Dedicated to all romanian geometers

ISOTOPIC LIFTINGS OF THE PYTHAGOREAN THEOREM, TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Ruggero Maria Santilli

Abstract

In this note we present the rudiments of the nonlinear, nonlocal- integral and noncanonical- nonhamiltonian, yet axiom- preserving isotopic liftings of the Pythagorean theorem, trigonometric and hyperbolic functions for the simplest possible liftings of Kadeisvill's Class I; we study some of their properties including the unification of trigonometric and hyperbolic functions via the isotopies of Class III; and we identify a number intriguing open geometrical problems. The note, written by a physicist, is intended to illustrate that the removal of the current restriction in effect since biblical times of the entire mathematical knowledge to the simplest conceivable unit $+1$, and the use of structurally more general units, imply a rather vast broadening of all mathematical, beginning with the simplest possible notions of angles, triangles, and ordinary functions, and then passing to all remaining mathematical structures, with basically novel applications in a variety of disciplines.

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1 Foreword

This note is devoted to the so-called *isotopies* which are nonlinear, nonlocal - integral and noncanonical-nonhamiltonian maps of any given linear, local and hamiltonian structure, yet they are *axiom-preserving* in the sense of being able to reconstruct linearity, locality and canonicity in certain generalized spaces and fields called isospaces

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and isofields. To avoid unnecessary length, we refer the reader to [1] for all background notions in isotopies and to [2, 3] for aspects pertaining to algebras, topologies, and manifolds.

It was pointed out in Vol. I, ref. [1], that the conventional Pythagorean theorem, the notion of angle, the trigonometric and hyperbolic functions and other familiar methopds are inapplicable under isotopies for numerous independent reasons, such as : the loss of the conventional unit I in favor of generalized units I with an arbitrary, nonlinear and integro-differential dependence on local quantities and their derivatives ; the inapplicability of the Euclidean distance; the generally curved character of the lines which prohibit the preservation of conventional angles; etc.

In this note, written by a physicist, we study the rudiments of the isotopic liftings of the Pythagorean theorem, trigonometric and hyperbolic functions which were preliminarily studied the first time in Appendix 6. A, Vol. I, ref. [1], under the respective names of Isopythagorean Theeorem, isotrigonometric and isohyperbolic. These generalizations are a necessary pre-requisite for : the isotopies of the Legendre functions, spherical harmonics, and other special functions; the study of the isorepresentation theory of the Lie-Santilli isogroup O(3) [2] (the most general known nonlinear, nonlocal and noncanonical realization of the rotation group; the application to a scattering theory capable of incorporating the conventional action-at-a-distance, potential interactions as well as additional contact, nonpotential effects due to the extended, nonspherical and deformable character of the colliding particles (for applications, see Vol. II, ref. [1]).

All symbols and conventions of [1] will be preserved for clarity in the comparison of the results. For instance, the symbols, etc. denote quantities computed in isospace and etc, denote their projection in the original space.

2 Isopythagorean Theorem.

Consider a conventional two-dimensional Euclidean space $E = E(r, \theta, \mathbf{R})$ with contravariant coordinates $r = (r^k) = (x, y)$ and metric $\delta = diag.(1, 1)$ over the field $\mathbf{R} = \mathbf{R}(n, +, *)$ of real numbers n with conventional sum $+$ and multiplication $*$ and respective additive unit 0 and multiplicative unit 1. The fundamental notion of this space is the assumption of the basic unit $1 = diag.\ (1, 1)$ wich implies the assumption of the same basic (dimensionless) unit $+1$ for both $x-$ and $y \not- axes$, resulting in the familiar *Euclidean distance* among two points $x, y \in E$

$$D = [(x_1 - x_2)(x_1 - x_2) + (y_1 - y_2)(y_1 - y_2)]^{1/2} \in \mathbf{R}(n, +, *) \tag{1}$$

The quantity $D^2 = D * D$, $* \in \mathbf{R}$, then represents the celebrated *Pythagorean theorem* expressing the hypothenuse D of a right triangle with sides A and B according to the familiar law $D^2 = A^2 + B^2$.

The flat geometry of the plane $\mathbf{R}(n, +, *)$ permits the introduction of the trigonometric notion of " angle α" between two intersecting straight vectors, and of "cosinus of α" which, for the case when the vectors initiate at the origin $0 \in E$ and go to two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by

$$\cos\alpha = \frac{x_1x_2 + y_1y_2}{(x_1x_1 + y_1y_1)^{1/2}\,(x_2x_2 + y_2y_2)^{1/2}} \tag{2}$$

From the above definition one can derive the entire conventional trigonometry. For instance, by assuming that the points are on a circle of unit radius $D = 1$, for $P_1(x_1, y_1)$ and $P_2(1, 0)$ we have $\cos\alpha = x_1$, for $P_1(x_1, y_1)$ and $P_2(0, 1)$ we have $\sin\alpha = y_1$, with consequential familiar properties, such as $\sin^2\alpha + \cos^2 = 1$, etc.

Consider now the *two-dimensional isoeuclidean space of Kadeisvili's Class 1*, $\hat{E} = \hat{E}(\hat{r}, \delta, \hat{R})$ (Sect 1.3.3 of [1]) over the isofield of isoreal numbers $\hat{n} = n \times \hat{1}$, where $n \in R$ and, from the Class 1 condition, $\hat{1}$ is a positive-definite matrix whose elements have a well behaved but otherwise arbitrary dependence on time t, the local coordinates r and their derivatives of arbitrary order $\hat{1} = \hat{1}(t, \dot{r}, \ddot{r}, \ldots)$ equipped with the conventional sum + and additive unit 0, and with the isomultiplication $\hat{n} \times \hat{m} = \hat{n} \times \hat{1} \times \hat{m}$. Under the condition assumed herein $\hat{1} = \hat{T}^{-1}$, $\hat{1}$ is the correct left and right unit of $\hat{R}$, called isounit, $\hat{1}\hat{\times}\hat{n} = \hat{n}\hat{\times}\hat{1} = \hat{n}$, $\forall\hat{n} \in \hat{R}$, $\hat{T}$ is called the isotopic element, and $\hat{R}$ satisfies all conditions to be a field [1].

The realization of $\hat{E}$ studied in this note is the simplest possible one of Class I, that with diagonal isounit, of the type

$$\begin{gathered} \hat{E} = \hat{E}(\hat{r}, \hat{\delta}, \hat{R}) = (\hat{r}^k) = (\hat{x}, \hat{y}) \equiv (r^k) = (x, y),\ \hat{r}_k = \\ \hat{\delta}_{ki}\hat{r}^i \neq r_k = \delta_{ki}r^i, \qquad (a) \\ \hat{\delta} = \hat{T}(t, \dot{r}, \ddot{r}, \ldots)\delta = diag.(b_1^2, b_2^2),\ b_k = b_k(t, \dot{r}, \ddot{r}, \ldots) > 0 \quad (b) \\ \hat{1} = \hat{T}^{-1} = diag.(b_1^{-2}, b_2^{-2}),\ k = 1, 2. \qquad (c) \end{gathered} \tag{3}$$

As one can see, the isospace is constructed via the most general possible signature-preserving deformation of the original metric δ via a positive-definite, but otherwise arbitrary 2×2 matrix $\hat{T}$, $\delta \to \hat{\delta} = \hat{T}\delta$, while jointly deforming the original two-dimensional unit by an amount which is the *inverse* of the deformation of the metric, $1 \to \hat{1} = \hat{T}^{-1}$. This mechanism permits the preservations under isotopies of all axioms of the Euclidean geometry, to such an extend that the Euclidean and isoeuclidean geometry coincide at the abstract level [1]. Alternatively, we can say that the isotopies permit the use of the most general possible functional dependence of the metric $\hat{\delta}(t, \dot{r}, \ddot{r}, \ldots)$ while preserving all Euclidean axioms, including that of flatness, caled *isoflatness*. Note also that the *isounit* of the *base isofield concides with that of the isospace.*

The central notion of the isoeuclidean plane is the assumption of new (dimensionless) units, the quantitties b_1^{-2} for the $\hat{x}$-axix and b_2^{-2} for the $\hat{y}$-axis. Thus, not only the unit is now different than +1, but different axes have different units and, in addition, each of them is a function of the local variables.

Consider now two points $P_1(\hat{x}_1, \hat{y}_1)$ and $P_2(\hat{x}_2, \hat{y}_2) \in \hat{E}(\hat{r}, \hat{\delta}, \hat{R})$. Then the conventional distance is (uniquely) generalized into the *isoeuclidean distance* [1]

$$\hat{D} = \left[(x_1 - x_2)b_1^2(x_1 - x_2) + (y_1 - y_2)b_2^2(y_1 - y_2)\right]^{1/2} * \hat{1} \in \hat{R} \tag{4}$$

were one should note the final (ordinary) multiplication by $\hat{1}$ as a necessary condition for $\hat{D}$ to be an element of the isofield $\hat{R}$.

Despite the visible difference between D and $\hat{D}$, all conventional notions in E are preserved under isotopies *provided* that they are computed in $\hat{E}$ over $\hat{R}$. in this way, we have the nitions of *isolines, isotraight line, isotriangle, isostraight triangle, etc.*

By using notation indicated in Sect. 1, we then have the following:

Theorem 2.1 *(* **Isopythagorean theorem** *) [1] : The following property holds in the isoeuclidean plane $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ of Class I, Eq. (3),*

$$\hat{D} = \hat{D}\hat{\times}\hat{D} = \hat{A}^2 + \hat{B}^2 = \hat{A}\hat{\times}\hat{A} + \hat{B}\hat{\times}\hat{B} \in \hat{R}, \tag{5}$$

with projection in the conventional plane $E(r,\delta,R)$

$$\hat{D}^2 = \left[Ab_1^2(t,\dot{r},\ddot{r},\ldots)A + Bb_2^2(t,\dot{r},\ddot{r},\ldots)B\right] \times \hat{1}, \tag{6}$$

that is, the isosquare of the isohypothenuse of an isoright isotriangle is the sum of the isosquare of the isosides.

To understand the geometric mwaning of the above theorem, we recall that all isotopic nitions have, in general, three different interpretations [1], the first in isospace $\hat{E}(\hat{r},\hat{\delta},\hat{R})$, the second via the projection in the original space $(E(r,\delta,R)$, and the third in a conventional Euclidean space $E(\bar{r},\delta,R)$ over the conventional reals $R(n,+,*)$ whose interval coincides with that in isospace. The latter condition is easily verified by the the assumption

$$\bar{x} = \hat{x}b_1(t,x,y,\dot{x},\dot{y},\ldots), \quad \bar{y} = \hat{x}b_2(t,x,y,\dot{x},\dot{y},\ldots)$$

under which

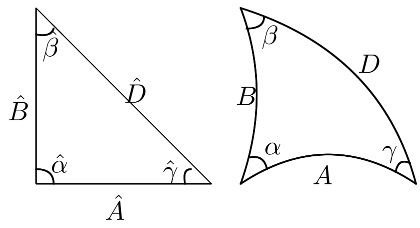
$$\left[(x_1-x_2)b_1^2(x_1-x_2) + (y_1-y_2)b_2^2(y_1-y_2)\right]^{1/2} \equiv$$
$$\left[(\bar{x}_1-\bar{x}_2)b_1^2(\bar{x}_1-\bar{x}_2) + (\bar{y}_1-\bar{y}_2)b_2^2(\bar{y}_1-\bar{y}_2)\right]^{1/2}.$$

The properties in isospace follow the general rules of all isotopies, that is, the preservation of all original properties, including their numerical values. Thus, straight lines in conventional space are mapped into *isostraight isolines* in isospace, i.e. lines which coincide with their tangent when computed in isospace; perpendicular lines in conventional space are mapped into *isoperpendicular isolines* whose angle is indeed 90° when measured in isospace, that is, with respect to its own isounit (see below); etc.

In this sense, a right triangle in the conventional plane remains so in isoplane, and the conventional Pythagorean Theorem holds also in isospace.

To understand the remaining geometric meaning we also have to consider the projection of Theorem 2.1 in the original Euclidean plane . Recall [1] that the isotopic lifting of the circle C in E yields the so-called isocircle $\hat{C}$ in $\hat{E}$ which preserves the original geometric character and value of the radius. This is due to the main mechanism of isotopies according to which the original semiaxes of the circle are lifted to arbitrary values $l_k \to b_k^2(t, r, \ddot{r}, \ldots)$, $k = 1, 2$, yielding an ellipse in conventional space E. But jointly the units of each axis are deformed in an amount *inverse* of the deformation of the semiaxes, $l_k \to b_k^{-2}(t, r, \ddot{r}, \ldots)$,, and this implies the preservation of the perfect circle *in isospace over isofields*.

ISOPYTHAGOREAN THEOREM



(a) (b)

Figure I. A schematic view of the *Isopythagorean Theorem*, first identified in ... for an *isoright isotriangle* as in Diag. (a) i.e, a triangle in isoeuclidean plane $\hat{E}(\hat{r}, \hat{\delta}, \hat{R})$ (isotriangle) with a 90° angle masured with respect to its own isounit (*isoright angle*-see below for irs identification, and its projection in the conventional plane $E(r, \delta, R)$ given by the Diag. (b).

We also recall that isotopic maps are not transitive, in the sense that the lifting of the circle C on into the isocircle $\hat{C}$ on $\hat{E}$ is axiom-preserving, but the projection of the isocircle $\hat{C}$ on the original space E is not, being in fact an ellipse, because such a projection does not imply the return to the original unit $1 = diag.(1, 1)$.

By using the reformulation in conventional space $\hat{E}$, it is easy to see that lines which are straight in become curved in $\hat{E}(\hat{r}, \hat{\delta}, \hat{R})$, according to the rule:

$$\hat{a}\hat{\times}\hat{x} + \hat{b}\hat{\times}\hat{y} + \hat{c} = 0 \to$$

$$\to a\bar{\times}b_1^{-1}(t, x, y, \ldots) + b\bar{y}b_2^{-1}(t, x, y, \ldots) = 0,\ \hat{a}, \hat{b}, \hat{c} \in \hat{R}. \qquad (7)$$

The projection of the Isopythagorean Theorem in a conventional plane then results in the map of a right triangle into a geometric figure in which the sides are curved, with one intersection per pair as in Figure I.

A conjecture on the *Inverse Isopythagorean Theorem* is presented in the concluding remarks.

3 Isotrigonometric functions.

Let us use again the convention according to which the symbols $\hat{a}$, $\hat{b}$, $\hat{c}$, etc, denote quantities computed in isospace $\hat{E}(\hat{r}, \hat{\delta}, \hat{R})$ the symbols $\bar{a}$, $\bar{b}$, $\bar{c}$, etc, denote corresponding quantities when computed in the plane $E(\bar{r}, \delta, R)$, and the symbols a, x, y, etc, denote the projection in the conventional space $E(r, \delta, R)$.

Suppose that the two points $P_1(\hat{x}_1,\hat{y}_1)$ and $P_2(\hat{x}_2,\hat{y}_2)$ represent isostraight *isovectors* initiating from the origin $\hat{0} \in \hat{E}(\hat{r},\hat{\delta},\hat{R})$. Let us denote with $\hat{\alpha}$ the *isoangle* between these two isovectors to be identified below. Consider their identical reformulation in the conventional space $E(\bar{r},\delta,R)$, in which case the angle persists. We can then introduce the conventional $\cos\hat{\alpha}$ in $\hat{E}(\hat{r},\hat{\delta},\hat{R})$

$$\cos\hat{\alpha} = \frac{\bar{x}_1\bar{x}_2 + \bar{y}_1\bar{y}_2}{(\bar{x}_1\bar{x}_1 + \bar{y}_1\bar{y}_1)^{1/2}\,(\bar{x}_2\bar{y}_2 + \bar{y}_2\bar{y}_2)^{1/2}} \tag{8}$$

with projection in $E(r,\delta,R)$

$$\cos\hat{\alpha} = \frac{x_1 b_1^2 x_2 + y_1 b_2^2 y_2}{(x_1 b_1^2 x_1 + y_1 b_2^2 y_1)^{1/2}\,(x_2 b_1^2 x_2 + y_2 b_2^2 y_2)^{1/2}} \tag{9}$$

We now assume that the points $P_1(\hat{x}_1,\hat{y}_1)$ and $P_2(\hat{x}_2,\hat{y}_2)$ are on the unit *isocircle*

$$\begin{array}{c} \hat{D} = (x b_1^2 x + y b_2^2 y)\times\hat{1} = \hat{1}, \quad i.e. \quad (a) \\ x b_1^2 x + y b_2^2 y = 1 \qquad (b) \end{array} \tag{10}$$

wich imply that for $y=0$, $x=b_1^{-1}$ and for $x=0$, $y=b_2^{-1}$.

Definition 3.1 By assuming the points $P_1(\hat{x}_1,\hat{y}_1)$ and $P_2(b_1^{-1},0)$, we have (for $0 < \hat{\alpha} < \pi/2$)

$$\cos\hat{\alpha} = x_1 b_1, \tag{11}$$

and for the points $P_1(\hat{x}_1,\hat{y}_1)$ and $P_2(0,b_2^{-1})$ we have

$$\sin\hat{\alpha} = y_1 b_2 \tag{12}$$

Definition 3.2 The "*isosinus*", "*isocosinus*" and other *isotrigonometric functions* on the isoeuclidian plane $E(\bar{r},\delta,R)$ are defined by (for $0 < \hat{\alpha} < \pi/2$)

$$\begin{array}{c} isosin\,\hat{\alpha} = b_2^{-1}\sin\hat{\alpha}, \quad (a) \\ isocos\,\hat{\alpha} = b_1^{-1}\cos\hat{\alpha}, \quad (b) \\ isotan\,\hat{\alpha} = \frac{isosin\,\hat{\alpha}}{isocos\,\hat{\alpha}}, \quad (d) \\ isocot\,\hat{\alpha} = \frac{isocos\,\hat{\alpha}}{isosin\,\hat{\alpha}}, \quad (e) \\ isosec\,\hat{\alpha} = 1/isocos\,\hat{\alpha},\ isocosec\,\hat{\alpha} = 1/isosin\,\hat{\alpha} \quad (f) \end{array} \tag{13}$$

with basic property

$$\begin{array}{c} isocos^2\hat{\alpha} + isosin^2\hat{\alpha} = b_1^2 isocos^2\hat{\alpha} + b_2^{-2} isosin^2\hat{\alpha} = \\ \cos^2\hat{\alpha} + \sin^2\hat{\alpha} = 1 \end{array} \tag{14}$$

and general rules for an isosquare isotriangle with isosides $\hat{A}$ and $\hat{B}$ and isohypothenuse $\hat{D}$ as in Diag.(a) of Fig.1

$$\hat{A} = \hat{D}\,isocos\,\hat{\gamma},\ \hat{B} = \hat{D}\,isosin\,\hat{\gamma},\ \hat{A}/\hat{B} = isotan\,\hat{\gamma},\ etc. \tag{15}$$

The isoangles have been identified from the representation theory of isorotations in a plane (see Vol. II, Ch.6[1]), and results to be given by

$$b_1 b_2 \alpha = \hat{\alpha} \tag{16}$$

where the factor $b_1 b_2$ *is fixed for all possible isoangles of a given isoeuclidian space.* This means that the *isotopy* of the trigonometric angles is given by

$$\alpha \rightarrow b_1 b_2 \alpha = \hat{\alpha}, \tag{17}$$

with consequetial angular isotopic element

$$\hat{T}_{\hat{\alpha}} = b_1 b_2 = (Det\, \hat{T})^{1/2} \tag{18}$$

and angular isounit

$$\hat{1}_{\hat{\alpha}} = b_1^{-1} b_2^{-1} = (Det\, \hat{1})^{1/2} \tag{19}$$

where $\hat{T}$ and $\hat{1}$ are the isotopic element and isounit, respectively, of the isoeuclidian plane, Eq.s (3).

Isoangles $\hat{\alpha}$ have a nonlinear and integro-differential dependence on the local plane with expression but they have constant values in isospace because measured with respect to the local coordinates and their derivates when projected in the original Euclidean plane with expression

$$\hat{\alpha} = b_1(t, x, y, \dot{x}, \dot{y}, \ldots) b_2(t, x, y, \dot{x}, \dot{y}, \ldots)\alpha, \tag{20}$$

but they have constant values in isospace because measured with respect to the angle unit $\hat{1}_{\hat{\alpha}} = b_1^{-1} b_2^{-1}$. We reach in this way the following property:

Proposition 3.1 *The isotopies of the plane geometry preserve the numerical value of the original angles, that is, if the original angle $\alpha = 90°$ is so is the value of the corresponding isoangle is isospace.*

In fact, a given isotopic deformation of the angle $\alpha \rightarrow b_1 b_2 \alpha$ occurs under the joint inverse deformation of the basic unit $1 \rightarrow \hat{1} = b_1^{-1} b_2^{-1}$, thus leaving the original numerical value a unchanged.

With respect to Fig. I we therefore have $\hat{\alpha} = 90°$ and $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = 180°$. However, after the lifting $\alpha = 90° \rightarrow \hat{\alpha} = 90°$, the projection of latter in the original plane does not yield back the angle $\alpha = 90°$, but an angle a such that $\hat{\alpha} = b_1 b_2 \alpha = 90°$ and similarly we have $\alpha + \beta + \gamma \neq 90°$ but $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = b_1 b_2(\alpha + \beta + \gamma) = 180°$. It is then easy to see that the isitrigonometric functions are periodic as in conventional case, i.e.

$$\begin{array}{c} isosin\,(\hat{\alpha} + 2k\pi) \equiv isosin\,\hat{\alpha} \quad (a) \\ isocos\,(\hat{\alpha} + 2k\pi) = isocos\,\hat{\alpha},\ k = 1, 2, 3, \ldots \quad (b) \end{array} \tag{21}$$

and preserve the conventional symmetry under the inversion of the angles

$$isocos\ -\hat{\alpha} = isocos\,\hat{\alpha}, \quad isosin\ -\hat{\alpha} = -isosin\,\hat{\alpha} \tag{22}$$

Similarly, we have the *Theorems of Isoaddition* [1]

$$\begin{aligned} isosin\left(\hat{\alpha}\pm\hat{\beta}\right) &= b_1^{-1}\left(isosin\,\hat{\alpha}\ isocos\,\hat{\beta}\pm\ isocos\,\hat{\alpha}\,isosin\,\hat{\beta}\right) \quad (a)\\ isocos\left(\hat{\alpha}+\hat{\beta}\right) &= b_1^2\left(b_2^{-2}\,isocos\,\hat{\alpha}\ isocos\,\hat{\beta}\pm b_1^{-2} isosin\,\hat{\alpha}\ isosin\,\hat{\beta}\right) \quad (b)\\ isosin\,\hat{\alpha}+\ isosin\,\hat{\beta} &= 2b_1^{-1}\,isosin\,\tfrac{1}{2}\left(\hat{\alpha}+\hat{\beta}\right)\ isocos\,\tfrac{1}{2}\left(\hat{\alpha}-\hat{\beta}\right) \quad (c)\end{aligned} \qquad (23)$$

The interested reader can then work out the isotopies of other trigonometric properties.

We are now equipped to introduce the following

Definition 3.3 The "*isopolar coordinates*" are the polar coordinates of the unit isocircle in the isoeuclidean plane $\hat{E}(\hat{r},\hat{\delta},\hat{R})$, and can be written

$$\hat{x} = isocos\,\hat{\alpha}, \quad \hat{y} = isosin\,\hat{\alpha}, \qquad (24)$$

with projection in the conventional Euclidean plane $E(r,\delta,R)$

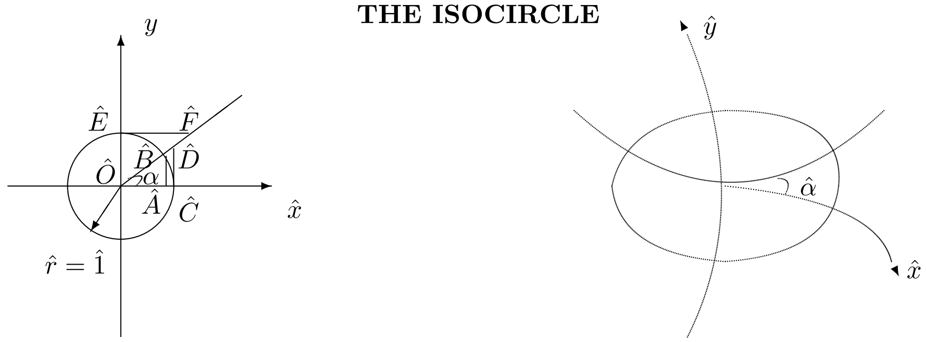
$$x = b_1^{-1}\cos\left(b_1 b_2 \alpha\right), \quad \hat{y} = b_2^{-1} isosin\ \left(b_1 b_2 \alpha\right) \qquad (25)$$

and property

$$\begin{aligned} \hat{x}^2+\hat{y}^2 = x b_1^2 x + y b_2^2 y = \\ = b_1^2 isocos^2\,\hat{\alpha} + b_2^2 isosin^2\,\hat{\alpha} = \cos\hat{\alpha}+\sin\hat{\alpha} = 1 \end{aligned} \qquad (26)$$

The exponential formulation of trigonometric functions also admits a simple, yet unique and effective isotopic image. It requires the lifting of the conventional enveloping associative algebras ξ and their infinite-dimensional basis with conventional unit 1 and product $*$ (the Poincaré-Birkhoff-Witt Theorem) into the *enveloping isoassociative algebras* ξ (or *isoenvelopes* for short) with isotopic image of the original infinite basis characterized by the isounit $\hat{1}$ and the isotopic product $\hat{*} = * \uparrow *$ (the *Poincaré-Birkhoff-Witt Theorem* [1]).

THE ISOCIRCLE



(a) (b)

FIGURE 2 : A schematic view of the isotrigonometric functions on the isocircle (Sect. 1.5.2. of [1]), that is, the circle in isospace, Diag. (a) and in its projection in conventional space, Diag. (b). Isotrigonometry shows that the geometric structure of the circle is indeed axiomatic in the sense that is persist under isotopies. This is illustrated by the preservation under isotopy of the polar coordinates on the conventional circle (Diag. (a))

$$x = \cos\alpha \rightarrow \hat{x} = isocos\,\hat{\alpha},$$
$$y = \sin\alpha \rightarrow \hat{y} = isosin\,\hat{\alpha}\,.$$

However, the projection of the above structure back to the conventional plane implies the deformation of the circle into the ellipse (Diag. (b)), with deformation of the polar coordinates

$$x = \cos\alpha \rightarrow x = b_1^{-1}\cos(b_1 b_2 \alpha),$$
$$y = \sin\alpha \rightarrow x = b_2^{-1}\sin(b_1 b_2 \alpha).$$

The reader is warned not to attempt the computation of *isotrigonometric* properties in the *conventional* Euclidean plane . This is due to the fact that the $\hat{x}$ and $\hat{y}$ isostraight axes in $\hat{E}$ are mapped into curves in E, as depicted in Diag. (b). Mathematical consistency of the isotrigonometry is then achieved only in isospace.

The isotrigonometric functions can then be expressed in term of the isoexponentiation according to the rule

$$\hat{e}^{i\hat{\alpha}} = \hat{1} + (i\hat{\alpha})\,/1! + (i\hat{\alpha}) \uparrow (i\hat{a})\,/2! + \cdots =$$

$$= \hat{1}_{\hat{\alpha}} * e^{i\hat{T}_{\hat{\alpha}}\alpha} = (b_1 b_2)^{-1} * e^{i(b_1 b_2)\alpha} =$$

$$= b_2^{-1} isocos\,\hat{\alpha} + i b_1^{-1} isosin\,\hat{\alpha}\,, \qquad (27)$$

where $\hat{e}$ denotes isoexponentiation and e conventional exponentiation .

The interested reader can then work out additional properties of the isotrigonometric functions.

4 Isohyperbolic functions.

The application of the preceding method to the lifting of the hyperbolic functions is straighforward, leading to the following:

Definition 4.1 The "*Isohyperbolic functions* " on isoeuclidean space $\hat{E}(\hat{r}, \hat{\delta}, \hat{R})$ of Class I are given by

$$\begin{array}{ll} isocosh\,\hat{\alpha} = b_1^{-1} cosh\ (b_1 b_2 \alpha) & (a) \\ isosinh\,\hat{\alpha} = b_2^{-1} sinh\ (b_1 b_2 \alpha)\,, & (b) \end{array} \tag{28}$$

with basic property

$$b_1^2 isosin^2 \hat{\alpha} - b_2^2 isosinh^2 = 1, \tag{29}$$

and derivation via the isoexponentiation

$$\hat{e}^{\hat{\alpha}} = \hat{1}_{\hat{\alpha}} e^{\hat{T}_{\hat{\alpha}} \alpha} = (b_1 b_2)^{-1} e^{(b_1 b_2)\alpha} \backslash =$$

$$= b_1^{-1} isocosh\,\hat{\alpha} + b_2^{-1} isosinh\,\hat{\alpha}. \tag{30}$$

The interested reader can then work out the remaining properties of the isohyperbolic functions.

We now show the property that the distinction between trigonometric and hyperbolic functions is essentially due to the excessive simplicity of the basic unit customarily used in contemporary mathematics, whille such a distinction is lost under more general units.

To understand this point we note that we have used until now for clarity the simplest possible isotopies, those of Kadeisvill's Class I, for which the isounit is smooth, bounded, nowhere null, Hermitean and positive-definite, $\hat{1} > 0$ (used for the representation of *matter* [1]). The isotopies of Class II are the same as those of Class I except that isounit is negative definite, $\hat{1} < 0$ (used for the representation of *antimatter* [loc. cit.]). The isotopies of Class III are those in which the isounit is the same as in Class I except that it has an indefine signature and can be either positive-or negative-definite (used for *mathematical unification of compact and noncompact structures* [loc.cit.]). The isotopies of Class IV are those of ClassIII plus the singular isounit (used for the representation of *gravitational collapse*, the value $\hat{1} = 0$ representing *gravitational singularities* [loc. cit.]). Finally, the isotopies of Class V are those of Class IV plus arbytrary isounits given by distributions, step-functions, lattices, etc. (used for novel treatment of deformable crystal, biological structures, etc. [loc. cit.]).

The unifying power of the isotopies is illustrated by the following:

Lemma 4.1 *[1]Isotrigonometric and isohyperbolic functions lose any distinction on isoeuclidean planes $\hat{E}(\hat{r}, \hat{\delta}, \hat{R})$ of Class III.*

Proof. Assume the realization of the isounits $\hat{1}$ and $\hat{1}_{\hat{\alpha}}$ of Class III,

$$\hat{1} = diag.\left(g_{11}^{-1}, g_{22}^{-1}\right), \quad \hat{1}_{\hat{\alpha}} = (g_{11}g_{22})^{-1/2}, \tag{31}$$

were the functions $g_{kk} = g_{kk}(t, x, y, \dot{x}, \dot{y}, \ldots)$ are smooth, real-valued and nowhere null but otherwise arbitrarily positive or negative. Then, the isoexponential relization of the isotrigonometric functions (27) and of the isohyperbolic functions (30) are unified into the form

$$\hat{e}^{\hat{\alpha}} = \hat{1}_{\hat{\alpha}} e^{\hat{T}_{\hat{\alpha}}\alpha} = (g_{11}g_{22})^{-1/2} e^{(g_{11}g_{22})^{1/2}\alpha}, \tag{32}$$

where the isotrigonometric functions occur when the product $g_{11}g_{22}$ is positive and the isohyperbolic functions occur when the same product is negative .$\square$

Lemma 4.1 also unifies the conventional trigonometric and hyperbolic functions, the former occurring for $\hat{1} = 1 = diag.(1, 1)$ and the latter for $\hat{1} = diag.(+1, -1)\, pr\, Diag.(-1, +1)$ (the latter being the *isodual* of the former [1]).

5 Open problems

In this note we have merely presented the rudiments of the isotopies of the Pythagorean Theorem, trigonometric and hyperbolic functions for the simplest possible isotopies of Class I in wich the isounit is positive-definite and diagonal, Eq. (3c). Numerous problems remain open for the interested reader, among which we indicate the study of the isopythagorean Theorem, isotrigonometric and isohyperbolic functions for :

1) Isotopies of Class II, requiring the study of the *isostraight lines*, *isoangles*, *isotriangle and isocircles with negative unit.*

2) Isotopies of Class III, requiring the study of *isostraight lines, isoangles, isotriangle and isocircles with units of undefined signature.*

3) Isotopies of Class IV, requiring the study of *isostraight lines, isoangles, isotriangle and isocircles with singular units.*

4) Isotopies of Class V, requiring the study of *isostraight lines, isoangles, isotriangle and isocircles with unrestricted-e.g., discontinuous- units.*

All the above studies are referred to *diagonal isounits* of the type

$$\hat{1} = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & g_{22}^{-1} \end{pmatrix}. \tag{33}$$

Additional open problems are given by the study of the isopythagorean Theorem, isotrigonometric and isohyperbolic functions of Classes I-V with *nondiagonal isounits* of the type

$$\hat{1} = \begin{pmatrix} 0 & g_{33}^{-1} \\ g_{33}^{-1} & 0 \end{pmatrix}, \tag{34}$$

as well as those with general isounits of the type

$$\hat{1} = \begin{pmatrix} g_{11}^{-1} & g_{33}^{-1} \\ g_{33}^{-1} & g_{22}^{-1} \end{pmatrix}, \tag{35}$$

which are unknown at this writing.

The study of the following conjecture may also be of some interest:

Conjecture 5.1 *(Inverse Isopythagorean Theorem) : Given a geometric figure consisting of of three smooth but otherwise arbitrary curves in a conventional Euclidean plane intersecting each other as per Diagram (b) of Fig.I, there always exists an isotopy of the unit of Class I,* $1 \to \hat{1}$ *under which said geometric figure is mapped into the isoright isotriangle in isoeuclidean space for which the isopythagorean theorem holds.*

If correct, the above conjecture would establish that the abstract geometric structure of the historical Pythagorean theorem applies to a much broader class of figures and it is in fact universal for all "triangles with" curved sides".

Note that the proof of Conjecture I appears to be possible for the case of nondiagonal isounits of type (35) because they contain *three* arbitrary functions $g_{kk}(t, x, y, \ldots)$ as needed to characterize the three independent curves of the "triangle". A more difficult case is whether the isotopic lifting of Disg. (b) into (a) of Fig.I exists also for a *diagonal* isounit with *two* independent functions g_{kk} while we have *three* independent curves.

The author hopes to have illustrated in this note that the removal of the current restriction of our entire mathematical knowledge to the trivial unit identified since biblical times, and the use of structurally more general units, implies a rather vast broadening of all of mathematics, beginning from the most elementary ones such as numbers and angles, and then following with all remaining structures, with for basically novel applications in a variety of fields.

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Author's address:

Ruggero Maria Santilli

Istituto per la Ricerca di Base

Castello Principe Pignatelli

1-86075 Monteroduni, Molise, Italy

ibrrms@pinet.alp.org