSLER CONNECTIONS

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Abstract

In this paper the definition of vectorial Finsler connection is extended. If we consider some general definitions of Finsler bundles and Finsler splittings, then nonlinear connections on vector subbundles can be induced. Thus, the induced vectorial Finsler connection on a submanifold can be obtained as a particular case.

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The main ideea of classical Finsler theories is to replace the base manifold of the tangent space τM (which is M itself) which TM, the total space of τM, obtaining thus the Finsler vector bundle $\mathcal{F}M$ of the manifold M (see e.g. [3]). So, the geometrical objects considered on the Finsler vector bundle depends on the directions of TM. Instead of τM it can be taken an arbitrary vector bundle (as in [4]), rising in this way a natural question: to investigate a theory of subspaces in such Finsler vector bundles. A first attempt was made by the author in [6], using some classical ideas concerning Finsler and Lagrange subspaces, as [1] and [3]. We generalise the previous results in this paper.

Definition 1 Let $\xi = (E, \pi, M)$ and $\eta = (F, p, E)$ be two vector bundles. We say that the vector bundle η is an *ξ-Finsler bundle* of ξ. The vector bundle $\pi^*\xi$ is the *Finsler bundle* of ξ.

A *vectorial Finsler connection* on (ξ, η) is a doublet (N, Γ), where N is a non-linear connection on ξ and Γ is a non-linear connection on η.

Observation 1 If Γ is a linear connection then the above definition gives the vectorial Finsler connection defined by A. Bejancu in [1], as a particular case.

In the sequel we use some definitions and results from [6].

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Definition 2 *[6]*Let $\xi = (E, \pi, M)$, η and η' be vector bundles which have the same base M, where η is a subbundle of η' and $i : \eta \to \eta'$ is the inclusion morphism. Then we say that a splitting S of the injective induced morphism $\pi^*\eta \xrightarrow{\pi^* i} \pi^*\eta'$ (i.e. $S \circ \pi^* i = id$) is a *Finsler ξ-splitting* of i. If there is a splitting $s : \eta' \to \eta$ of i such that. $S = \pi^* s$, then we say that the Finsler splitting S is *induced* by the splitting s.

Using the following sequence of vector bundles morphisms over the same base E:

$$\tau E \xrightarrow{i_*} \tau E'_{|E} = i^*\tau E' \xrightarrow{i^* N'} i^* V\xi' \cong i^*\pi_1^*\xi' = \pi^*\xi' \xrightarrow{S} \pi^*\xi \cong V\xi \tag{1}$$

it is easy to verify that we have:

Proposition 1 *[6]Let $\xi = (E, \pi, M)$ be a vector subbundle of the vector bundle $\xi' = (E', \pi_1, M)$, let $i : \xi \to \xi'$ be the inclusion morphism and N' be a non-linear connection on ξ'. Then every Finsler splitting S of i induces a non-linear connection on ξ.*

If the Finsler splitting S is induced by a spliting s and N' is a linear connection, then the induced connection N is linear, too.

The non-linear connection N' induces the decomposition

$$T_{u'}E' = V_{u'}E' \oplus H_{u'}E'.$$

in every point $u \in E'$.

If $u \in E$, then the Finsler splitting S gives the decomposition:

$$V_u E' = V_u E \oplus (\ker S_u) \oplus H_u E' .$$

An interpretation of the result proved in [1, Theorem 3.3] for Finsler subspaces can be given. If we use the definition of the splitting given by (1), using that for every $u \in E$ we have:

$$\dim(T_u E \cap (\ker S_u \oplus H_u E')) = \dim M.. \tag{2}$$

and taking into account of dimensions it can be proved:

Proposition 2 *[6]Let N be the connection given by Proposition 1. Then, for every $u \in E$, we have:*

$$H_u E \subset \ker S_u \oplus H_u E'. \tag{3}$$

The connection N is the only connection which has this property.

Proposition 3 *[6] Let $\xi = (E, \pi, M)$ and $\xi' = (E', \pi_1, M')$ be two vector bundles such that $M \xrightarrow{i} M'$ is the inclusion of the submanifold M and $\xi \xrightarrow{I} \xi'$ is the i-morphism of the inclusion of ξ. Let N' be a non-linear connection on ξ'. Then every ξ-splitting S of the inclusion $\xi \xrightarrow{i^* I} i^*\xi'$ induces a non-linear connection N on ξ.*

If the Finsler splitting S is induced by a splitting of the inclusion $\xi \xrightarrow{i^* I} i^*\xi'$ and N' is a linear connection, then N is a linear connection, too.

The splitting which defines the nonlinear connection on ξ is obtained if we make the composition of the following vector bundle morphisms:

$$\tau E \xrightarrow{I_*} \tau E'_{|E} = i^*\tau E' \xrightarrow{i^*N} i^*V\xi' \xrightarrow{S} V\xi.$$

Theorem 1 *Let $\xi = (E, \pi, M)$ be a vector subbundle of the vector bundle $\xi' = (E', \pi_1, M)$, where $\xi \xrightarrow{I} \xi'$ is the inclusion morphism. Let $\eta = (F, p, E)$ and $\eta' = (F', p_1, E')$ be vector bundles such that there is an I-morphism $\eta \xrightarrow{J} \eta'$ which is an injection on fibres. Consider a vectorial Finsler connection (N', Γ') on (ξ', η').*

Then every Finsler ξ-splitting of I and every Finsler ξ'-splitting of I^*J induce a vectorial Finsler connection (N, Γ) on (ξ, η). If the Finsler ξ'-splitting is induced by a splitting of I^*J and Γ' is a linear connection, then Γ is a linear connection, too.

Proof. Using Propositions 1 and 3 then N, respectively Γ, can be constructed and the assertions are proved. Q.e.d.

Given a vector bundle $\xi = (E, \pi, M)$, consider the vector bundles τE and $V\xi \cong \pi^*\xi$, over the same base E.

Definition 3 Let $\xi = (E, \pi, M)$ be a vector bundle. A *Finsler connection* on ξ is a vectorial Finsler connection (N, Γ) on $(\xi, \pi^*\xi)$.

Corollary 1 *Let $\xi = (E, \pi, M)$ be a vector subbundle of the vector bundle $\xi' = (E', \pi_1, M)$ and let $\xi \xrightarrow{I} \xi'$ be the inclusion morphism. Then every Finsler ξ-splitting of I and every vectorial Finsler connection (N', Γ') on ξ' define canonically a vectorial Finsler connection (N, Γ) on ξ, such that Γ' linear implies Γ linear, too.*

Definition 4 Let $\xi = (E, \pi, M)$ be a vector subbundle of the vector bundle $\xi' = (E', \pi_1, M)$, $\xi \xrightarrow{I} \xi'$ be the inclusion morphism and S be a Finsler ξ-splitting of I. The *horizontal subbundle* of S is the vector subbundle $\ker S = (\eta, q, E)$ of the vector bundle $\pi^*\xi'$.

Observation 2 The above definition is an extension of the definition of the normal vector subbundle to a Finsler subspace [1, Secttion 2], or to a Lagrange subspace [3, Chap. VII].

Theorem 2 *If $\xi \xrightarrow{I} \xi'$ is the inclusion morphism of the vector subbundle ξ and S is a Finsler splitting of I, then every vectorial Finsler connection (N', Γ') on the vector bundle ξ' induces a vectorial Finsler connection $(N, \widetilde{\Gamma})$ on $(\xi, \ker S)$.*

Proof. Consider the morphism $\xi \xrightarrow{I} \xi'$ and the I-morphism J which is the composition of the canonical inclusions: $\ker S \to \pi^*\xi' \to \pi_1^*\xi'$. The inclusion morphism

$$\ker S \to I^*(\pi_1^*\xi') = (\pi_1 \circ I)^*\xi' = \pi^*\xi'$$

has as splitting the complementary splitting of S. Using Theorem 1 the assertion follows. Q.e.d.

Observation 3 When M is a Finsler subspace of M', if we take $\xi = \tau M$, $\xi' = \tau M'$ and S the splitting induced by the Finsler metric, then we obtain [1, Theorem 4.1]. An analogous result holds for a Lagrange subspace, as in [3, Chap. VII].

We give now an example, which generalise the cases of the previous Observation, using generalised Finsler metrics.

Definition 5 A *Finsler metric* g on a vector bundle ξ is a metric tensor on the vector bundle $\pi^*\xi$.

Observation 4 Every Finsler metric g on a vector bundle ξ induces a Finsler metric g' on every subbundle ξ' of ξ, provided that g is non-degenerate on (the sub-fibres of) ξ'. It follows that a Finsler splitting of the inclusion of ξ' in ξ is induced in this case.

Thus we have the following:

Proposition 4 *Let ξ' be a vector subbundle of the vector bundle ξ, let g be a Finsler metric on ξ, which is non-degenerate on ξ' and let (N, Γ) be a Finsler connection on ξ. Then a Finsler connection (N', Γ') is induced on ξ'.*

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