HOMOMORPHISMS IN THE THEORY OF THE M-POLYSYMMETRICAL HYPERGROUPS AND MONOGENE M-POLYSYMMETRICAL HYPERGROUPS

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Abstract

J. Mittas in his paper [6], which has been announced in the French Academy of Sciences, has introduced a special type of hypergroup that he has named *polysymmetrical*. Also, in the same paper J.Mittas has given certain fundamental properties of this hyperstructure. Staring from the above paper and having called Mittas' structure M-*polysymmetrical hypergroup* (in order to distinguish this polysymmetrical hypergroup from other types of polysymmetrical hypergroups [2], [8], [9] we have proceeded to a profound analysis of this hypergroup [11] and its subhypergroups [12]. In this paper appear the initial results that have been reached during the study of the homomorphisms of the M-polysymmetrical hypergroups and also of the monogene M-polysymmetrical hypergroups.

AMS Subject Classification: 20N20
Key words: hypergroup, subhypergroup, hyperoperation, homomorphism of the hypergroups, monogene

1 Introduction

Let's see, as introductory elements, two definitions and some important properties, which derive from [6],[11], [12].

A set H is called a **M-polysymmetrical hypergroup** (M-P-H) if is is endowed with a hyperoperation $x + y$ that satisfies the following axioms:

1. $(x + y + +z = x + (y + z)$ for every $x, y, z \in H$
2. $x + y = y + x$ for every $x, y \in H$
3. $(\exists 0 \in H)\, (\forall x \in H)\ [x \in 0 + x]$
4. $(\forall x \in H)\, (\exists x' \in H)\ [x + x' = 0]$

Editor Gr.Tsagas *Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, 1995,* 155-165

(x' is an *opposite* or *symmetrical* of x, with regard to the considered 0, and the set of the opposites $S(x) = \{x' \in H : x + x' = 0\}$ is the *symmetrical* set of x).

5. For every $x, y, z \in H$, $x' \in S(x)$, $y' \in S(y)$, $z' \in S(z)$ we have

$$z \in x + y \Rightarrow z' \in x' + y'$$

We remind that on such a hypergroup, wher x runs in H, the sets $C_0(x) = 0 + x$ form a partition of H, which is denoted by $mod(0)$, or simply (0) and for which we have $C_0(x) = C_0(y) \Leftrightarrow 0 + x = 0 + y$. Also, fopr every $x \in H$, $x' \in S(x)$ we have $S(x) = C(x')$ and the set of classes, $H/(0) = G(H)$ is an abelian group which is named *group of reduction* of H. A subhypergroup of a M-P.H. $(H, +)$ which is also M-P.H. with regard to the hyperoperation of H and which has the same zero 0 with H, is called *M-polysymmetrical subhypergroup* (M-P. SH) of H.

Every subhypergroup of a M-P.H. H is M-P. SH of H. Also, a non void subset h of H is a subhypergroup of H if and only if, for all $x, y \in h$ we have $x + S(y) \subseteq h$. If h is a subhypergroup of H then $y \equiv x \ mod\ h \Leftrightarrow y + x' \subseteq h$ holds for every $x, y \in H$. It must be noted that for every normal equivalence relation R, the class $C_R(0)$ is a subhypergroup of H and moreover $R \equiv mod(C_R(0))$.

2 Homomorphisms of M-P.Hs

Having as a starting point the general definition of the homomorphism [1], [3] and especially the definition of the homomorphism in the theory of the canonical hypergroups [5], [7] we proceed to the study of the homomorphisms of M-P.Hs.

Let's suppose that H and H_1 are two M-P.Hs and let φ be a normal homomorphism[1] from H to H_1. Also let 0, 0_1 be the zeros of H, H_1 respectively.

Regarding the above, we have the propositions

Proposition 2.1 *i)* $\varphi(0) = 0_1$,

ii) $\varphi((C_0(x)) = C_{0_1}(\varphi(x))$ *thus* $\varphi(S(x)) = S(\varphi(x))$ *for every* $x \in H$ *(where* $(C_0(x)$, $C_{0_1}(\varphi(x))$ *are the classes* $mod(0)$ *of* x *and* $mod(0_1)$ *of* $\varphi(x)$*, respectively).*

iii) The homomorphic image $\varphi(H)$ *of* H *is a subhypergroup of* H*, (thus M-P.SH of* H_1*).*

iv The Kernel $\mathcal{N}_\varphi = \varphi^{-1}(\varphi(0)) = \varphi^{-1}(0_1)$ *of the homomorphism* φ*,is a subhypergroup of* H*.*

[1] Let H and H_1 be hypergroups. A mapping $\varphi : H \to P(H_1)$ is named *homomorphism from* H to $P(H_1)$, if the relation $\varphi(x \cdot y) \subseteq \varphi(x) \cdot \varphi(y)$ is valid for every $x, y \in H$.
φ is named *strong* homomorphism if the above relation hold as an equality, i.e. $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$.
A homomorphism $\varphi : H \to H_1$, from H to H_1 is named *strict*.
A strong and strict homomorphism is named *normal*.
A homomorphism $\varphi : H \to H_1$, which is one to one mapping from H onto H_1 is named *isomorphism*. Generally, in the case of H, H_1 being hypergroupoids, we define, in the same way as above, the homomorphism from H to H_1 [1], [3], [5].

Proof. i) Let $x \in H$, so $\varphi(x) \in \varphi(H)$. Then $\varphi(x) \in \varphi(x+0) = \varphi(x) + \varphi(0)$ thus $\varphi(0) = 0_1$ (it has been prooved that if $y \in x + y \Rightarrow y = 0$ [11]).

ii) $\varphi(C_0(x)) = \varphi(0+x) = \varphi(0) + \varphi(x) = o_1 + \varphi(x) = C_{0_1}(\varphi(x))$

Also:

$\varphi(S(x)) = \varphi(C_0(x')) = C_{0_1}(\varphi(x')) = 0_1 + \varphi(x') = S(\varphi(x))$, since
$\varphi(0) = 0_1 \Rightarrow \varphi(x+x') = 0_1 \Rightarrow \varphi(x) + \varphi(x') = 0_1 \Rightarrow \varphi(x') \in S(\varphi(x))$
and thus $0_1 + \varphi(x') = S(\varphi(x))$.

iii) There exists $x \in H$ such that $\varphi(x) = a$ for every $a \in \varphi(H)$ and also $\varphi(H) \subseteq H_1$. Moreover it holds that $a + \varphi(H) = \varphi(H)$.

Indeed $a + \varphi(H) = \varphi(x) + \varphi(H) = \bigcup_{y\in H} [\varphi(x) + \varphi(y)] = \bigcup_{y\in H} \varphi(x+y) = \varphi(x+H) = \varphi(H)$.

iv) For every $x, y \in \mathcal{N}_{(\varphi)}$ we have:

$$\varphi(x+y) = \varphi(x) + \varphi(y) = 0_1 + 0_1$$

thus $x + y \subseteq \mathcal{N}_{(\varphi)}$.

Also, for every $x \in \mathcal{N}_{(\varphi)}$, we have:

$0_1 = \varphi(0) = \varphi(x+x') = \varphi(x) + \varphi(x') = 0_1 + \varphi(x') \Rightarrow \varphi(x') = 0 \Rightarrow x' \in \mathcal{N}_{(\varphi)}$ [for every $x' \in \mathcal{N}_{(\varphi)}$] $\Rightarrow$S(x)$\subseteq \mathcal{N}_{(\varphi)}$. Thus $\mathcal{N}_{(\varphi)}$ is M-P.SH of H. Q.e.d.

Proposition 2.2 *i) The homomorphic image $\varphi(h)$ of every subhypergroup h of H is a subhypergroup of $\varphi(H)$ (thus a subhypergroup of H_1).*

ii) The inverse image $\varphi^{-1}(h)$ of every subhypergroup h of $\varphi(H)$ is a subhypergroup of H and $\mathcal{N}_{(\varphi)} \subseteq \varphi^{-1}(h)$.

Proof. i) Because of proposition 2.1 (iii).

ii) Let $x, y \in \varphi^{-1}(h)$. So $\varphi(x), \varphi(y) \in h$ and since $\varphi(S(y)) = S(\varphi(y)) \subseteq h$ we have that $\varphi(x + S(y)) = \varphi(x) + \varphi(S(y)) = \varphi(x) + S(\varphi(Y)) \subseteq h$ thus $x + S(y) \subseteq \varphi^{-1}(h)$, hence $\varphi^{-1}(h)$ is a subhypergroup of H.

Finally, it is obvious that $\mathcal{N}_{(\varphi)} \subseteq \varphi^{-1}(h)$, since $\mathcal{N}_{(\varphi)} = \varphi^{-1}(0_1)$ and $0_1 \in h$.Q.e.d.

Proposition 2.3 *The mapping $x + \mathcal{N}_{(\varphi)} \to \varphi(x) + 0_1$ is an isomorphism from the group $H/\mathcal{N}_{(\varphi)}$ onto $\varphi(H)/(0_1)$ [where $\varphi(H)/(0_1)$ is the group of reduction of $\varphi(H)$].*

Proof. Let ψ be the mapping

$$H/\mathcal{N}_{(\varphi)} \to \varphi(H)/(0_1)$$

such that $\psi(x + \mathcal{N}_{(\varphi)}) = f(x) + 0_1$ for every $x + \mathcal{N}_{(\varphi)} = C_{\mathcal{N}_{(\varphi)}}(x) \in H/\mathcal{N}_{(\varphi)}$.

For the ψ we have:

i) $\psi[(x+\mathcal{N}_{(\varphi)}) + (y+\mathcal{N}_{(\varphi)})] = \{\psi z \in \mathcal{N}_{(\varphi)} : z = x+y\} = \{\varphi(z) + 0_1 : z = x+y\} = \varphi(x+y) + 0_1 = \varphi(x) + \varphi(y) + 0_1 = [\varphi(x) + 0_1] + [\varphi(y) + 0_1] = \psi(x+\mathcal{N}_{(\varphi)}) + \psi(y+\mathcal{N}_{(\varphi)})$.

Thus ψ is a homomorphism between groups.

ii) It is obvious that ψ is a mapping onto $\varphi(H)/(0_1)$.

iii) ψ is one to one, that is the following condition holds:

$$C_{\mathcal{N}_{(\varphi)}}(x) \neq C_{\mathcal{N}_{(\varphi)}}(y) \Rightarrow \psi\left(C_{\mathcal{N}_{(\varphi)}}(x)\right) \neq \psi\left(C_{\mathcal{N}_{(\varphi)}}(y)\right)$$

Since, if it were $\psi\left(C_{\mathcal{N}_{(\varphi)}}(x)\right) = \psi\left(C_{\mathcal{N}_{(\varphi)}}(y)\right)$, that is $\psi\left(x+\mathcal{N}_{(\varphi)}\right) = \psi\left(y+\mathcal{N}_{(\varphi)}\right)$, then it would be $\varphi(x)+0_1 = \varphi(x)+0_1$. So, for every x'$\in S(x)$ we would have: $\varphi(x)+\varphi(x')+0_1 = \varphi(y)+\varphi(x')+0_1$, that is $\varphi(x+x')+0_1 = \varphi(y+x')+0_1 \Rightarrow \varphi(0)+0_1 = \varphi(y+x')+0_1 \Rightarrow \varphi(y+x') = 0$, thus $y+x' \subseteq \mathcal{N}_{(\varphi)}$.

Consequently, if it would be $y = x \ mod(\mathcal{N}_{(\varphi)})$ [12] which means that $x+\mathcal{N}_{(\varphi)} = y+\mathcal{N}_{(\varphi)}$, which is absurd. Therefore ψ is an isomorphism. Q.e.d.

Corollary 2.1 *For every normal monomorphism*[1] $\varphi : H \to H_1$, *the groups* $H/\mathcal{N}_{(\varphi)}$ *and* $H_1/(0_1)$ *are isomorphic.*

Corollary 2.2 *If* $\mathcal{N}_{(\varphi)} = \{0\}$ *then the groups* $H/(0)$ *and* $\varphi(H)/(0_1)$ *are isomorphic.*

Now, having as a starting point a normal homomorphysm $\varphi : H \to H_1$ we consider the mapping:

$$\bar{\varphi} : H \to \varphi(H)/(0_1)$$

such that $\bar{\varphi}(x) = \varphi(x)+0_1$ for every $x \in H$.

For every $x, y \in H$ we have for the $\bar{\varphi}$:

$$\bar{\varphi}(x+y) = \{\bar{\varphi}(z) : z \in x+y\} = \{\varphi(z)+0_1 : z \in x+y\} =$$

$$\varphi(x+y)+0_1 = [\varphi(x)+0_1]+[\varphi(y)+0_1] = \bar{\varphi}(x)+\bar{\varphi}(y).$$

Thus $\bar{\varphi}$ is a normal homorphism from H onto the group of reduction of image $\varphi(H)$ of H.

So we have the proposition:

Proposition 2.4 *To every normal homomorphism* $\varphi : H \to H_1$ *corresponds the mapping* $\bar{\varphi} : H \to \varphi(H)/(0_1)$, *from* H *onto* $\varphi(H)/(0_1)$ *such that* $\bar{\varphi}(x) = \varphi(x)+0_1$ *for every* $x \in H$ *which is also a normal homomorphism.*

We have the mapping $\bar{\varphi}$ *homomorphism of reduction* of the homomorphism φ and because of proposition 2.3 we have:

Proposition 2.5 *For every normal homomorphism* $\varphi : H \to H_1$ *the homomorphism of reduction* $\bar{\varphi}$ *is being factorized as follows:* $\bar{\varphi} = \psi n$, *where* n *is the canonical mapping* $x \to x+\mathcal{N}_{(\varphi)}$ *from* H *onto* $H/\mathcal{N}_{(\varphi)}$, *and* ψ *is the isomorphism from* $H/\mathcal{N}_{(\varphi)}$ *onto* $\varphi(H)/(0_1)$.

Proposition 2.6 *For every subhypergroup* h *of* H, *the canonical homomorphism* $\varphi : x \to x+h$ *is a normal homomorphism from* H *onto* H/h *and obviously* $\mathcal{N}_{(\varphi)} = h$.

[1]i.e. a homomorphism from H onto H_1.

Proof. It is obvious that φ is strict. Also we have:

$$\varphi(x+y) = x+y+h = (x+h)+(y+h) = \varphi(x)+\varphi(y)$$

thus φ is also strong. Therefore φ is normal. Q.e.d.

Corollary 2.3 *For every normal equivalence relation R in H, the mapping $\varphi : H \to H/R$ such that $\varphi(x) = C_R(x)$ for every $x \in H$, is a normal homomorphism.*

[it has been proved that $C_R(0)$ is a subhypergroup of H for every normal equivalence relation in H and it is also valid that $C_R(x) = x + C_R(0)$ [12]].

Proposition 2.7 *For every subhypergroup h of H holds: $h = \mathcal{N}^h_{(\varphi)}$ where $\mathcal{N}^h_{(\varphi)}$ is the kernel of the canonical mapping $\varphi : H \to H/h$.*

Proof. Indeed the kernel of φ is exactly the h because $x \in \mathcal{N}_{(\varphi)} \Leftrightarrow f(x) = h$, also $\varphi(x) = x+h$ and so $x \in \mathcal{N}_{(\varphi)} \Leftrightarrow x+h = h$. Thus $\mathcal{N}_{(\varphi)} = h$.Q.e.d.

In the theory of homomorphisms between M-P.Hs. except the equivalence relation of the homomorphism, we also have the following equivalence relation, which appears due to the feature or the M-P.H:

Definition 2.1 Let $\varphi : H \to H_1$ be a normal homomorphism. We name *equivalence relation* (in H) of the normal homomorphism, related with the $0_1 \in H_1$, the equivalence relation R which is defined as follows:

$$xRy \Leftrightarrow \varphi(x) \equiv \varphi(y)\ mod(0_1).$$

Now, let R' be the equivalence relation of the homomorphism, that is:

$$xR'y \Leftrightarrow \varphi(x) = \varphi(y).$$

We can see that, for every $x, y \in H$, we have:

$$(x,y) \in R' \Rightarrow \varphi(x)+0_1 = \varphi(y)+0_1 \Rightarrow \varphi(x) \equiv \varphi(y)\ mod(0_1) \Rightarrow (x,y) \in R.$$

[The converse generally is not valid because the cancellation law does not hold for the M-P.Hs, i.e. $\varphi(x)+0_1 = \varphi(y)+0_1 \not\Rightarrow \varphi(x) \equiv \varphi(y)$ [11]]. Thus $R' \subseteq R$ and so the proposition:

Proposition 2.8 *Every class $mod(R)$ of H is saturated with respect to the equivalence R'.*

Proposition 2.9 *For every $x, y \in H$ we have:*

$$xRy \Leftrightarrow x \equiv y\ mod(\mathcal{N}_{(\varphi)})$$

or, with other words, starting with the kernel $\mathcal{N}_{(\varphi)}$ of φ, we define the relatin R as follows:

$$xRy \Leftrightarrow x + S(y) \subseteq \mathcal{N}_{(\varphi)}.$$

Proof. Let xRy. Then we have:
$\varphi(x) \equiv \varphi(y)\ mod(0_1) \Leftrightarrow \varphi(x)+0_1=\varphi(y)+0_1 \Leftrightarrow \varphi(x)+\varphi(y')+0_1=\varphi(y)+\varphi(y')+0_1 \Leftrightarrow \varphi(x+y')+0_1=\varphi(y+y')+0_1 \Leftrightarrow \varphi(x+y')+\varphi(0)=\varphi(0)+0_1=0_1+0_1=0_1 \Leftrightarrow \varphi(x+y'+0)=0_1 \Leftrightarrow \varphi(x+y')=0_1 \Leftrightarrow x+y' \subseteq \mathcal{N}_{(\varphi)} \Leftrightarrow x+S(y) \subseteq \mathcal{N}_{(\varphi)}$ and thus $x \equiv y\ mod(\mathcal{N}_{(\varphi)})$ [12]. Q.e.d.

Corollary 2.4 *The equivalence relation R is normal and the class $C_R(0)$ is the kernel $\mathcal{N}_{(\varphi)}$ of the homomorphism φ and for every $x \in H$ holds:*

$$C_R(x) = x + \mathcal{N}_{(\varphi)}.$$

In the end of the paragraph of the homomorphisms we give the following example:

Example 2.1 : Let's consider the M-P.H $(H,+)$ of the example 2.1 of [12], where $H_1 = \underset{X^j \in \bar{G}_1}{\cup} X^j$, $\bar{G} = \{X^0, X^1, X^2, X^3\}$ and the M-P.H $(H_1,+)$ of the example 3.1 of [11] by taking $X = Y'$, so $\bar{G}_1 = \{O^1, Y^1\}$, $H_1 = \underset{Y^i \in \bar{G}_1}{\cup} Y^i$. Both M-P.Hs are equipped with the hyperoperations of the above mentioned examples. We also consider that the sets X^j, Y^i have the same cardinality. Let, for instance be $X^j = \{x_1^j, x_2^j, x_3^j\}$, $j \in \{1,2,3\}$, $Y^i = Y^1 = \{y_1^i, y_2^i, y_3^i\}$ and $X^0 = O = \{0\}$, $Y^0 = O^1 = \{0_1\}$. Then, the mapping $\varphi : H \to H_1$ such that: $\varphi(0) = 0_1$, $\varphi(x_k^2) = 0_1$ and $\varphi(x_k^1) = \varphi(x_k^3) = y_k^1$ for every $k \in \{1,2,3\}$ is a normal homomorphism with kernel $\mathcal{N}_{(\varphi)} = X^0 \cup X^2$ subhypergroup of H [12]. According to corollary 2.1 and proposition 2.3 the groups $H/\mathcal{N}_{(\varphi)}$ and $H_1/(0_1)$ are isomorphic. Indeed
$x_k^1 + \mathcal{N}_{(\varphi)} = x_k^1 + (X^0 \cup X^2) = (x_k^1 + X^0) \cup (x_k^1 + X^2) = X^1 \cup X^3$,
$x_k^2 + (X^0 \cup X^2) = (x_k^2 + X^0) \cup (x_k^2 + X^2) = X^2 \cup X^0$,
$x_k^3 + (X^0 \cup X^2) = (x_k^3 + X^0) \cup (x_k^3 + X^2) = X^3 \cup X^1$ and $0 + \mathcal{N}_{(\varphi)} = X^0 \cup X^2$. So $H/\mathcal{N}_{(\varphi)} = \{X^0 \cup X^2, X^1 \cup X^3\}$ and $H_1/(0_1) = \{Y^0, Y^1\}$.

Remark 2.1 : If we reconsider the above example, taking as M-P.H $(H,+)$ the one which appears on the example 3.2 of [11] and if we take $X = Y^1$, $Y = Y^2$, so $\bar{G} = \{O, Y^1, Y^2\}$, then the proposition 2.3 does not hold for none of the candidate homomorphisms, with $\mathcal{N}_{(\varphi)} = \{0\} = X^0$ or $\mathcal{N}_{(\varphi)} = X^0 \cup X^2$ [X^0, $X^0 \cup X^2$ are the only proper subhypergroups of H] because *card* $H/\mathcal{N}_{(\varphi_1)} = 4$, $H/\mathcal{N}_{(\varphi_2)} = 2$, while $H/(\prime_1) = 3$. Thus the hypergroups H and H_1 are not homomorphic

3 Monogene M-P.Hs

Let H be a M.P.H and $x \in H$. The subhypergroup $\overline{\{x\}}$ of H (thus M-P.SH of H) which is generated by x is named, (in an analogous way to the classical case of the groups and the case of the canonical hypergroups [5], [7],), *monogene* subhypergroup of H generates by x (see also [10]). A subhypergroup h of H is named *monogene*, if there is $x \in H$ such that $h = \overline{\{x\}}$. A M-P.H is named *monogene* M-P.H if there is $x \in H$ such that $h = \overline{\{x\}}$.

First af all, for every $x_1, \ldots, x_n \in C_0(x)$, $x'_1, \ldots, x'_n \in S(x)$ for $n > 1$, we have (see [11]):

$$x_1 + x_2 + \cdots + x_n = \underbrace{x + x + \cdots + x}_{n\ times} = C_0(x) + \cdots + C_0(x)$$

$$x'_1 + x'_2 + \cdots + x'_n = \underbrace{x' + x' + \cdots + x'}_{n\ times} = S(x) + \cdots + S(x) =$$

$$= C_0(x') + \cdots + C_0(x')$$

Also for $n = 1$ we have:

$$0 + x_1 = 0 + x = C_0(x),\ 0 + x'_1 = 0 + x' = S(x) = C_0(x')$$

So, we introduce the definition of a multiplication of an integer n with an element x$\in H$ as follows:

$$n \cdot x = \begin{cases} x + x + \cdots + x & n\ times\ for\ n > 0,\ n \neq 1 \\ 0 & for\ n = 0 \\ x' + x' + \cdots + x' & n\ times\ for\ n < 0,\ n \neq -1 \end{cases}$$

Also for $n = 1$ we define $1 \cdot x = 0 + x$
and for $n = -1$ we define $(-1) \cdot x = 0 + x'$, $x' \in S(x)$, arbitrary.
We easy conclude that the following relations are valid:

$$n \cdot (x + y) = n \cdot x + n \cdot y$$

$$m \cdot x + n \cdot x = (m + n) \cdot x$$

Hence, based on the proposition 2.6 of [12], we have for the subhypergroup $\overline{\{x\}}$ which is generated by x:

$$\overline{\{x\}} = \bigcup_{k,l \in Z^+} (k \cdot x - l \cdot x) = \bigcup_{n \in Z} n \cdot x = \left(\bigcup_{m \in Z - \{1,-1\}} n \cdot x \right) \cup C_0(x) \cup S(x)$$

And so the proposition:

Proposition 3.1 *For every $x \in H$ we have:*

$$\overline{\{x\}} = \bigcup_{n \in Z} n \cdot x = \left(\bigcup_{m \in Z - \{1,-1\}} n \cdot x \right) \cup C_0(x) \cup S(x)$$

Remarks 3.1 :
a) If $0 \in k \cdot x$, then $k \cdot x = k \cdot x' = 0$ for every $k \in \mathbf{Z}$, x$\in H$. Indeed, $0 \in k \cdot x \Rightarrow 0 + k \cdot x' \subseteq k \cdot x + k \cdot x' = k(x + x') = 0 \Rightarrow 0 + k \cdot x' = 0 \Rightarrow k \cdot x' = 0 \Rightarrow k \cdot x' + k \cdot x = 0 + k \cdot x \Rightarrow 0 + k \cdot x = k \cdot (x + x') = 0 \Rightarrow k \cdot x = 0$. b) $m \cdot x \cap n \cdot x \neq \emptyset$ for every $x \in H$ and $m, n \in \mathbf{Z}$. Because, $m \cdot x \cap n \cdot x \neq \emptyset \Leftrightarrow 0 \in m \cdot x - n \cdot x = (m - n) \cdot x$.

Now, we distinguish between two cases which are contradictory to each other, that is:

I. either, for every $m, n \in \mathbf{Z}$, $m \neq n$ is $m \cdot x \cap n \cdot x = \emptyset$, thus $0 \in m \cdot x - n \cdot x = (m-n) \cdot x$ and therefore $0 \notin h \cdot x$ for none of the $h \in \mathbf{Z} - \{0\}$.

   In thisd case the element x and the monogene subhypergroup $\overline{\{x\}}$, is said to have *infinite* $(+\infty)$ *order*.

II. or, there exist, $m, n \in \mathbf{Z}$, $m \neq n$ such that: $m \cdot x \cap n \cdot x \neq \emptyset$ and thus there exists $h \in \mathbf{Z} - \{0\}$ such that $0 \in n \cdot x$ (so $h \cdot x = 0$).

   Because of the previous remark 3.1,a when $h < 0$ then we have for $-h > 0$, that $0 \in -h \cdot x$. Therefore, in this case, there exists a minimum positive integer h, hence the element x and the subhypergroup $\overline{\{x\}}$ is said to have *order h*. In this case the subhypergroup $\overline{\{x\}}$ is named *cyclic*.

Denoting the order of an element x and also the order of the subhypergroup $\overline{\{x\}}$ by $\omega(x)$, we give in briefly the above cases in the following proposition:

Proposition 3.2 *We say that:*

i) $\omega(x) = +\infty$, *when for every* $m, n \in \mathbf{Z}$, $m \neq n$ *is valid that* $m \cdot x \cap n \cdot x = \emptyset$ *or, equivalently, when for every* $h \in \mathbf{Z} - \{0\}$ *we have* $h \cdot x \neq 0$.

ii) $\omega(x) = l \in \mathbf{N} - \{0\}$, *when there exist* $m, n \in \mathbf{Z}$, $m \neq n$ *such that* $m \cdot x \cap n \cdot x \neq \emptyset$ *or, equivalently, when there exist* $h \in \mathbf{Z} - \{0\}$ *such that* $h \cdot x = 0$. l *is the minimum positive integer which has the previous mentioned property.*

Remarks 3.2 :
a) The zero of H has order 1 and it is the only element oh H which has order 1 (Indeed, let $x \in H$, $x \neq 0$ and $\omega(x) = 1$, then $1 \cdot x = 0$ but $1 \cdot x = x \cdot 0$ and $0 \notin 0 + x$). b)Obviously, the elements $x, x' \in H$, $x' \in S(x)$ generate the same monogene subhypergroup.

Now let's consider a monogene M-P.SH, with $\omega(x) = \lambda$, so $\lambda \cdot x = 0$. Also let $m \in \mathbf{Z} - \{0\}$ such that $m \cdot x = 0$. Then $m = k \cdot \lambda + u$, $k \in \mathbf{Z}$, $0 \leq u < \lambda$.
So: $(k \cdot \lambda + u) \cdot x = 0 \Leftrightarrow k \cdot \lambda \cdot x + u \cdot x = 0 \Leftrightarrow k \cdot 0 + u \cdot x = 0 \Leftrightarrow 0 + u \cdot x = 0 \Leftrightarrow u \cdot x = 0$. But $u < \lambda$ and λ is the minimum non zero positive integer which has this property, consequently $u = 0$.
And so the proposition:

Proposition 3.3 *If* $\omega(x) = \lambda$ *then* $m \cdot x = 0$, $m \in \mathbf{Z} - \{0\}$ *if and only if* $m = \lambda \cdot x$, $x \in \mathbf{Z}$.

Further on, we easily observe that $\Omega(x) = \Omega\left(\overline{\{x\}}\right) = 0$[1], that is the quotient set $\overline{\{x\}}/\Omega(x) = \overline{\{x\}}/(0)$ is the group of reduction of $\overline{\{x\}}$ [11]. Obviously $\overline{\{x\}}/(0) \subseteq H/(0)$ is a subgroup of $H/(0)$.

[1]In the theory of M-P.Hs holds that $\Omega(x) = \{0\}$, $X \subseteq H$, $X \neq \emptyset$ where H is a M-P.H. See also [4], [7], [8].

Taking into consideration the above, we come to the conclusion that for every $x \in H$ the sets $m \cdot x$, when m runz into $\mathbf{Z}$ are either disjoint sets or coincide to each other. Specifically, when $\omega(x) = +\infty$, for every m_1, $m_2 \in \mathbf{Z}$, $m_1 \neq m_2$ then the sets $m_1 \cdot x$, $m_2 \cdot x$ are disjoint sets, but when $\omega(x) = \lambda$, $\lambda \in \mathbf{N} - \{0\}$, then every set $m \cdot x$ coincides with one of the sets $m \cdot x$, $m \in \{0, 1, \ldots, \lambda - 1\}$. (Because $m = k \cdot \lambda + u \Rightarrow m \cdot x = k \cdot \lambda \cdot x + u \cdot x = k \cdot 0 + u \cdot x$, $0 \leq u < \lambda$). So the sets $m \cdot x$ form a partition in $\overline{\{x\}}$, denoted by $mod(\omega(x))$. For the partition $mod(\omega(x))$ we have:

$$z_1 \equiv z_2 \ (mod(\omega(x)) \Leftrightarrow (\exists k \in \mathbf{Z})[(z_1 \in k \cdot x) \wedge (z_2 \in k \cdot x)] \Leftrightarrow$$

$$\Leftrightarrow z_1 + z_2' \subseteq k \cdot x + k \cdot x' = k \cdot (x + x') = k \cdot 0 = 0 \Leftrightarrow$$

$$z_1 + z_2' = 0 \Leftrightarrow z_1 + z_2' + z_2 = 0 + z_1 = 0 + z_2 \Leftrightarrow z_1 \equiv z_2 \ mod(0).$$

Thus we deduce to the proposition:

Proposition 3.4 *For every $x \in H$, the sets $m \cdot x$, $m \in \mathbf{Z}$ when m runs into $\mathbf{Z}$, form a partition of the monogene M-P.SH $\overline{\{x\}}$, which coincides with the partition $mod(0)$ of $\overline{\{x\}}$.*

Corollary 3.1 *When $z \in m \cdot x$ then it is valid that $m \cdot x = 0 + z = C_0(z)$.*

Proposition 3.5 *If $\omega(x) = +\infty$ then the group of reduction $\overline{\{x\}}/(0)$ of $\overline{\{x\}}$ is isomorphic to the additive group $\mathbf{Z}$ of integers. If $\omega(x) = \lambda$, $\lambda \in \mathbf{N} - \{0\}$ then the group $\overline{\{x\}}/(0)$ is isomorphic to the additive group $\mathbf{Z}/(\lambda)$ of the classes mod (λ) of $\mathbf{Z}$.*

As it is in the theory of the groups, a M-P.H is said to be *without torsion* if everyone of its elements, except zero, has infinite order and a M-P.H is said to be *with torsion* or *periodic* if everyone of its elements has finite order.

The above can be seen in two examples of monogene M-P.Hs (one without torsion and one with torsion):

Example 3.1 : 1. Taking into consideration the proposition 2.3 of [11] and starting with the example 2.1 of [12] we consider as the group G, the group $\mathbf{Z}$ of integers and an union $H = \bigcup_{i \in \mathbf{Z}} X^i$ of disjoint sets, with $X^0 = O = \{X_l^0\} = \{0\}$. The set H, endowed with the hyperoperation

$$x_l^k \dot{+} x_n^m = \varphi^{-1}\left[\varphi\left(x_l^k\right) + \varphi\left(x_n^m\right)\right],$$

where $\varphi : H \to \mathbf{Z}$ from H onto $\mathbf{Z}$, such that $\varphi(x_l^k) = k$ for every $x_l^k \in X^k$, $k \in \mathbf{Z}$ (thus $\varphi(X^k) = k$), is a monogene M-P.H without torsion, according to proposition 3.4, having as generator an arbitrary $e \in X'$ (thus $\varphi(e) = 1$). Indeed, for every $k \in \mathbf{Z}^+$, k$\neq$ 1, we have:

$$k \cdot e = \underbrace{e \dot{+} \cdots \dot{+} e}_{k \ times} = \varphi^{-1} \underbrace{[\varphi(e) + \cdots + \varphi(e)]}_{k \ times} = \varphi^{-1} \underbrace{(1 + \cdots + 1)}_{k \ times} \varphi^{-1}(k) =$$

$$= X^k \neq \emptyset.$$

Analogous are the cases, for $k \in \mathbf{Z}$, $k \neq -1$ and for $k = 0, 1, -1$.

2. Based again on the proposition 2.3 of [11] and on example 2.1 of [12] we consider as the group G, the additive group $\mathbf{Z}/(n)$ of the classes $mod(n)$ of $\mathbf{Z}$, $n \in \mathbf{N} - \{0.1\}$ and an union $H = \bigcup_{i \in E} X^i$ of disjoint sets, where $E = \{0, 1, \ldots, (n-1)\}$, with $X^0 = O = \{X_l^0\} = \{0\}$. Here the set of H which is endowed with the same hyperoperation $x_l^k \dot{+} x_n^m$ as above, but with $\varphi : H \to \mathbf{Z}/(n)$ from H onto $\mathbf{Z}/(n)$ such that $\varphi(x_l^k) = \bar{k}$ for every $x_l^k \in X^j$, $k \in E$ (thus $\varphi(X^k) = \bar{k}$), is a monogene M-P.H with torsion. According to proposition 3.4, H has order n and the generator of H is an arbitrary element $q \in X^\delta$, $\delta \in E - \{0\}$ with n being the order of $\bar{\delta}$, thus $n \cdot \bar{\delta} = \bar{0}$ and $\varphi(q) = \bar{\delta}$. Indeed $n \cdot q = q \dot{+} \cdots \dot{+} q = \varphi^{-1}[\varphi(q) + \cdots + \varphi(q)] = \varphi^{-1}(\bar{\delta} + \cdots + \bar{\delta}) = \varphi^{-1}(n\bar{\delta}) = \varphi^{-1}(\bar{0}) = \{0\}$ thus $\omega(q) = n$. Moreover, for every $\tau \in X^\rho$ (thus $\varphi(\tau) = \bar{\rho}$) and for every $k \in E$ we have:

$$k \cdot \tau = \underbrace{\tau \dot{+} \cdots \dot{+} \tau}_{k\ times} = \varphi^{-1}\ \underbrace{[\varphi(\tau) + \cdots + \varphi(\tau)]}_{k\ times} = \varphi^{-1}\ \underbrace{(\bar{\rho} + \cdots + \bar{\rho})} =$$

$$= \varphi^{-1}(k\rho) = X^{k\rho}$$

and also for some $k_1 \in E$ it will be $k_1 \cdot \bar{\rho} = 0$. Therefore $X^{k_1\rho} = X^0 = \{0\}$. Hence $\omega(\tau) = k_1$, that is τ has finite order.

Remark 3.3 : Resulting from the above examples there derives the way of construction of all the monogene M-P.Hs with or without torsion.

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