LOCALLY ANISOTROPIC STOCHASTIC PROCESSES IN FIBER BUNDLES

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Abstract

We formulate the theory of stochastic differential equations for spaces with local anisotropy (vector bundles provided with compatible nonlinear and distinguished connections and metric structures and generalized Lagrange and Finsler spaces).

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1 Introduction

Modern geometric methods are applied in various branches of science and economy [1-6]. Modeling of diffusion processes in nonhomogerneous media and formulation of nonlinear thermodynamics in physics, or of dynamics of evolution of species in biology, requires a more extended geometrical background than that used in the theory of stochastic differential equations and diffusion processes on Riemann and Lorentz manifolds [7-10].

Our purpose is to consider the geometrical basis of the theory of diffusion processes on spaces with local anisotropy (in brief, we shall call them la-spaces and denote as $\mathcal{E}_N$). In general, such spaces are modeled as vector bundles (v-bundles) on space-times provided with nonlinear and distinguished connections (respectively, N- and d-connections) and metric structures [11,12]. Transferring our considerations on tangent bundles we can formulate the theory of stochastic differential equations on generalized Lagrange spaces which contain as particular cases Lagrange and Finsler spaces [13-17].

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2 Stochastic Differential Equations in V-Bundles

We assume that the reader is familiar with the concepts and basic results on stochastic calculus, Brownian motion and diffusion processes (an excellent presentation can be found in [7-9, 18-20]) and with the theory of stochastic differential equations on Riemannian spaces [7-9]. On la-spaces we shall follow Miron and Anastesiei conventions [11,12].

Let $A_{\widehat{0}}, A_{\widehat{1}}, ..., A_{\widehat{r}} \in \mathbf{X}(\mathcal{E}_{\mathbf{N}})$ (by $\mathbf{X}(\mathcal{E}_N)$ we denote the set of C^∞-vector fields on $\mathcal{E}_N$) and consider stochastic differential equations

$$\delta U(t) = A_{\widehat{\alpha}} \circ \delta B^{\widehat{\alpha}}(t) + A_{\widehat{0}}(U(t))\,\delta t, \tag{1}$$

where $\widehat{\alpha} = 1, 2, ..., r$ and $\circ$ is the symmetric Q-product and t is (if necessary a timelike) a parameter. We shall use the point compactification of space $\mathcal{E}_N$ and write $\widehat{\mathcal{E}}_N = \mathcal{E}$ or $\widehat{\mathcal{E}}_N = \mathcal{E} \cup\{\Delta\}$ in dependence of that if $\mathcal{E}_N$ is compact or noncompact. By $\widehat{W}(\mathcal{E}_N)$ we denote the space of paths in $\mathcal{E}_N$, defined as

$\widehat{W}(\mathcal{E}_N) = \{w : w$ is a smooth map $[0,\infty) \to \widehat{\mathcal{E}}_N$ with the property that $w(0) \in \mathcal{E}_N$ and $w(t) = \Delta, w(t') = \Delta$ for all $t' \geq t\}$

and by $\mathcal{B}\left(\widehat{W}(\mathcal{E}_N)\right)$ the σ-field generated by Borel cylindrical sets.

The explosion moment $e(w)$ is defined as $e(w) = \inf\{t, w(t) = \Delta\}$.

Definition 2.1. The solution $U = U(t)$ of equation (1) in v-bundle space $\mathcal{E}_N$ is defined as such a $(\mathcal{F}_t)$-*compatible* $\widehat{W}(\mathcal{E}_N)$-valued random element (i.e. as a smooth process in $\mathcal{E}_N$ with the trap Δ), given on the probability space with a filtration $(\mathcal{F}_t)$ and r-dimensional $(\mathcal{F}_t)$-Brownian motion $B = B(t)$, when $B(0) = 0$, for which

$$f(U(t)) - f(U(0)) =$$
$$\int_0^t A_{\widehat{\alpha}}(t)(U(s))\,\delta B^{\widehat{\alpha}}(s) + \int_0^t (A \circ f)(U(s))\,ds \tag{2}$$

for every $f \in F_0(\mathcal{E}_N)$ (we consider $f(\Delta) = 0$), where the first term is understood as a Fisk-Stratonovich integral.

In the equations (1) and (2) we use $\delta B^{\widehat{\alpha}}(s)$ and δt instead of respectively $dB^{\widehat{\alpha}}(s)$ and dt because on $\mathcal{E}_N$ the Brownian motion must be compatible with the N-connection structure (see details in [11,12] with respect to locally adapted to the N-connection bases and operators of partial derivation).

In a manner similar to that for stochastic equations on Riemannian spaces [7] we can construct the unique strong solution to the equations (1). To do this we have to use the space of paths in $\mathcal{R}^r$ starting in point 0, denoted as W_0^r, the Wiener measure P^W on W_0^r, σ-field $\mathcal{B}_t$ (W_0^r)-generated by Borel cylindrical sets up to the moment t and the similarly defined σ-field.

Theorem 2.1. *There is a function* $F : \mathcal{E}_N \times W_0^r \to \widehat{W}(\mathcal{E}_N)$ *being* $\bigcap_\mu \mathcal{B}(\mathcal{E}_N) \times \mathcal{B}_t$ $(W_0^r)^{\mu \times P^W} / \mathcal{B}_t$ $\left(\widehat{W}(\mathcal{F}_t)\right)$*-measurable (index* μ *runs all probabilities in* $(\mathcal{E}_N, B(E_N))$

) for every $t \geq 0$ and having properties:

1)*For every $U(t)$ and Brownian motion $B = B(t)$ the equality $U = F(U(0), B)$ almost sure, in brief a.s., is satisfied.*

2)*For every r-dimensional $(\mathcal{F}_t)$-Brownian motion $B = B(t)$ with $B = B(0)$, defined on the probability space with filtration $\mathcal{F}_t$, and $\mathcal{E}_{\mathcal{N}}$-valued $\mathcal{F}_0$-measurable random element ξ, the function $U = F(\xi, B)$ is the solution of the differential equation (1) with $U(0) = \xi$, a.s.*

Proofs of the theorems presented in this paper by using methods developed in [7] are contained in [34].

Let P_u be a probability law on $\widehat{W}(\mathcal{E}_N)$ of a solution $U = U(t)$ of the equation (1) with initial conditions $U_{(0)} = u$. Taking into account the uniqueness of the mentioned solution we can prove that $U = U(t)$ is an A-diffusion and satisfy the Markov property [7]. Really, because for every $f \in F_0(\mathcal{E}_N)$

$$\delta f(U(t)) = \left(A_{\widehat{\alpha}} f\right)(U(t)) \circ \delta w^{\widehat{\alpha}} + (A_0 f)(U(t))\,\delta t =$$

$$\left(A_{\widehat{\alpha}} f\right)(U(t))\,\delta w^{\widehat{\alpha}} + (A_0 f)(U(t))\,\delta t + \frac{1}{2} d\left(A_{\widehat{\alpha}} f\right)(U(t)) \cdot \delta w^{\widehat{\alpha}}(t)$$

and

$$\delta\left(A_{\widehat{\beta}} f\right)(U(t)) = A_{\widehat{\alpha}}\left(A_{\widehat{\beta}} f\right)(U(t)) \circ \delta w^{\widehat{\alpha}}(t) + \left(A_{\widehat{0}} A_{\widehat{\beta}} f\right)(U(t))\,\delta t,$$

we have

$$\delta\left(A_{\widehat{\alpha}} f\right)(U(t)) \cdot \delta w^{\widehat{\alpha}}(t) = \sum_{\widehat{\alpha}=1}^{r} A_{\widehat{\alpha}}\left(A_{\widehat{\alpha}} f\right)(U(t))\,\delta t.$$

Consequently, it follows that

$$\delta f(U(t)) = \left(A_{\widehat{\alpha}} f\right)(U(t))\,\delta w^{\widehat{\alpha}}(t) + (Af)(U(t))\,\delta t,$$

i.e. the operator (Af) defined by the equality

$$Af = \frac{1}{2}\sum_{\widehat{\alpha}=1}^{r} A_{\widehat{\alpha}}\left(A_{\widehat{\alpha}} f\right) + A_0 f \tag{3}$$

generates a diffusion process $\{P_u\}, u \in \mathcal{E}_N$.

The above presented results are summarized in this form:

Theorem 2.2. *A second order differential operator Af generates an A-diffusion on $\widehat{W}(\mathcal{E}_N)$ of a solution $U = U(t)$ of the equation (3) with initial condition $U(0) = u$.*

Using similar considerations as in flat spaces [7], on carts covering $\mathcal{E}_{N.}$, we can prove the uniqueness of A-diffusion$\{P_u\}, u \in \mathcal{E}_N$ on $\widehat{W}(\mathcal{E}_N)$.

3 Heat Equations in V-Bundles

Let v-bundle $\mathcal{E}_N$ be a compact manifold of class C^∞. We consider operators

$$A_0, A_1, ..., A_r \in \mathbf{X}\left(\mathcal{E}_\mathbf{N}\right)$$

and suppose that the property

$$E[Sup_{t\in[0,1]} Sup_{u\in\mathcal{U}}\ |D^{\underline{\alpha}}\{f\left(U\left(t,u,w\right)\right)\}|] < \infty$$

is satisfied for all $f \in F_0\left(\mathcal{E}_N\right)$ and every multiindex $\underline{\alpha}$ in the coordinate vicinity $\mathcal{U}$ with $\overline{\mathcal{U}}$ being compact for every $T > 0$. The heat equation in $F_0\left(\mathcal{E}_N\right)\mathcal{E}_N$ is written as

$$\frac{\delta\nu}{\delta t}(t,u) = A\nu(t,u), \tag{4}$$

$$\lim_{t\downarrow 0, \overline{u}\to u} \nu(t,\overline{u}) = f\left(u\right),$$

where operator A acting on $F\left(F_0\left(\mathcal{E}_N\right)\right)$ is defined by expression (3). We not that in out considerations of random processes on la-spaces it is useful to use locally adapted to the N-connection structure bases (or, equivalently, operators of local partial derivations, see details in [11,12]):

$$\delta_\alpha = (\delta_i, \partial_a) = \frac{\delta}{\delta u^\alpha} = (\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^a(u)\frac{\partial}{\partial y^a}, \frac{\delta}{\delta y^b} = \frac{\partial}{\partial y^b}),$$

where $u^\alpha = (x^i, y^b)$ are local coordinates on $\mathcal{E}_N$ and $N_i^a(u)$ are local components of the N-connection on $\mathcal{E}_N$. The dual locally adapted bases (or "differentials") are written as

$$\delta^\alpha = (d^i, \delta^a) = \delta u^\alpha = (dx^i, \delta y^a = dy^a + N_i^a(u)dx^i).$$

We denote by $C^{1,2}\left([0,\infty) \times F_0\left(\mathcal{E}_N\right)\right)$ the set of all functions $f(t,u)$ on $[0,\infty)\times\mathcal{E}_N$ being smoothly differentiable on t and twice differentiable on u.

The existence and properties of solutions of the equation (4) are stated according the theorem:

Theorem 3.1. *The function*

$$\zeta(t,u) = E[f(U(t,u,w)] \in C^\infty\left([0.\infty) \times \mathcal{E}_N\right) \tag{5}$$

$f \in F_0\left(\mathcal{E}_N\right)$ satisfies the heat equation (4). Inversely, if a bounded function $\nu(t,u) \in C^{1,2}\left([0,\infty) \times \mathcal{E}_N\right)$ solves the equation (4) and satisfies the condition

$$\lim_{k\uparrow\infty} E[\nu(t-\sigma_k, U\left(\sigma_k,u,w\right)) : \sigma_k \leq t] = 0 \tag{6}$$

for every $t > 0$ and $u \in \mathcal{E}_N$, where $\sigma_k = \inf\{t, U(t,u,w) \in D_k\}$ and D_k is an increasing sequence with respect to closed sets in $\mathcal{E}_N$, $\bigcup_k D_k = \mathcal{E}_N$.

Remarks

1. The conditions (6) are necessary in order to select a unique solution of (4).
2. Defining

$$\zeta(t,u) = E[\exp\{\int_0^t C(U(s,u,w))\,ds\} f(U(t,u,w))]$$

instead of (5) we generate the solution of the generalized heat equation in $\mathcal{E}_N$:

$$\frac{\delta\nu}{\delta t}(t,u) = (A\nu)(t,u) + C(u)\,\nu(t,u),$$

$$\lim_{t\downarrow 0,\overline{u}\to u} \nu(t,\overline{u}) = f(u).$$

For given vector fields $A_{(\alpha)} \in \mathbf{X}(E_N), (\alpha) = 0,1,...,r$ in Section II [34] we have constructed the map (here we note that for such type of geometric constructions sometimes it is more useful to use partial derivations and unadapted to the N-connection bases and then to consider reparametizations of coefficients of equations with respect to transforms to locally adapted geometrical objects):

$$U = (U(t,u,w)) : \mathcal{E}_N \times W_0^r \ni (u,w) \to U(\cdot,u,w) \in \widehat{W}(E_N),$$

which is a map of type

$$[0,\infty) \times \mathcal{E}_N \times W_0^r \ni (u,w) \to U(t,u,w) \in \widehat{\mathcal{E}_N}.$$

Let us show that map $u \in \mathcal{E}_N \to U(t,u,w) \in \widehat{\mathcal{E}_N}$ is a local diffeomorphism of the manifold $\mathcal{E}_N$ for every fixed $t \geq 0$ and almost every w that $\in \mathcal{E}_N$.

We first consider the case when $\mathcal{E}_N \cong \mathcal{R}^{n+m}$,

$$\sigma(u) = \left(\sigma^\alpha_\beta(u)\right) \in \mathcal{R}^{n+m} \otimes \mathcal{R}^{n+m}$$

and $b(u) = (b^\alpha(u)) \in \mathcal{R}^{n+m}$ are given smooth functions (i.e. C^∞-functions) on $\mathcal{R}^{n+m}$, $\parallel \sigma(u) \parallel + \parallel b(u) \parallel \leq K(1+|u|)$ for a constant $K > 0$ and all derivations of σ^α and b^α are bounded. It is known [7] that there is a unique solution $U = U(t,u,w)$, with the property that $E[(U(t))^p] < \infty$ for all $p > 1$, of the equation

$$dU_t^\alpha = \sigma^\alpha_{\widehat{\alpha}}(U_t)\,dw^{\widehat{\alpha}}(t) + b^\alpha(U_t)\,dt, \quad (7)$$

$$U_0 = u, (\alpha = 1,2,...,m+n-1),$$

defined on the space $\left(W_0^r, P^W\right)$ with the flow $\left(\mathcal{F}_t^0\right)$.

In order to show that the map $u \to U(t,u,w)$ is a diffeomorphism of $\mathcal{R}^{n+m}$ it is more convenient to use the Fisk-Stratonovich differential and to write the equation (7) equivalently as

$$\delta U_t^\alpha = \sigma^\alpha_{\widehat{\alpha}}(U_t) \circ \delta w^{\widehat{\alpha}}(t) + \overline{b}^\alpha(U_t)\,\delta t, \quad (8)$$

$$U_0 = u,$$

by considering that

$$\overline{b}^{\alpha}(u) = b^{\alpha}(u) + \frac{1}{2}\sum_{\widehat{\alpha}=1}^{r}\left(\delta_{\beta}\sigma^{\alpha}_{\widehat{\alpha}}\right)\sigma^{\beta}_{\widehat{\alpha}}(u). \tag{9}$$

We emphasize that for solutions of equations of type (7) one holds the usual derivation rules as in mathematical analysis.

Let introduce matrices

$$\sigma'_{\widehat{\alpha}} = \left(\sigma'(u)^{\alpha}_{\widehat{\alpha}\beta} = \frac{\delta}{\delta u^{\beta}}\sigma^{\alpha}_{\widehat{\alpha}}(u)\right), b'(u) = \left(b'(u)^{\alpha}_{\beta} = \frac{\delta b^{\alpha}}{\delta u^{\beta}}\right), I = \delta^{\alpha}_{\beta}$$

and the Jacobi matrix $Y(t) = \left(Y^{\alpha}_{\beta}(t) = \frac{\delta U^{\alpha}}{\delta u^{\beta}}(t,u,w)\right)$, which satisfy the matrix equation

$$Y(t) = I + \int_0^t \sigma'_{\widehat{\alpha}}(U(s))\,Y(s)\circ dw^{\widehat{\alpha}}(s) + \int_0^t b'(U(s))\,Y(s)\,ds. \tag{10}$$

As a modification of a process $U(t,u,w)$ one means a such process $\widehat{U}(t,u,w)$ that $P^{W}\{\widehat{U}(t,u,w) = U(t,u,w) \text{ for all } t \geq 0\} = 1$ a.s.

It is known this result for flows of diffeomorphisms of flat spaces [24-26,7]:

Theorem 3.2. *Let $U(t,u,w)$ be the solution of the equation (8) (or (7)) on Wiener space $\left(W^{r}_{0}, P^{W}\right)$. Then we can choose a modification $\widehat{U}(t,u,w)$ of this solution when the map $u \to U(t,u,w)$ is a diffeomorphism $\mathcal{R}^{n+m}$ a.s. for every $t \in [0,\infty)$.*

Process $u = \widehat{U}(t,u,w)$ is constructed by using the equations

$$\delta U^{\alpha}_{t} = \sigma^{\alpha}_{\widehat{\alpha}}(U_t)\circ\delta w^{\widehat{\alpha}}(t) - b^{\alpha}(U_t)\,\delta t,$$

$$U_0 = u.$$

Then for every fixed $T > 0$ we have

$$U(T-t,u,w) = \widehat{U}(t, U(T,u,w), \widehat{w})$$

for every $0 \leq t \leq T$ and u P^{W}-a.s., where the Wiener process $\widehat{w}$ is defined as $\widehat{w}(t) = w(T-t) - w(T), 0 \leq t \leq T$.

Now we can extend the results on flows of diffeomorphisms of stochastic processes to v-bundles. The solution $U(t,u,w)$ of the equation (4) can be considered as the set of maps $U_t : u \to U(t,u,w)$ from $\mathcal{E}_N$ to $\hat{\mathcal{E}_N} = \mathcal{E}_N \cup \{\triangle\}$.

Theorem 3.3. *A process $|U|(t,u,w)$ has such a modification, for simplicity let denote it also as $U(t,u,w)$, that the map $U_t(w) : u \to U(t,u,w)$ belongs to the class C^{∞} for every $f \in F_0(\mathcal{E}_N)$ and all fixed $t \in [0,\infty)$ a.s. In addition, for every $u \in U$ and $t \in [0,\infty)$ the differential of map $u \to U(t,u,w)$,*

$$U(t,u,w)_{*} : T_u\left(U(t,u,w)\right) \to T_{U(t,u,w)}\left(\mathcal{E}_N\right),$$

is an isomorphism, a.s., in the set $\{w : U(t,u,w) \in \mathcal{E}_N \}$.

Let $A_0, A_1, ..., A_r \in \mathbf{X}(\mathcal{E}_\mathbf{N})$ and $U_t = (U(t,u,w))$ is a flow of diffeomorphisms on $\mathcal{E}_\mathbf{N}$. Then

$$\widetilde{A}_0, \widetilde{A}_1, ..., \widetilde{A}_r \in \mathbf{X}(GL(\mathcal{E}_\mathbf{N}))$$

define a flow of diffeomorphisms $r_t = (r(t,r,w))$ on $GL(\mathcal{E}_\mathbf{N})$ with

$$(r(t,r,w)) = (U(t,u,w), e(t,u,w)),$$

where $r = (u,e)$ and $e(t,r,u) = U(t,u,w)_* e$ is the differential of the map $u \to U(t,u,w)$ satisfying the property

$$U(t,u,w)_* e = [U(t,u,w)_* e_0, U(t,u,w)_* e_1, ..., U(t,u,w)_* e_{q-1}].$$

In local coordinates

$$A_{\widehat{\alpha}}(u) = \sigma^{\alpha}_{\widehat{\alpha}} \delta_\alpha, (\alpha = 1,2,...,r), A_0(u) = b^\alpha(u)\delta_\alpha,$$

and

$$e^{\alpha}_{\underline{\beta}}(t,u,w) = Y^{\alpha}_{\underline{\gamma}}(t,u,w) e^{\gamma}_{\underline{\beta}},$$

where $Y^{\alpha}_{\underline{\gamma}}(t,u,w)$ is defined from (10). So we can construct flows of diffeomorphisms of the bundle $\mathcal{E}_\mathbf{N}$.

4 Nondegenerate Diffusion in V-Bundles

Let the v-bundle $\mathcal{E}_\mathbf{N}$ is provided with a positively defined d-metric being compatible with a d- connection $D = \{\Gamma^{\alpha}_{\beta\gamma}\}$ (see details in [11,12]). The connection D allows us to roll $\mathcal{E}_\mathbf{N}$ along a curve $\gamma(t) \subset \mathcal{R}^{n+m}$ in order to draw the curve $c(t)$ on $\mathcal{E}_\mathbf{N}$ as the trace of $\gamma(t)$. More exactly, let $\gamma : [0,\infty) \ni t \to \gamma(t) \subset \mathcal{R}^{n+m}$ be a smooth curve in $\mathcal{R}^{n+m}$, $r = (u,e) \in O(\mathcal{E}_\mathbf{N})$. We define a curve $\widetilde{c}(t) = (c(t), e(t))$ in $O(\mathcal{E}_\mathbf{N})$ by using the equalities

$$\frac{dc^\alpha(t)}{dt} = e^{\alpha}_{\underline{\alpha}}(t)\frac{d\gamma^{\underline{\alpha}}}{dt},$$
$$\frac{de^{\alpha}_{\underline{\alpha}}(t)}{dt} = -\Gamma^{\alpha}_{\beta\gamma}(c(t)) e^{\gamma}_{\underline{\alpha}}(t)\frac{dc^\beta}{dt}, \tag{11}$$
$$c^\alpha(0) = u^\alpha, e^{\alpha}_{\underline{\alpha}}(0) = e^{\alpha}_{\underline{\alpha}}.$$

Equations (11) can be written as

$$\frac{d\widetilde{c}(t)}{dt} = \widetilde{L}_\alpha(\widetilde{c}(t)) d\gamma^\alpha,$$

$$\widetilde{c}(0) = r,$$

where$\{\widetilde{L}_\alpha\}$ is the system of canonical horizontal vector fields (see a similar construction for Riemannian spaces in [7]). Curve $c(t) = \pi(\widetilde{c}(t))$ on $\mathcal{E}_N$ depends on the fixing of a frame p in a point u; this curve is parametrized as $c(t) = c(t,r,\gamma), r = r(u,e)$.

Let $w(t) = (w^{\underline{\alpha}}(t))$ is the canonical realization of a n+m-dimensional Wiener process. We can define the random curve $U(t) \subset \mathcal{E}_N$ in a similar manner. Consider $r(t) = (r(t, r, w))$ as the solution of stochastic differential equations

$$\delta r(t) = \widetilde{L}_{\underline{\alpha}}(r(t)) \circ \delta w^{\underline{\alpha}}(t), \quad (12)$$

$$r(0) = r,$$

where $r(t, r, w)$ is the flow of diffeomorphisms on $O(\mathcal{E}_N)$ corresponding to the canonical horizontal vector fields $\widetilde{L}_1, \widetilde{L}_2, ..., \widetilde{L}_{q-1}$ and vanishing drift field $\widetilde{L}_0 = 0$. In local coordinates the equations (12) are written as

$$\delta U^{\alpha}(t) = e^{\alpha}_{\underline{\alpha}}(t) \circ \delta w^{\underline{\alpha}}(t),$$

$$\delta e^{\alpha}_{\underline{\alpha}}(t) = -\Gamma^{\alpha}_{\beta\gamma}(U(t))\, e^{\gamma}_{\underline{\alpha}} \circ \delta u^{\beta},$$

where $r(t) = \left(U^{\alpha}(t), e^{\alpha}_{\underline{\alpha}}(t)\right)$. It is obvious that $r(t) = \left(U^{\alpha}(t), e^{\alpha}_{\underline{\alpha}}(t)\right) \in O(\mathcal{E}_N)$ if $r(0) \in O(\mathcal{E}_N)$ because $\widetilde{L}_{\underline{\alpha}}$ are vector fields on $O(\mathcal{E}_N)$. The random curve $\{U^{\alpha}(t)\}$ on $\mathcal{E}_N$ is defined as $U(t) = \pi[r(t)]$. We point out that $aw = (aw(t))$ is another (n+m)-dimensional Wiener process and as a consequence the probability law $U(\cdot, r, w)$ does not depend on $a \in O(n+m)$. It depends only on $u = \pi(r)$. This law is denoted as P_w and should be mentioned that it is a Markov process because a similar property has $r(\cdot, r, w)$.

Remark 3.1. We can define $r(t, r, w)$ as a flow of diffeomorphisms on $GL(\mathcal{E}_N)$ for every d-connection on $\mathcal{E}_N$. In this case $\pi[r(\cdot, r, w)]$ does not depend only on $u = \pi(t)$ and in consequence we do not obtain a Markov process by projecting on $\mathcal{E}_N$. The Markov property of diffusion processes on $\mathcal{E}_N$ is assumed by the conditions of compatibility of metric and d-connection and of vanishing of torsion.

Now let us show that a diffusion $\{P_u\}$ on $\mathcal{E}_N$ can be considered as an A-diffusion process with the differential operator

$$A = \frac{1}{2}\Delta_{\mathcal{E}} + b, \quad (13)$$

where $\Delta_{\mathcal{E}}$ is the Laplace-Beltrami operator on $\mathcal{E}_N$,

$$\Delta_{\mathcal{E}} f = G^{\alpha\beta}\overrightarrow{D}_{\alpha}\overrightarrow{D}_{\beta} f = G^{\alpha\beta}\frac{\delta^2 f}{\delta u^{\alpha}\delta u^{\beta}} - \{\frac{\alpha}{\gamma\beta}\}\frac{\delta f}{\delta u^{\alpha}}, \quad (14)$$

where operator $\overrightarrow{D}_{\alpha}$ is constructed by using Christoffel d-symbols and b is the vector d-field with components

$$b^{\alpha} = \frac{1}{2}G^{\beta\gamma}\left(\{\frac{\alpha}{\beta\gamma}\} - \Gamma^{\alpha}_{\beta\gamma}\right) \quad (15)$$

Theorem 4.1. *The solution of stochastic differential equation (4.2) on $O(\mathcal{E}_N)$ defines a flow of diffeomorphisms $r(t) = (r(t, r, w))$ on $O(\mathcal{E}_N)$ and its projection*

$U(t) = \pi(r(t))$ defines a diffusion process on $\mathcal{E}_N$ corresponding to the differential operator (13).

Definition 4.1. The process $r(t) = (r(t, r, w))$ from the theorem 4.1 is called the *horizontal lift* of the A-diffusion $U(t)$ on $\mathcal{E}_N$.

Proposition 4.1. *For every d-vector field $b = b^{\alpha}(u)\delta_{\alpha}$ on $\mathcal{E}_N$ provided with the canonical d-connection structure there is a d-connection $D = \{\Gamma^{\alpha}_{\beta\gamma}\}$ on $\mathcal{E}_N$, compatible with d-metric $G_{\alpha\beta}$, which satisfies the equality (15).*

We note that a similar proposition is proved in [7] for, respectively, metric and affine connections on Riemannian and affine connected manifolds: M. Anastasiei proposed [28] to define Laplace-Beltrami operator (14) by using the canonical d-connection in generalized Lagrange spaces. Taking into account a corresponding redefinition of components of d-vector fields (15), because of the existence of multiconnection structure on the space H^{2n} , we conclude that we can equivalently formulate the theory of d-diffusion on H^{2n}-spaces by using both variants of Christoffel d-symbols and canonical d-connection.

Definition 4.2. For $A = \frac{1}{2}\Delta_{\mathcal{E}}$ an A-diffusion $U(t)$ is called a Riemannian motion on $\mathcal{E}_N$.

Let an A-differential operator on $\mathcal{E}_N$ is expressed locally as

$$Af(u) = \frac{1}{2}a^{\alpha\beta}(u)\frac{\delta^2 f}{\delta u^{\alpha}\delta u^{\beta}}(u) + b^{\alpha}(u)\frac{\delta f}{\delta u^{\alpha}}(u),$$

where $f \in F(\mathcal{E}_N)$, matrix $a^{\alpha\beta}$ is symmetric and nonegatively defined . If $a^{\alpha\beta}(u)$ $\xi_{\alpha}\xi_{\beta} > 0$ for all u and $\xi = (\xi_{\alpha}) \in \mathcal{R}^q\backslash\{0\}$, than the operator A is nondegenerate and the corresponding diffusion is called nondegenerate.

By using a vector d-field b_{α} we can define the 1-form

$$\omega_{(b)} = b_{\alpha}(u)\delta u^{\alpha},$$

where $b = b^{\alpha}\delta_{\alpha}$ and $b_{\alpha} = G_{\alpha\beta}b^{\beta}$ in local coordinates. According the de Rham-Codaira theorem [27] we can write

$$\omega_{(b)} = dF + \widehat{\delta}\beta + \alpha \tag{16}$$

where $F \in F(\mathcal{E}_N), \beta$ is a 2-form and α is a harmonic 1-form. The scalar product of p-forms $\Lambda_p(\mathcal{E}_N)$ on $\mathcal{E}_N$ is introduced as

$$(\alpha, \beta)_B = \int_{\mathcal{E}_N} < \alpha, \beta > \delta u,$$

where

$$\alpha = \sum_{\gamma_1<\gamma_2<...<\gamma_p} \alpha_{\gamma_1\gamma_2...\gamma_p}\delta u^{\gamma_1}\bigwedge\delta u^{\gamma_2}\bigwedge...\bigwedge\delta u^{\gamma_p},$$

$$\beta = \sum_{\gamma_1<\gamma_2<...<\gamma_p} \beta_{\gamma_1\gamma_2...\gamma_p}\delta u^{\gamma_1}\bigwedge\delta u^{\gamma_2}\bigwedge...\bigwedge\delta u^{\gamma_p},$$

$$\beta^{\gamma_1\gamma_2...\gamma_p} = G^{\gamma_1\tau_1}G^{\gamma_2\tau_2}...G^{\gamma_p\tau_p}\beta_{\tau_1\tau_2...\tau_p},$$
$$<\alpha,\beta>= \sum_{\gamma_1<\gamma_2<...<\gamma_p} \alpha_{\gamma_1\gamma_2...\gamma_p}(u)\,\beta^{\gamma_1\gamma_2...\gamma_p}(u),$$
$$\delta u = \sqrt{|\det G_{\alpha\beta}|}\delta u^0\delta u^1...\delta u^{q-1}.$$

The operator $\widehat{\delta}:\Lambda_p(\mathcal{E}_N)\to\Lambda_{p-1}(\mathcal{E}_N)$ from (16) is defined by the equality

$$(d\alpha,\beta)_p = \left(\alpha,\widehat{\delta}\beta\right)_{p-1}, \alpha\in\Lambda_{p-1}(\mathcal{E}_N), \beta\in\Lambda_p(\mathcal{E}_N).$$

De Rham-Codaira Laplacian $\Box:\Lambda_p(\mathcal{E}_N)\to\Lambda_p(\mathcal{E}_N)$ is defined by the equality

$$\Box = -\left(d\widehat{\delta}+\widehat{\delta}d\right). \qquad (17)$$

A form $\alpha\in\Lambda_p(\mathcal{E}_N)$ is called as harmonic if $\Box\alpha = 0$. It is known that $\Box\alpha = 0$ if and only if $d\alpha = 0$ and $\widehat{\delta}\alpha = 0$.For $f\in F(\mathcal{E}_N)$ and $U\in\mathbf{X}(\mathcal{E}_N)$ we can define the operators *gradf* $\in\mathbf{X}(\mathcal{E}_N)$ and *divU* $\in F(\mathcal{E}_N)$ by using correspondingly the equalities

$$gradf = G^{\alpha\beta}\delta_\alpha\delta_\beta f \qquad (18)$$

and

$$divU = -\widehat{\delta}\omega_U = \frac{1}{\sqrt{|\det G|}}\delta_\alpha\left(U^\alpha\sqrt{|\det G|}\right). \qquad (19)$$

The Laplace-Beltrami operator (13) can be also written as

$$\triangle_{\mathcal{E}}\ f = div(gradf) = -\widehat{\delta}\widehat{\delta}f \qquad (20)$$

for $F(M)$.

Let suggest that $\mathcal{E}_N$ is compact and oriented and $\{P_u\}$ be the system of diffusion measures defined by an A-operator (13). Because $\mathcal{E}_N$ is compact P_u is the probability measure on the set $\widehat{W}(\mathcal{E}_N) = W(\mathcal{E}_N)$ of all continuous paths in $\mathcal{E}_N$.

Definition 4.3. The *transition semigroup* T_t of A-diffusion is defined by the equality

$$(T_tf)(u) = \int_{W(\mathcal{E}_N)} f(w(t))\,P_u(dw), f\in C(\mathcal{E}_N).$$

For a connected open region $\Omega\subset\mathcal{E}_N$ we define $\rho^\Omega w\in\widehat{W}(\Omega), w\in\widehat{W}(\mathcal{E}_N)$ by the equality

$$(\rho^\Omega w)(t) = \langle^{w(t),if\ \ t<\tau_\Omega(w),}_{\Delta,if\ \ t\geq\tau_\Omega(w),}$$

where $\tau_\Omega(w) = \inf\{t : w(t)\notin\Omega\}$. We denote the image-measure $P_u(u\in\Omega)$ on map ρ^Ω as P_u^Ω ; this way we define a probability measure on $\widehat{W}(\Omega)$ which will be called as the minimal A-diffusion on Ω. The transition group of this diffusion is introduced as

$$\left(T_t^\Omega f\right)(u) = \int_{W(\Omega)} f(w(t))\,P_u^\Omega(dw) =$$

$$\int\limits_{W(\mathcal{E}_N\ )} f(w(t)) I_{\{\tau_\Omega(w)>t\}} P_u(dw), f \in C_p(\Omega).$$

Definition 4.4. The Borel measure $\mu(du)$ on $\mathcal{E}_N$ is called an *invariant measure* on A-diffusion $\{P_u\}$ if $\int\limits_{\mathcal{E}_N} T_t f(u) \mu(du) = \int\limits_{\mathcal{E}_N} f(u) \mu(du)$ for all $f \in C(\mathcal{E}_N)$.

Definition 4.5. An A-diffusion $\{P_u\}$ is called *symmetrizable* (locally symmetrizable) if there is a Borel measure $\nu(du)$ on $\mathcal{E}_N$ ($\nu^\Omega(du)$ on Ω) that

$$\int\limits_{\mathcal{E}_N} T_t f(u) g(u) \nu(du) = \int\limits_{\mathcal{E}_N} f(u) T_t g(u) \nu(du) \tag{21}$$

for all $f, g \in C(\mathcal{E}_N)$

$$(\int\limits_{\Omega} T_t^\Omega f(u) g(u) \nu^\Omega(du) = \int\limits_{\Omega} f(u) T_t^\Omega g(u) \nu^\Omega(du)$$

for all $f, g \in C(\Omega)$).

The fundamental properties of A-diffusion measures are satisfied by the following theorem and corollary:

Theorem 4.2. *a) An A-diffusion is symmetrizable if and only if $\widehat{\delta}\beta = \alpha = 0$ (see (18)); this condition is equivalent to the condition that $b = gradF, F \in F(\mathcal{E}_N)$ and in this case the invariant measures are of type $C\exp[2F(u)]du$, where $C = const$.*

b) An A-diffusion is locally symmetrizable if and only if $\widehat{\delta}\beta = 0$ (see (18)) or, equivalently, $dw_{(b)} = 0$.

c) A measure cdu (constant $c > 0$) is an invariant measure of an A-diffusion if and only if $dF = 0$ (see (18)) or, equivalently, $\widehat{\delta}w_{(b)} = -divb = 0$.

Corollary 4.1. *An A-diffusion is symmetric with respect to a Riemannian volume du (i.e. is symmetrizable and the measure ν in (14) coincides with du) if and only if it is a Brownian motion on $\mathcal{E}_N$.*

We omit the proofs of the theorem 4.2 and corollary 4.1 because they are similar to those presented in [7] for Riemannian manifolds. In our case we have to change differential forms and measures on Riemannian spaces into similar objects on $\mathcal{E}_N$

5 Heat Equations for D-Tensors in V- Bundles

To generalize the results presented in Section 3 to the case of d-tensor fields in $\mathcal{E}_N$ we use the Ito idea of stochastic parallel transport [27,30] (correspondingly adapted to transports in vector bundles provided with N-connection structure).

5.1 Scalarized Tensor d-fields and Heat Equations

Consider a compact bundle $\mathcal{E}_N$and the bundle of orthonormalized adapted frames on $\mathcal{E}_N$ denoted as $O(\mathcal{E}_N)$ (see details in [34-37]). Let

$$\{\widetilde{L}_0, \widetilde{L}_1, ..., \widetilde{L}_{q-1}\}$$

be a system of canonical horizontal vector fields on $O(\mathcal{E}_N)$ (with respect to canonical d-connection $\overrightarrow{\Gamma}^{\alpha}_{\beta\gamma}$. The flow of diffeomorphisms $r(t) = r(t, r, w)$ on $O(\mathcal{E}_N)$ is defined through the solution of equations

$$\delta r(t) = \widetilde{L}_{\underline{\alpha}}(r(t)) \circ \delta w^{\underline{\alpha}}(t),$$

$$r(0) = r,$$

and this flow defines a diffusion process, the horizontal Brownian motion on $O(\mathcal{E}_N)$, which corresponds to the differential operator

$$\frac{1}{2}\Delta_{O(\mathcal{E}_N)} = \frac{1}{2}\sum_{\underline{\alpha}} \widetilde{L}_{\underline{\alpha}}\left(\widetilde{L}_{\underline{\alpha}}\right). \tag{22}$$

For a tensor d-field $S(u) = S^{\alpha_1\alpha_2\ldots\alpha_p}_{\beta_1\beta_2\ldots\beta_q}(u)$ we can define its scalarization $F_S(r) = F^{\underline{\alpha}_1\underline{\alpha}_2\ldots\underline{\alpha}_p}_{S\underline{\beta}_1\underline{\beta}_2\ldots\underline{\beta}_q}$ (a system of smooth functions on $O(\mathcal{E}_N)$), defined in our case by using orthonormalized frames in order to deal with bundle $O(\mathcal{E}_N)$.

The action of Laplace-Beltrami operator on d-tensor fields is defined as

$$(\Delta T)^{\alpha_1\alpha_2\ldots\alpha_p}_{\beta_1\beta_2\ldots\beta_q} = G^{\alpha\beta}\left(\overrightarrow{D}_\alpha\left(\overrightarrow{D}_\beta T\right)\right)^{\alpha_1\alpha_2\ldots\alpha_p}_{\beta_1\beta_2\ldots\beta_q} = G^{\alpha\beta}T^{\alpha_1\alpha_2\ldots\alpha_p}_{\beta_1\beta_2\ldots\beta_q;\alpha\beta},$$

where $\overrightarrow{D}T$ is the covariant derivation with respect to $\overrightarrow{\Gamma}^{\alpha}_{\beta\gamma}$. We can calculate that

$$\Delta_{O(\mathcal{E}_N)}(F^{\alpha_1\alpha_2\ldots\alpha_p}_{S\beta_1\beta_2\ldots\beta_q}) = (F_{\Delta S})^{\alpha_1\alpha_2\ldots\alpha_p}_{\beta_1\beta_2\ldots\beta_q}.$$

For a given d-tensor field $S = S(u)$ let be defined this system of functions on $[0,\infty) \times O(\mathcal{E}_N)$:

$$V^{\alpha_1\alpha_2\ldots\alpha_p}_{\beta_1\beta_2\ldots\beta_q}(t, r) = E\left[F^{\alpha_1\alpha_2\ldots\alpha_p}_{S\beta_1\beta_2\ldots\beta_q}(r(t, r, w))\right].$$

According to the theorem 3.1 $V^{\alpha_1\alpha_2\ldots\alpha_p}_{\beta_1\beta_2\ldots\beta_q}$ is a unique solution of heat equation

$$\frac{\delta V}{\delta t} = \frac{1}{2}\Delta_{O(\mathcal{E}_N)}V, \tag{23}$$

$$V_{|t=0} = F^{\alpha_1\alpha_2\ldots\alpha_p}_{S\beta_1\beta_2\ldots\beta_q}.$$

In a similar manner we can construct unique solutions of heat equations (22) for the case when instead of differential forms one considers $\mathcal{R}^{m+n}$-tensors (see [7] for details concerning Riemannian manifolds). We have to take into account the torsion components of the canonical d-connection on $\mathcal{E}_N$.

5.2 Boundary Conditions

We analyze the heat equations for differential forms on a bounded space $\mathcal{E}_N$:

$$\frac{\delta\alpha}{\delta t}=\frac{1}{2}\Box\alpha, \tag{24}$$

$$\alpha_{|t=0}=f,$$

$$\alpha_{norm}=0,(d\alpha)_{norm}=0, \tag{25}$$

where $\Box$ is the de Rham-Codaira Laplacian (17),

$$\alpha_{norm}=\theta_{q-1}(u)\,\delta u^{q-1},$$

$$(d\alpha)_{norm}=\sum_{\gamma=1}^{q}\left(\frac{\delta\alpha}{\delta u^{q-1}}-\frac{\delta\alpha_{q-1}}{\delta u^{\gamma}}\right)\delta u^{q-1}\bigwedge\delta u^{\gamma}.$$

We consider the boundary of $\mathcal{E}_N$ to be a manifold of dimension $q=m+n$ and denote by $\widehat{\mathcal{E}_N}$ the interior part of $\mathcal{E}_N$ and as $\partial\mathcal{E}_N$ the boundary of $\mathcal{E}_N$. In the vicinity $\widehat{\mathcal{U}}$ of the boundary we introduce the system of local coordinates $u=\{(u^{\alpha}),u^{q-1}\geq 0\}$ for every $u\in\mathcal{U}$ and $u\in\mathcal{U}\cap\partial\mathcal{E}_N$ if and only if $u^{q-1}=0$.

The scalarization of 1-form α is defined as

$$[F_{\alpha}]_{\underline{\beta}}(r)=\theta_{\beta}(u)\,e^{\beta}_{\underline{\beta}},r=\left(u^{\beta},e^{\beta}_{\underline{\beta}}\right)\in\mathcal{O}(\mathcal{E}_{\mathcal{N}}).$$

Conditions (25) are satisfied if and only if

$$e^{\alpha}_{q-1}[F_{\alpha}]_{\underline{\alpha}}(r)=0$$

and

$$e^{\underline{\beta}}_{\beta}\frac{\delta}{\delta u^{q-1}}[F_{\alpha}]_{\underline{\beta}}(r)=0,$$

$\underline{\alpha}$=0,1,2,...,q-1, where $e^{\underline{\alpha}}_{\beta}$ is inverse to $e^{\alpha}_{\underline{\beta}}$.

Now we can formulate the Cauchy problem for differential 1-forms (23) and (24) as a corresponding problem for $\mathcal{R}^{n+m}$-valued equivariant functions $V_{\underline{\alpha}}(t,r)$ on $\mathcal{O}(\mathcal{E}_{\mathcal{N}})$:

$$\frac{\delta V_{\underline{\alpha}}}{\delta r}(t,r)=\frac{1}{2}\{\Delta_{\mathcal{O}(\mathcal{E}_{\mathcal{N}})}\,V_{\underline{\alpha}}(t,r)+R^{\underline{\beta}}_{\underline{\alpha}}(r)\,V_{\underline{\beta}}(t,r), \tag{26}$$

$$V_{\underline{\alpha}}(0,r)=(F_f)_{\underline{\alpha}}(r),(\beta=0,1,...,q-2),(\underline{\alpha},\underline{\beta}=0,1,...,q-1),$$

$$e^{\underline{\beta}}_{\beta}\frac{\delta}{\delta u^{q-1}}V_{\underline{\beta}}(t,r)_{|\partial\mathcal{O}(\mathcal{E}_{\mathcal{N}})}=0,f^{\underline{\beta}}_{q-1}V_{\underline{\beta}}(t,r)_{|\partial\mathcal{O}(\mathcal{E}_{\mathcal{N}})}\;=0,$$

where $R^{\underline{\beta}}_{\underline{\alpha}}(r)$ is the scalarization of the Ricci d-tensor and $\partial\mathcal{O}(\mathcal{E}_{\mathcal{N}})=\{r=(u,e)\in\mathcal{O}(\mathcal{E}_{\mathcal{N}}),\,u\in\partial\mathcal{E}_N\,\}$.

The Cauchy problem (25) can be solved by using the stochastic differential equations for the process $(U(t),c(t))$ on $\mathcal{R}^{n+m}_{+}\times\mathcal{R}^{(n+m)^2}$:

$$\delta U_t^\alpha = e^\alpha_{\widehat{\beta}}(t) \circ \delta B^{\widehat{\beta}}(t) + \delta^\alpha_{q-1}\delta\varphi(t),$$

$$\delta e^\alpha_{\underline{\beta}}(t) = -\overrightarrow{\Gamma}^\alpha_{\beta\gamma}(U(t))\, e^\gamma_{\underline{\beta}}(t) \circ \delta U^{\underline{\beta}}(t) =$$

$$-\overrightarrow{\Gamma}^\alpha_{\beta\gamma}(U(t))\, e^\gamma_{\underline{\beta}}(t)\, e^\beta_{\widehat{\tau}}(t) \circ \delta B^{\widehat{\tau}}(t) - \overrightarrow{\Gamma}^\alpha_{q-1\tau}(U(t))\, e^\tau_{\underline{\beta}}(t)\,\delta\varphi(t), \quad (27)$$

$$\left(\widehat{\beta}, \widehat{\tau} = 1, 2, ..., q-1\right)$$

where $B^\alpha(t)$ is a $(n+m)$-dimensional Brownian motion, $U(t)$ is a nondecreasing process which increase only if $U(t) \in \partial\mathcal{E}_N$. In [7] (Chapter IV,7) it is proved that for every Borel probability measure μ on $\mathcal{R}^{n+m}_+ \times \mathcal{R}^{(n+m)^2}$ there is a unique solution $(U(t), c(t))$ of equations (27) with initial distribution μ. Because if

$$G_{\alpha\beta}(U(0))\, e^\alpha_{\underline{\alpha}}(0)\, e^\beta_{\underline{\beta}}(0) = \delta_{\underline{\alpha}\underline{\beta}}$$

then for every $t \geq 0$

$$G_{\alpha\beta}(U(t))\, e^\alpha_{\underline{\alpha}}(t)\, e^\beta_{\underline{\beta}}(t) = \delta_{\underline{\alpha}\underline{\beta}} \ a.s.$$

(this is a consequence of the metric compatibility criterions) we obtain a diffusion process $r(t) = (U(t), c(t))$ on $\mathcal{O}(\mathcal{E}_\mathcal{N})$. Such processes are called horizontal Brownian motions on the bundle $\mathcal{O}(\mathcal{E}_\mathcal{N})$ with a reflecting bound . Let introduce the canonical horizontal fields

$$\left(\widetilde{L}_{\underline{\alpha}}F\right)(r) = e^\alpha_{\underline{\alpha}}\frac{\partial F(r)}{\partial u^\alpha} - \overrightarrow{\Gamma}^\alpha_{\beta\gamma}(u)\, e^\gamma_{\underline{\alpha}} e^\beta_{\underline{\tau}}\frac{\partial F(r)}{\partial e^\alpha_{\underline{\tau}}}, r = (u, e),$$

define the Bochner Laplacian as

$$\Delta_{\mathcal{O}(\mathcal{E}_\mathcal{N})} = \sum_{\underline{\alpha}=1}^{q} \widetilde{L}_{\underline{\alpha}}\left(\widetilde{L}_{\underline{\alpha}}\right)$$

and put

$$\alpha^{q-1q-1}(r) = G^{q-1q-1}(u), \alpha^{q-1\beta}_{\underline{\beta}} = -e^\tau_{\underline{\beta}}\overrightarrow{\Gamma}^\beta_{q-1\tau}(u)\, G^{q-1q-1}.$$

Theorem 5.1. *Let* $r(t) = (U(t), c(t))$ *be a horizontal Brownian motion with reflecting bound given as a solution of equations (26). Then for every smooth function* $S(t, r)$ *on* $[0, \infty) \times \mathcal{O}(\mathcal{E}_\mathcal{N})$ *we have*

$$dS(t, r(t)) = \widetilde{L}_{\widehat{\alpha}} S(t, r(t))\, \delta B^{\widehat{\alpha}} +$$

$$\{\frac{1}{2}\left(\Delta_{\mathcal{O}(\mathcal{E}_\mathcal{N})}S\right)(t, r(t)) + \frac{\partial S}{\partial t}(t, r(t))\}dt + \left(\widetilde{U}_{q-1}S\right)(t, r(t))\,\delta\varphi(t),$$

where $\widetilde{U}_{q-1}$ *is the horizontal lift of the vector field* $U_{q-1} = \frac{\delta}{\delta u^{q-1}}$ *defined as*

$$\left(\widetilde{U}_{q-1}S\right)(t, r) = \frac{\delta S}{\delta u^{q-1}}(t, r) + \frac{\alpha^{q-1\beta}_{\underline{\beta}}}{\alpha^{q-1q-1}(r)}\frac{\partial S}{\partial e^\beta_{\underline{\beta}}}(t, r)$$

and

$$\delta U^{q-1}(t) = \alpha^{q-1q-1}(r(t))\,\delta t, \delta e^{\beta}_{\underline{\beta}}(t) = \alpha^{q-1\beta}_{\underline{\beta}}(r(t))\,\delta t.$$

Finally, in this subsection, we point out that for diffusion processes we are also dealing with the so-called (A,L)-diffusion for bounded manifolds (see, for example, [7]) which is defined by second order operators A and L given correspondingly on $\mathcal{E}_N$ and $\partial\mathcal{E}_N$.

6 Outlook and Conclusions

In the present paper we have given a geometric evidence for a generalization of stochastic calculus on spaces with local anisotropy . It was possible a consideration rather similar to that for Riemannian manifolds by using adapted to nonlinear connection lifts to tangent bundles [31] and restricting our analysis to the case of v-bundles provided with compatible N-connection, d-connection and metric structures. We emphasize that in the so-called almost Hermitian model of generalized Lagrange geometry [11,12] this condition is naturally satisfied. As a matter of principle we can construct diffusion processes on every space $\mathcal{E}_N$ provided with arbitrary d-connection structure. In this case we can formulate all results with respect to an auxiliary convenient d-connection, for instance, induced by the Christoffel d-symbols, and then by using deformations of connections we shall find the deformed analogous of stochastic differential equations and theirs solutions.

When some the results of this paper have been communicated during the Iasi Academic Days, Romania, October 1994 [32] Academician R. Miron and Professor M. Anastasiei pointed our attention to the pioneer works on the theory of diffusion on Finsler manifolds with applications in biology by P.L. Antonelli and T.J.Zastavniak [1,33]. This is a modern brunch of geometry and stochastic calculus applications in physics and biology. Here we remark that because on Finsler spaces the metric in general is not compatible with connection the definition of stochastic processes is very sophisticate.Perhaps,the uncompatible metric and connection structures are more convenient for modeling of stochastic processes in biology and this is successfully exploited by the mentioned authors in spite of the fact that in general it is still unclear the possibility and manner of definition of metric relations in biology. As for formulation of physical models of diffusion in anisotropic media and on locally anisotropic spaces we have to pay a due attention to the mutual concordance of the laws of transport (i.e. of connections) and of metric properties of the space, which in physics plays a crucial role. This allows us to define the Laplace-Beltrami, gradient and divergence operators and in consequence to give the mathematical definition of diffusion process on la-spaces in a standard manner.

Finally, we remark possible extensions of the results of this paper and of [32,34,35] in order to study locally anisotropic random processes with Yang-Mills and gravitational gauge-like interactions [36-38] and the diffusion of locally anisotropic spinor fields [39,40] or of stochastic calculus on supermanifolds for membranes and superstrings [41-43].

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