A CONTRAVARIANT APPROACH IN (NON)COMMUTATIVE GEOMETRY

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Abstract

A non-commutative version of the algebraic theory which arise from the study of vector bundles is considered in this paper. The categories of modules with differentials over associative algebras are defined.

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The geometry gives rise to several algebraic theories. Sometimes, the algebraic version is dificult to be constructed, since the geometrical functors are not pure algebraic. A such theory is schetch below in the non-commutative case, following the commutative case from [6]. It seems to be an alternative of the well-known non-commutative geometry of A. Connes and M. Dubois-Violette. More precisely, our treatement is a contravariant approach and the classical one is covariant.

We begin with some algebraic facts from the geometry. In [3] it is constructed the algebraic form of a morphism of vector bundles (see also [4]): Denoting as $\theta = (E, p, M)$ a vector bundle and considering a vector bundle morphism $\theta' \stackrel{(f,g)}{\longrightarrow} \theta$, where $f : M' \to M$ and $g : E' \to E$, then a canonical map $\Gamma(\theta') \to \Gamma(\theta)$ can not be induced. But, considering the induced morphism of vector bundles over the same base $g^* \stackrel{not}{=} f^*g : f^*\theta' \to \theta$ and a canonical isomorphism $\Gamma(f^*\theta') \cong \mathcal{F}(M') \otimes_{\mathcal{F}(M)} \Gamma(\theta')$, then a morphism of $\mathcal{F}(M')$-modules $\Gamma(\theta') \to \mathcal{F}(M') \otimes_{\mathcal{F}(M)} \Gamma(\theta')$ is induced. For a vector bundle θ consider the module $(\mathcal{F}(M), \Gamma(\theta))$ and for a vector bundle morphism $\theta' \stackrel{(f,g)}{\longrightarrow} \theta$, consider $(\mathcal{F}(M'), \Gamma(\theta')) \stackrel{(f^*,g^*)}{\longrightarrow} (\mathcal{F}(M), \Gamma(\theta))$, which is a morphism from an unusual category of modules, called in [6] a *contravariant category of modules.* This category was considerated in [5], in order to study the algebraic properties of Lie algebroids and Lie pseudoalgebras. Using [4] these ideas were continued in [6, 7], where the algebraic study is extended to categories of vector bundles with differentials

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and categories of modules with differentials. The goal of this paper is to show that considering the modules not only on commutative, but over associative algebras, the basic definitions at least are almost the same. In the non-commutative case we must consider not only modules, but also bimodules (see [2]).

We define now the contravariant categories of modules and bimodules.

The modules with arrows, the (pre)infinitesimal modules and Lie pseudoalgebras over commutative algebras are studied by de Barros and J. Pradines in the seventh decade and recently by K. Mackenzie, P.J. Higgins (see [5]) and the author [6, 7]. These modules appear naturally, in a functorial manner, in the study of the algebroids and groupoids.

Consider an associative algebra A over a field k. If M is an A-module, we say that (A, M) is a module (on left or on right). If M is an A-bimodule, we say that (A, M) is a bimodule. We denote as $Z(A)$ the center of A.

Consider two left modules (A', M') and (A, M). A *contravariant morphism of left modules* is a couple (φ, ψ), where $\varphi : A \to A'$ is a morphism of algebras such that $\varphi(Z(A)) \subset Z(A')$ and $\psi : M' \to A' \otimes_{Z(A)} M$ is a morphism of left A'-modules.

In an analogous way is defined the *contravariant morphism of right modules.*

Consider two bimodules (A', M') and (A, M). A *contravariant morphism of bimodules* is a couple (φ, ψ), where $\varphi : A \to A'$ is a morphism of algebras and $\psi : M' \to A' \otimes_{Z(A)} M \ {}_{Z(A)} \otimes A'$is a morphism of A'-bimodules.

In all the above cases, for every $m' \in M$, we say that $\psi(m')$ is the ψ-decomposition of m.

For commutative algebras, the definition of a contravariant morphism of left or right modules (which are the same) is the very definition from [4, 6].

In the sequel we denote as $\mathcal{A}$ the category of associative algebras, or one of its subcategories. As in [6] it can be proved

Theorem 1 *The left modules (right modules, respectively bimodules) (A, M) with A an object from $\mathcal{A}$ and their contravariant morphisms are the objects and the morphisms of a category $\mathcal{M}^l_A$ ($\mathcal{M}^r_A$, respectively $\mathcal{M}^b_A$).*

We say that a left module (A, M) is a *left module with arrow* (l.m.w.a.) if there is given a morphisme of left modules $p^M : M \to Der(A)$, called an *anchor*. We denote $p^M(m)(a) = [m, a]_M$ for every $m \in M$ and $a \in A$.

In an analogous way is defined the right module with arrow (r.m.w.a.), respectively bimodule with arrow (b.w.a.).

Let (A', M') and (A, M) be l.m.w.a.'s. A contravariant morphism of left module (φ, ψ) is a *morphism of l.m.w.a.* if it is a contravariant morphism of left modules, such that for every $X' \in M'$ which has the ψ-decomposition

$$\psi(X') = \sum_i a'_i \otimes_{Z(A)} X_i$$

then, for every $a \in A$, we have $[X', \varphi(a)] = \sum_i a'_i \varphi([X_i, a])$.

In an analogous way it can be defined the morphism of r.m.w.a. and of b.m.w.a.. The definitions are correct; it can be checked up as in [6, Lemma 3.1].

A *preinfinitesimal left module* (p.l.m.) is a l.m.w.a. (A, M) together with a bracket $[\cdot,\cdot]_M : M \times M \to M$ which is k-bilinear, antisymmetric and

$$[X, aY]_M = [X, a]_M Y + a [X, Y]_M \quad , \quad (\forall) X, Y \in M,\ a \in A.$$

Let (A', M') and (A, M) be p.l.m.'s. A contravariant morphism of l.m.w.a. (φ, ψ) is a *morphism of p.l.m.* if it is a contravariant morphism of left modules and for every $x', y' \in M'$ which has the ψ-decompositions

$$\psi(X') = \sum_i a'_i \otimes_{Z(A)} X_i \quad , \quad \psi(Y') = \sum_\alpha b'_\alpha \otimes_{Z(A)} Y_\alpha$$

then we have

$$\psi([X', Y']_M) = \sum_\alpha [X', b'_\alpha]_M \otimes_{Z(A)} Y_\alpha - \sum_i [Y', a'_i]_M \otimes_{Z(A)} X_i + \sum_{i,\alpha} a'_i b'_\alpha \otimes_{Z(A)} [X_i, Y_\alpha]_M$$

In an analogous way it can be defined the morphism of p.l.m. and of p.b.m.. The definitions are correct; it can be checked up as in [6, Lemmas 4.1, 4.2].

Let (A', M') and (A, M) be p.l.m.'s and $(A', M') \xrightarrow{(\varphi,\psi)} (A, M)$ a morphism of l.m.w.a.. The *curvature* of (φ, ψ) is the map $K : M' \times M' \to A' \otimes_{Z(A)} M$ defined by

$$K(X', Y') = \psi([X', Y']_M) - \sum_\alpha [X', b'_\alpha]_M \otimes_{Z(A)} Y_\alpha + \sum_i [Y', a'_i]_M \otimes_{Z(A)} X_i - \sum_{i,\alpha} a'_i b'_\alpha \otimes_{Z(A)} [X_i, Y_\alpha]_M \ .$$

It is clear that (φ, ψ) is a morphism of preinfinitesimal module iff K vanish. The curvature of a morphism of r.m.w.a. (or b.m.w.a.) of two p.r.m.'s (respectively p.b.m.'s) can be defined in a similar way. It vanishes iff it is a morphism of p.r.m. (p.b.m. respectively). In the case of commutative algebras the definition agrees with [6, Proposition 4.1].

In the case of $A = A'$, a morphism of left A-modules $\psi_0 : M' \to M$ induces a morphism $\psi : M' \to A \otimes_{Z(A)} M$, $\psi(X') = 1_A \otimes_{Z(A)} \psi_0(X')$. If (A, M') and (A, M) are l.m.w.a.'s and $[X', \varphi(a)] = \varphi([\psi_0(X'), a])$ we say that ψ is a strong morphism of A-l.m.w.a.. It induces a morphism of l.m.w.a. as above. The curvature of a strong morphism of A-l.m.w.a. as above is $K_0(X', Y') = [\psi(X'), \psi(Y')]_M - \psi([X', Y']_{M'})$. An *infinitesimal left module* (i.l.m.) is a p.l.m. (A, M), such that the anchor p^M : $M \to Der(A)$ is a morphism of p.i.m. with a vanishimg curvature (i.e. $p^M([X, Y]_M) - \left[p^M(X), p^M(Y)\right]_{Der(A)} = 0$). A *left Lie pseudoalgebra* (l.L.p.a.) is an i.l.m. which has the property $\mathcal{J}(X, Y, Z) \stackrel{def}{=} [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$. $\mathcal{J}$ is called the Jacobi map.

In an analogous way we can define the infinitesimal right module (i.r.m.), the infinitesimal bimodule (i.b.m.), the right Lie pseudoalgebra (r.L.p.a.) and the Lie bi-pseudoalgebra (L.b.p.a.).

All these modules, together with their morphisms, are the objects and the morphisms of some categories, called as in [6] *categories of modules with differentials*. The study of these categories will be done in subsequent papers. Almost all the results from [6, 7], stated for the associative and commutative algebras, can be extended with care for only associative algebras.

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