ON THE GEOMETRY OF THE SHEAF OF FRAMES OF A VECTOR SHEAF

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Abstract

Given a vector sheaf $\mathcal{E}$ we define a corresponding principal sheaf of frames $\mathcal{P}(\mathcal{E})$ and we show that $\mathcal{A}$–connections of $\mathcal{E}$ (in the sence of [3], [4]) correspond bijectively to principal connections of $\mathcal{P}(\mathcal{E})$ (in the sense of [5], [6]). Details and complete proofs will be given in [7].

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1 Introduction

In [3] and [4], A. Mallios has considered $\mathcal{A}$–connections on vector sheaves. Roughly speaking, a vector sheaf $\mathcal{E}$ is a locally free $\mathcal{A}$–module of finite rank over a topological space X, where $\mathcal{A}$ is sheaf of commutative, unital and associative C–algebras. An $\mathcal{A}$–connection on $\mathcal{E}$ is defined, accordingly, by an appropriate morphism on $\mathcal{E}$, as explained in Section 2.

On the other hand, the present author has defined connections on principal sheaves, i.e. connections on sheaves which locally look like certain "geometric" sheaves of groups.

In both cases, the approach is purely algebrotopological, *without any notion of differentiability* (which is, of course, meaningless in this context), a fact which allows to extend a great part of the classical differential geometry to non–smooth spaces.

Motivated by the relationship between linear connections on vector bundles and their counterparts on the corresponding frame bundles, we associate a vector sheaf with a principal sheaf of frames and we show that $\mathcal{A}$–connections on the former correspond bijectively to principal connections on the latter.

The paper contains an outline of the main results in this direction, whereas full details and proofs will appear in [7].

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2 Preliminaries

The starting point is an algebraized space $(X, \mathcal{A})$, where X is a topological space and $\mathcal{A}$ a sheaf of commutative, associative and unital (linear) C–algebras over X. This is an abstraction of the notion of differential manifolds containing also many cases of differential spaces.

In order to develop geometric structures along the main lines of traditional differential geometry, we also attach to X a differential triad $(\mathcal{A}, \partial, \Omega^1)$, where Ω^1 is an $\mathcal{A}$–module over X and $\partial : \mathcal{A} \longrightarrow \Omega^1$ is a C–linear morphism satisfying condition

$$\partial(s \cdot t) = s \cdot \partial(t) + t \cdot \partial(s), \tag{1}$$

for every $s, t \in \mathcal{A}(U)$ and $U \subseteq X$ open.

Remark. The morphism ∂ in (1.1) is the one induced between sections. In fact, we apply the identification of a sheaf with the sheaf of germs of its sections, as well as the identification of a sheaf morphism with that between the corresponding presheaves of sections (see e.g. [1], [2] for relevant details).

If we denote by $\mathcal{A}^{\cdot}$ the *sheaf of units* (: invertible elements) of $\mathcal{A}$, i.e. the sheaf generated by the (complete) presheaf of abelian groups $U \longmapsto (\mathcal{A}(U))^{\cdot}$, U running over the open sets of X, then ∂ induces the morphism of sheaves of (abelian) groups $\widetilde{\partial} : \mathcal{A}^{\cdot} \longrightarrow \Omega^1$ given by

$$\widetilde{\partial}(s) := s^{-1} \cdot \partial(s), \quad s \in \mathcal{A}^{\cdot}(U), \tag{2}$$

for every open $U \subseteq X$.

If $\mathcal{M}_n(\mathcal{A})$ $(n \geq 1)$ denotes the *matrix algebra sheaf* generated by the complete presheaf $U \longmapsto \mathcal{M}_n(\mathcal{A}(U))$ of matrices with entries in the algebra $\mathcal{A}(U)$, for all $U \subseteq X$ open, i.e.

$$\mathcal{M}_n(\mathcal{A})(U) \cong M_n(\mathcal{A}(U)),$$

we may define the $\mathcal{A}$–module

$$\mathcal{M}_n(\Omega^1) := \mathcal{M}_n(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1 \cong (\Omega^1)^{n^2},$$

on which ∂ induces the morphism (using the same symbol)

$$\partial : \mathcal{M}_n(\mathcal{A}) \longrightarrow \mathcal{M}_n(\Omega^1)$$

given by

$$\partial(a) := (\partial(a_{ij})),$$

if $a = (a_{ij}) \in \mathcal{M}_n(\mathcal{A})(U) \cong \mathcal{M}_n(\mathcal{A}(U))$, $U \subseteq X$ open.

Analogously, $\widetilde{\partial}$ extends to a logarithmic differential

$$\widetilde{\partial} : \mathcal{GL}(n, \mathcal{A}) \longrightarrow \mathcal{M}_n(\Omega^1)$$

by setting

$$\widetilde{\partial}(a) := a^{-1} \cdot \partial(a), \quad a = (a_{ij}) \in \mathcal{GL}(n, \mathcal{A})(U).$$

Here $\mathcal{GL}(n,\mathcal{A})$, is the sheaf of units of $\mathcal{M}_n(\mathcal{A})$, generated by the (complete) presheaf

$$U \longmapsto GL(n,\mathcal{A}(U)) = M_n(\mathcal{A}(U))^{\cdot}, \quad U \subseteq X \text{ open.}$$

Hence, $\mathcal{GL}(n,\mathcal{A})(U) \cong GL(n,\mathcal{A}(U))$ and $\mathcal{GL}(n,\mathcal{A}) = \mathcal{M}_n(\mathcal{A})^{\cdot}$, $n \geq 1$.

Since the *adjoint representation* $\mathcal{A}d : \mathcal{GL}(n,\mathcal{A}) \longrightarrow End\,(\mathcal{M}_n(\mathcal{A}))^{\cdot}$, given by $[\mathcal{A}d(g)](a) := g \cdot a \cdot g^{-1}$, induces the representation

$$\mathcal{A}d : \mathcal{GL}(n,\mathcal{A}) \longrightarrow \mathcal{A}ut\,(\mathcal{M}_n(\Omega^1)) := \mathcal{E}nd\,(\mathcal{M}_n(\Omega^1))^{\cdot}$$

with

$$\mathcal{A}d(g)(a \otimes \theta) := (g \cdot a \cdot g^{-1}) \otimes \theta,$$

for every $g \in \mathcal{GL}(n,\mathcal{A})(U)$, $a \in \mathcal{M}_n(\mathcal{A})(U)$, $\theta \in \Omega^1(U)$ and $U \subseteq X$ open, we easily verify that

$$\widetilde{\partial}(a \cdot b) = \mathcal{A}d(b^{-1}) \cdot \widetilde{\partial}(a) + \widetilde{\partial}(b),$$

for every $a, b \in \mathcal{GL}(n,\mathcal{A})(U)$, $\omega \in \mathcal{M}_n(\Omega^1)(U)$ and $U \subseteq X$ open.

3 Connections on Vector Sheaves

In this section we consider a vector sheaf $\mathcal{E} \equiv (\mathcal{E}, X, \pi_E)$ of finite rank, say n. This means, by definition, that $\mathcal{E}$ is an $\mathcal{A}$–module such that there exist $\mathcal{A}$–isomorphisms

$$\phi_\alpha : \mathcal{E}|_{U_\alpha} \xrightarrow{\simeq} \mathcal{A}^n|_{U_\alpha} \cong (\mathcal{A}|_{U_\alpha})^n\,, \tag{3}$$

over an open cover $\mathcal{C} = \{U_\alpha |\ \alpha \in I\}$ of X. An $\mathcal{A}$–*connection* on $\mathcal{E}$ is a C–linear morphism

$$\bigtriangledown : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 =: \Omega^1(\mathcal{E})$$

satisfying the *Leibniz* condition

$$\bigtriangledown(\alpha \cdot s) = \alpha \cdot \bigtriangledown s + s \otimes \partial\alpha, \tag{4}$$

for every $\alpha \in \mathcal{A}(U)$, $s \in \mathcal{E}(U)$ and $U \subseteq X$ open.

For the existence of $\mathcal{A}$–connections, examples and related topics we refer to [3] and [4], where $\bigtriangledown$ is denoted by D (the latter is used here for the principal connections below).

Now, each isomorphism (2.1) determines a natural basis $e^\alpha := (e_1^\alpha, \ldots, e_n^\alpha)$ of $\mathcal{E}(U_\alpha)$ by

$$e_i^\alpha(x) := \phi_\alpha^{-1}(0_x, \ldots, 1_x, \ldots, 0_x); \quad x \in U_\alpha,$$

if 0_x and 1_x (in the i–th entry) are the zero and unit elements of the stalk $\mathcal{A}_x$ respectively. As a result, any section $s \in \mathcal{E}(U_\alpha)$ can be written in the form

$$s = \sum_{i=1}^{n} s_i^\alpha \cdot e_i^\alpha; \quad s_i^\alpha \in \mathcal{A}(U_\alpha)$$

and (2.2) leads to the expression

$$\bigtriangledown(s) = \sum_{i=1}^{n} (s_i^\alpha \cdot \bigtriangledown(e_i^\alpha) + e_i \otimes \partial s_i^\alpha) = \sum_{i=1}^{n} e_i^\alpha \otimes \left(\partial s_i^\alpha + \sum_{j=1}^{n} s_j^\alpha \cdot \omega_{ij}^\alpha \right),$$

from which we determine the *local connection matrix* (with respect to U_α)

$$\omega^\alpha := (\omega_{ij}^\alpha) \in M_n(\Omega^1(U_\alpha)) \cong M_n(\mathcal{A}(U_\alpha)) \otimes_{\mathcal{A}(U_\alpha)} \Omega^1(U_\alpha).$$

As a matter of fact, the previous matrix completely determines the restriction of $\bigtriangledown$ on $\mathcal{E}|_{U_\alpha}$.

The main result here is the following

Theorem 1 *An $\mathcal{A}$–connection ∇ on $\mathcal{E}$ corresponds bijectively to a family of local matrices $(\omega^\alpha)_{\alpha\in I}$, with repect to $\mathcal{C}$, satisfying the compatibility condition (: local gauge equivalence)*

$$\omega^\beta = \mathcal{A}d(g_{\alpha\beta}^{-1}) \cdot \omega^\alpha + \tilde{\partial} g_{\alpha\beta}.$$

4 The Sheaf of Frames

For a given vector sheaf $(\mathcal{E}, X, \pi)$ as in the previous section, we consider the basis of topology $\mathcal{B}$ on X the elements of which are the open subsets V of X such that $V \subseteq U_\alpha$, for some $\alpha \in I$ (recall that $\mathcal{C} = \{U_\alpha | \ \alpha \in I\}$ is an open cover of X over which (2.1) holds).

We verify that if $Iso_{\mathcal{A}|U_\alpha}(\mathcal{A}^n|U_\alpha, \mathcal{E}|U_\alpha)$ is the group of all $\mathcal{A}|U_\alpha$–module isomorphisms, then

$$U \longmapsto Iso_{\mathcal{A}|U}(\mathcal{A}^n|U, \mathcal{E}|U),$$

U running in $\mathcal{B}$, is a complete presheaf. The resulting sheaf $\mathcal{P}(\mathcal{E})$ is defined to be *the sheaf of frames* of $\mathcal{E}$.

$\mathcal{GL}(n, \mathcal{A})$ acts on the right of $\mathcal{P}(\mathcal{E})$ by means of the local actions

$$\delta_U : Iso_{\mathcal{A}|U}(\mathcal{A}^n|U, \mathcal{E}|U) \times GL(n, \mathcal{A}|U) \longrightarrow Iso_{\mathcal{A}|U}(\mathcal{A}^n|U, \mathcal{E}|U),$$

given by

$$\delta_U(f, g) \equiv f \cdot g := f \circ g,$$

for every $U \in \mathcal{B}$.
Moreover, the mappings

$$\Phi_\alpha : \mathcal{P}(\mathcal{E})(U_\alpha) \cong Iso_{\mathcal{A}|U_\alpha}(\mathcal{A}^n|U_\alpha, \mathcal{E}|U_\alpha) \longrightarrow \mathcal{GL}(n, \mathcal{A})(U_\alpha)$$

defined by $\Phi_\alpha(f) := \phi_\alpha \circ f$, are $\mathcal{GL}(n, \mathcal{A})(U_\alpha)$–equivariant isomorphism inducing the corresponding $\mathcal{GL}(n, \mathcal{A})|U_\alpha$–equivariant sheaf isomorphisms

$$\mathcal{P}(\mathcal{E})|U_\alpha \cong \mathcal{GL}(n, \mathcal{A})|U_\alpha, \quad U_\alpha \in \mathcal{C}.$$

As a result, we conclude that

the sheaf of frames $\mathcal{P}(\mathcal{E})$ of $\mathcal{E}$ is a principal sheaf (in the sense of [2]), of structure type $\mathcal{GL}(n, \mathcal{A})$ and with structure sheaf also $\mathcal{GL}(\mathcal{A})$.

It is quite standard to show that $\mathcal{P}(\mathcal{E})$ is fully determined by the 1–cocycle $(g_{\alpha\beta}) \in Z^1(\mathcal{C}, \mathcal{GL}(n, \mathcal{A}))$ coinciding with the coordinate cocycle of $\mathcal{E}$. Moreover, $\mathcal{E}$ is associated with $\mathcal{P}(\mathcal{E})$, i.e.

$$\mathcal{E} \cong \mathcal{P}(\mathcal{E}) \times_X \mathcal{A}^n / \mathcal{GL}(n, \mathcal{A}).$$

According to the general definition of a connection on a principal sheaf (see [5] and [6]), a connection on $\mathcal{P}(\mathcal{E})$ is a morphism of sheaves of sets $D : \mathcal{P}(\mathcal{E}) \longrightarrow \mathcal{M}_n(\Omega^1)$ such that

$$D(\sigma \cdot g) = \mathcal{A}d(g^{-1}) \cdot D(\sigma) + \widetilde{\partial} g, \tag{5}$$

for every $\sigma \in \mathcal{P}(\mathcal{E})(U)$, $g \in \mathcal{GL}(n, \mathcal{A})(U)$ and any $U \subseteq X$ open.

If $\sigma_\alpha := \Phi_\alpha^{-1}(id|\mathcal{A}^n(U_\alpha))$ are the *natural sections* of $\mathcal{P}(\mathcal{E})$, with respect to $\mathcal{C}$, then the evaluation of D on them yields the corresponding *local connection matrices* of D

$$\theta^\alpha := D(\sigma_\alpha), \quad \alpha \in I.$$

As a direct consequence of (3.1) and $\sigma_\beta = \sigma_\alpha \cdot g_{\alpha\beta}$, under appropriate restrictions on $U_{\alpha\beta} \neq \emptyset$, we see that the local connection matrices of D satisfy the compatibility condition

$$\theta^\beta = \mathcal{A}d(g_{\alpha\beta}^{-1}) \cdot \theta^\alpha + \widetilde{\partial} g_{\alpha\beta}. \tag{6}$$

The following result may be thought of as the principal sheaf counterpart of Theorem 1.

Theorem 2 *A connection D on $\mathcal{P}(\mathcal{E})$ corresponds bijectively to a family of local matrices $\{\theta^\alpha \in \mathcal{M}_n(\Omega^1)(U_\alpha) |\ \alpha \in I\}$ satisfying the compatibility condition (3.2).*

Comparing conditions (2.3) and (3.2), the previous theorems lead to the last main result of the note, namely we have

Theorem 3 *There exists a bijection between $\mathcal{A}$–connections on $\mathcal{E}$ and connections D on $\mathcal{P}(\mathcal{E})$.*

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