Dedicated to the 65'th birthday

of Professor Vasile Crucianu

ON THE SUPPLEMENTARY VECTOR SUBBUNDLES

Marcela Popescu

Abstract

In this paper, two supplementary vector subbundles E' and E'' of a vector bundle E, are studied. Given a non-linear connection C on E, a canonical method to induce non-linear connections C' on E' and C'' on E'' is indicated. Kinds of Gauss and Codazzi equations are given. In the particular case of a linear connection C on E, the method and the equations of Gauss and Codazzi given in [2] are found. The vertical and horizontal lifts defined in the present paper extend the classical ones.

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All the manifolds and maps are C^∞, the manifolds are paracompact and all the vector bundles have finite dimensional vector spaces as fibers. $\mathcal{F}(M)$ is the real algebra of C^∞-real functions on the manifold M, $\mathcal{X}(M)$ and $S(\xi)$ are the $\mathcal{F}(M)$-modules of vector fields on M and of sections on the vector bundle $\xi = (E, \pi, M)$ respectively. $V\xi = \ker \tau\pi$ is the vertical bundle of ξ (where $\tau\pi : \tau E \longrightarrow \tau M$ is the differential map of π) and there is a canonical isomorphism $VE \simeq \pi^* E$.

First we show the basic constructions and results from [3] which are used in the sequel.

Let $\xi = (E, \pi, M)$ be a vector bundle, $\xi' = (E', \pi', M)$ and $\xi'' = (E'', \pi'', M)$ be two supplementary vector subbundles, P' and P'' be the projections of ξ on ξ', ξ''. $I' : \xi' \longrightarrow \xi$ and $I'' : \xi'' \longrightarrow \xi$ be the inclusion morphisms. Consider the

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vector bundles $\eta' = (E, P', E')$ and $\eta'' = (E, P'', E'')$ and their vertical bundles $\mathcal{V}'\xi = \ker \tau P''$ and $\mathcal{V}''\xi = \ker \tau P'$. Since $\pi = \pi' \circ P' = \pi'' \circ P''$, it easily follows that $\mathcal{V}'\xi$ and $\mathcal{V}''\xi$ are vector subbundles of $V\xi$.

Throughout the paper we consider vectorial coordinates on E which are adapted(i.e., local coordinates on E adapted to the vector bundle structure) which induce also on E' and E'' adapted vectorial coordinates. More precisely, around every $y \in E$, $\pi(y) = x$, $P'(y) = y'$, $P''(y) = y''$ we have as adapted coordinates: $x : (x^i)$, $y' : (x^i, y^\alpha)$, $y'' : (x^i, y^u)$, $y : (x^i, y^\alpha, y^u)$, where $i = \overline{1, m}$, $\alpha = \overline{1, k_1}$, $u = \overline{1, k_2}$ and $k = k_1 + k_2$. The change rules are: $x^{i'} = x^{i'}(x^i)$, $y^{\alpha'} = h^{\alpha'}_\alpha(x^i) y^\alpha$, $y^{u'} = h^{u'}_u(x^i) y^u$.

Proposition 1 [3] *a) There are canonical isomorphisms*

$$\mathcal{V}'E \simeq \pi^* E' \quad \text{and} \quad \mathcal{V}''E \simeq \pi^* E''$$

.

b) In every point $u \in E$ we have $(VE)_u = (\mathcal{V}'E)_u \oplus (\mathcal{V}''E)_u$.

c) In every $u' \in E'$ and $u'' \in E''$ we have

$$(\tau I')_{u'}(VE')_{u'} = (\mathcal{V}'E)_{u'}, \quad (\tau I'')_{u''}(VE'')_{u''} = (\mathcal{V}''E)_{u''} .$$

According to b) from Proposition 1, it follows that $\mathcal{V}'E$ and $\mathcal{V}''E$ are supplementary vector subbundles of $V\xi$, and the projectors of these subbundles on $V\xi$ are denoted as Q' and Q''.

Let $C : \tau E \longrightarrow V\xi$ be a non-linear connection on E, i.e. (cf. [2]), a vector bundle morphism such that $C \circ i = \mathrm{id}_{VE}$ where $i : V\xi \longrightarrow \tau E$ is the inclusion morphism and consider the following sequence of vector bundle morphisms:

$$T\xi' \xrightarrow{\tau I'} T\xi \xrightarrow{C} V\xi \xrightarrow{Q'} \mathcal{V}'\xi \xrightarrow{P_1'} V\xi'. \tag{1.1}$$

where $P_1' = \tau P'_{|\mathcal{V}'\xi} : \mathcal{V}'\xi \longrightarrow V\xi'$ is a left inverse of $\tau I'_{|V\xi'}$.

Proposition 2 [3] *$C' = P_1' \circ Q' \circ C \circ \tau I' : TE' \longrightarrow VE'$ is a non-linear connection on the vector bundle E'.*

It is easy to see that, in an adapted vectorial system of coordinates, the local components of C are $(N^\alpha_i(x^j, y^\beta, y^v), N^u_i(x^j, y^\beta, y^v))$. In [3], it is proved that the local components of the induced non-linear connection C' are: $\widetilde{N}^\alpha_j(x^i, y^\beta) = N^\alpha_j(x^i, y^\beta, 0)$. Notice that a non-linear connection C'' can be induced in the same way on ξ'', and the local components of C'' are: $\widetilde{\widetilde{N}}{}^u_j(x^i, y^v) = N^u_j(x^i, 0, y^v)$.

Giving the non-linear connection C and the supplementary vector subbundles ξ' and ξ'' on ξ, it follows that for every $u \in E$ we have:

$$(TE)_u = (HE)_u \oplus (VE)_u = (HE)_u \oplus (\mathcal{V}'E)_u \oplus (\mathcal{V}''E)_u \tag{1.2}$$

Denoting as:

$$(\mathcal{H}'E)_u = (HE)_u \oplus (\mathcal{V}''E)_u \ , \ (\mathcal{H}''E)_u = (HE)_u \oplus (\mathcal{V}'E)_u$$

we have

$$(TE)_u = (\mathcal{H}'E)_u \oplus (\mathcal{V}'E)_u\ ,\ (TE)_u = (\mathcal{H}''E)_u \oplus (\mathcal{V}''E)_u\ .$$

There are defined the vector bundles $\mathcal{H}'\xi$ and $\mathcal{H}''\xi$ of τE which have as supplementary vector subbundles $\mathcal{V}'\xi$ and $\mathcal{V}''\xi$ respectively. Denote the supplementary projectors as $\mathcal{H}'$ and $\mathcal{V}'$, respectively $\mathcal{H}''$ and $\mathcal{V}''$. It is easy to see that $\mathcal{V}' = Q' \circ v,\ \mathcal{V}'' = Q'' \circ v$ where v is the vertical projector associated to the non-linear connection C. In an adapted vectorial system of coordinates, these projectors have the forms:

$$\mathcal{V}'(X) = (X^\alpha + X^i N_i^\alpha)\frac{\partial}{\partial y^\alpha}(= Q' \circ v(X)),$$

$$\mathcal{H}'(X) = X^i\left(\frac{\partial}{\partial x^i} - N_i^\alpha\ \frac{\partial}{\partial y^\alpha}\right) + X^u\ \frac{\partial}{\partial y^u}\ (= X - \mathcal{V}'(X));$$

$$\mathcal{V}''(X) = (X^u + X^i N_i^u)\ \frac{\partial}{\partial y^u}(= Q'' \circ v(X)),$$

$$\mathcal{H}''(X) = X^i\left(\frac{\partial}{\partial x^i} - N_i^u\ \frac{\partial}{\partial y^u}\right) + X^\alpha\ \frac{\partial}{\partial y^\alpha}(= X - \mathcal{V}''(X)),$$

where $X = X^i\ \dfrac{\partial}{\partial x^i} + X^\alpha\ \dfrac{\partial}{\partial y^\alpha} + X^u\ \dfrac{\partial}{\partial y^u}$.

Proposition 3 *[3]Every non-linear connection C on the vector bundle ξ induces non-linear connections on the vector bundles η' and η'' such that the vertical bundle of one of these vector bundles is a subbundle of the horizontal bundle of the connection on the other vector bundle.*

Conversely, every two non-linear connections which have this property, induce a non-linear connection C on ξ.

We shall define now the vertical and horizontal lifts associated to sections on ξ', ξ'', $\tau E'$ and $\tau E''$.

Let $s' \in S(\xi')$. Since the π'-morphism $P' : E \longrightarrow E'$ of vector bundles $E \xrightarrow{P''} E''$ and $E' \xrightarrow{\pi'} M$ is an epimorphism and an isomorphism on fibers, it follows, using this isomorphism, that there is an unique section $\tilde{s}'$ on the vector bundle $E \xrightarrow{P''} E''$ such that $P'(\tilde{s}') = s'$.

We can consider the vertical lift of $\tilde{s}'$ in the vector bundle $E \xrightarrow{P''} E''$ denoted as $(s')^{\mathcal{V}'} \in S(\mathcal{V}'\xi)$ and called the ξ''-vertical lift of the section s'.

In the same way we can define the ξ'-vertical lift $(s'')^{\mathcal{V}''} \in S(\mathcal{V}''\xi)$ of a section $s'' \in S(\xi'')$.

In an adapted vectorial system of coordinates, the sections and the vertical lifts have the same components in the adapted bases.

It is easy to see that for $s \in S(\xi)$, denoting as $s' = P'(s)$ and $s'' = P''(s)$ $(s = s' + s'')$ and considering the vertical lift s^V of s (cf. [2]), we have: $s^V = (s')^{\mathcal{V}'} + (s'')^{\mathcal{V}''}$.

In the particular case when $\xi = \xi'$ and ξ'' is the null vector bundle, then the ξ'-vertical lift of a section $s \in S(\xi)$ is the same as the vertical lift of s.

We define now horizontal lifts of vector fields on E' and E'' with respect to the non-linear connections defined in the first part of Proposition 3. For every $X'' \in \mathcal{X}(E'')$, $X' \in \mathcal{X}(E')$, we denote $(X'')^{\mathcal{H}'} \in S(\mathcal{H}'\xi)$, $(X')^{\mathcal{H}''} \in S(\mathcal{H}''\xi)$, and we call them the ξ'-horizontal lift and ξ''-horizontal lift of X'' and X', respectively. In an adapted system of coordinates we have:

$$(X'')^{\mathcal{H}'} = X^i(x^j, y^v)\left(\frac{\partial}{\partial x^i} - N_i^\alpha(x^j, y^\beta, y^v)\,\frac{\partial}{\partial y^\alpha}\right) +$$

$$+X^u(x^j, y^v)\,\frac{\partial}{\partial y^u} \in S(\mathcal{H}'E)$$

where

$$X'' = X^i(x^j, y^v)\,\frac{\partial}{\partial x^i} + X^u(x^j, y^v)\,\frac{\partial}{\partial y^u} \in \mathcal{X}(E'')$$

and

$$(X')^{\mathcal{H}''} = X^i(x^j, y^\beta)\cdot\left(\frac{\partial}{\partial x^i} - N_i^u(x^j, y^\beta, y^v)\,\frac{\partial}{\partial y^u}\right) +$$

$$+X^\alpha(x^j, y^\beta)\,\frac{\partial}{\partial y^\alpha} \in S(\mathcal{H}''E),$$

where

$$X' = X^i(x^j, y^\beta)\,\frac{\partial}{\partial x^i} + X^\alpha(x^j, y^\beta)\,\frac{\partial}{\partial y^\alpha} \in \mathcal{X}(E').$$

In the particulary case when $\xi = \xi'$ and ξ'' is the null vector bundle, then the ξ''-horizontal lift of $X' \in \mathcal{X}(M)$ is the same as the horizontal lift of X', and the ξ'-horizontal lift of $X'' \in \mathcal{X}(E)$ is X''.

Generally, for a vector bundle $\xi = (E, \pi, M)$ and $M' \subset M$ a submanifold of M, we denote as $\xi_{|M'} = i^*\xi$, and for $s \in S(\xi)$ we denote as $s_{|M'}$ the induced section on $\xi_{|M'}$. With these notations we have:

Proposition 4 . *a) If $Y'' \in S(V\xi'')$ then*

$$(Y'')^{\mathcal{H}'}_{|E''} = \tau I''(Y'').$$

Particularly if $Y \in S(\xi'')$ then

$$(Y^{V''})^{\mathcal{H}'}_{|E''} = \tau I''(Y^{V''})$$

.

b) If $X \in \mathcal{X}(M)$, then

$$\left((X^{h''})^{\mathcal{H}'}\right)_{|E''} = X^h_{|E''}$$

.

c) If $X'' \in \mathcal{X}(E'')$, then for every $u'' \in E''$ we have

$$\mathcal{V}''_{u''}\left((X'')^{\mathcal{H}'}\right)_{u''} = (\tau I'')_{u''}(V''X'')_{u''} \ .$$

Proof. Using an adapted vectorial system of coordinates, the local expression of $Y'' = X^u(x^j, y^v)\ \dfrac{\partial}{\partial y^u}$ we have $(Y'')^{\mathcal{H}'} = X^u(x^j, y^v)\ \dfrac{\partial}{\partial y^u}$ and the first equality follows since $\tau I''$ sends $\dfrac{\partial}{\partial y^u}$ in $\dfrac{\partial}{\partial y^u}$. If $X = X^i(x^j)\dfrac{\partial}{\partial x^i}$ both sides of b) are equal to

$$X^{h''} = X^i(x^j)\left(\frac{\partial}{\partial x^i} - N_i^\alpha(x^j, y^\alpha, 0)\ \frac{\partial}{\partial y^\alpha} - N_i^u(x^j, y^\alpha, 0)\ \frac{\partial}{\partial y^u}\right),$$

because on E'' we have $y^\alpha = 0$. For the last assertion, taking $X'' = X^i(x^j, y^v)\dfrac{\partial}{\partial x^i} + X^u(x^j, y^v)\ \dfrac{\partial}{\partial y^u} \in \mathcal{X}(E'')$ and using the local form of V'', we have:

$$\mathcal{V}''_{u''}\left((X'')^{\mathcal{H}'}\right)_{u''} = \left(X^u(x^j, y^v) + X^i(x^j, y^v)N_i^u(x^j, 0, y^v)\right)\ \frac{\partial}{\partial y^u} =$$

$$= (\tau I'')_{u''}(V''X'')_{u''}\ .$$

(q.e.d.)

Proposition 5 . *For every $X, Y \in \mathcal{X}(M)$ we have:*

$$\Omega(X^h, Y^h)_{|E''} = \tau I''(\Omega''(X^{h''}, Y^{h''})) + ([X^{h''}, Y^{h''}]^{\mathcal{H}'} - [X^h, Y^h])_{|E''} \qquad (1)$$

$$\Omega(X^h, Y^h)_{|E'} = \tau I'(\Omega'(X^{h'}, Y^{h'})) + ([X^{h'}, Y^{h'}]^{\mathcal{H}''} - [X^h, Y^h])_{|E'}\ , \qquad (2)$$

where Ω, Ω' and Ω'' are the curvatures of the connections C, C' and C'' respectively.

Proof. Using

$$\Omega(X^h, Y^h) = [X, Y]^h - [X^h, Y^h]\ ,$$

$(\forall)\ X, Y \in \mathcal{X}(M)$ (see [2]) for C'' we have:

$$\Omega''(X^{h''}, Y^{h''})^{\mathcal{H}'} = ([X, Y]^{h''})^{\mathcal{H}'} - [X^{h''}, Y^{h''}]^{\mathcal{H}'}$$

Using Proposition 4, it follows:

$$\tau I''(\Omega''(X^{h''}, Y^{h''}) = [X, Y]^h_{|E''} - ([X^{h''}, Y^{h''}]^{\mathcal{H}'})_{|E''} =$$

$$= \Omega(X^h, Y^h)_{|E''} + ([X^h, Y^h] - [X^{h''}, Y^{h''}]^{\mathcal{H}'})_{|E''}$$

(this holds on the fibers of $VE_{|E''}$). (q.e.d.)

If we apply $\mathcal{V}''_{|E''}$ in (1) and using Proposition 1 b) and c), it follows:

$$(\mathcal{V}''\Omega(X^h, Y^h))_{|E''} = \tau I''(\Omega''(X^{h''}, Y^{h''})) + \mathcal{V}''([X^{h''}, Y^{h''}]^{\mathcal{H}'} - [X^h, Y^h])_{|E''}\ ,$$

which we call the vectorial equation of Gauss on E''. If we apply $\mathcal{V}'_{|E''}$, then we obtain the relation:

$$(\mathcal{V}'\Omega(X^h, Y^h))_{|E''} = -\mathcal{V}'([X^h, Y^h])_{|E''}$$

which we call the vectorial equation of Codazzi on E''. In an analogous way, we define the vectorial equations of Gauss and Codazzi on E' as:

$$(\mathcal{V}'\Omega(X^h,Y^h))_{|E'} = \tau I'(\Omega'(X^{h'},Y^{h'})) + \mathcal{V}'([X^{h'},Y^{h'}]^{\mathcal{H}''} - [X^h,Y^h])_{|E'}$$

$$(\mathcal{V}''\Omega(X^h,Y^h))_{|E'} = -\mathcal{V}''([X^h,Y^h])_{|E'}.$$

In adapted vectorial coordinates we give them a more simple form and we show that these equations are a natural extension of those of [1], denoting as in [2]:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^\alpha\,\frac{\partial}{\partial y^\alpha} - N_i^u\,\frac{\partial}{\partial y^u}$$

$$\frac{\delta'}{\delta' x^i} = \frac{\partial}{\partial x^i} - \widetilde{N}_i^\alpha\,\frac{\partial}{\partial y^\alpha},\ \widetilde{N}_i^\alpha(x^j,y^\beta) = N_i^\alpha(x^j,y^\beta,0)$$

$$\frac{\delta''}{\delta'' x^i} = \frac{\partial}{\partial x^i} - \widetilde{\widetilde{N}}{}_i^u\,\frac{\partial}{\partial y^u},\ \widetilde{\widetilde{N}}{}_i^u(x^j,y^v) = N_i^u(x^j,0,y^v)$$

$$\Omega\left(\frac{\delta}{\delta x^i},\,\frac{\delta}{\delta x^j}\right) = \Omega_{ij}^\alpha\,\frac{\partial}{\partial y^\alpha} + \Omega_{ij}^u\,\frac{\partial}{\partial y^u}\,;\ \widetilde{\Omega}\left(\frac{\delta'}{\delta' x^i},\,\frac{\delta'}{\delta' x^j}\right) = \widetilde{\Omega}_{ij}^\alpha\,\frac{\partial}{\partial y^\alpha}$$

$$\widetilde{\widetilde{\Omega}}\left(\frac{\delta''}{\delta'' x^i},\,\frac{\delta''}{\delta'' x^j}\right) = \widetilde{\widetilde{\Omega}}{}_{ij}^u\,\frac{\partial}{\partial y^u}.$$

By a straightforward computation we obtain:

$$\Omega_{ij}^u(x^k,0,y^v) = \widetilde{\widetilde{\Omega}}{}_{ij}^u(x^k,y^v) + N_j^\alpha(x^k,0,y^v)\cdot N_{i,\alpha}^u(x^k,0,y^v) -$$
$$-N_i^\alpha(x^k,0,y^v)\cdot N_{j,\alpha}^u(x^k,0,y^v) \qquad \text{(Gauss)}$$

$$\Omega_{ij}^\alpha(x^k,0,y^v) = \frac{\delta}{\delta x^i}(N_j^\alpha)(x^k,0,y^v) - \frac{\delta}{\delta x^j}\,(N_i^\alpha)(x^k,0,y^v) \qquad \text{(Codazzi)}$$

and the analogous ones for E'.

If C is a linear connection, then C' and C'' are also linear connections, and the above equations of Gauss and Codazzi agree with those of [1].

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Author's address:

Marcela Popescu
Department of Mathematics, University of Craiova,
13, Al. I. Cuza street, Craiova, 1100, Romania,