THE INVARIANCE OF THE EINSTEIN EQUATION ON GENERALIZED FLAG MANIFOLDS UNDER INNER AUTOMORPHISMS

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Abstract

For a certain homogeneous space G/H, called *generalized flag manifold*, let $\mathbf{g}$ and $\mathbf{k}$ be the Lie algebras of G and K respectively, and $\mathbf{h}$ a fixed Cartan subalgebra of $\mathbf{k}$. In this paper we show that the Einstein equation for a generalized flag manifold is invariant under the group of inner automorphisms of $\mathbf{g}$ that preserve $\mathbf{h}$ and $\mathbf{k}$.

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1 Introduction

A metric g on a Riemannian manifold M is called an *Einstein metric* if $Ric(g) = cg$. In [A] we presented new Einstein metrics for certain homogeneous spaces G/K called *generalized flag manifolds*. These metrics were non-Kähler [Be] and different from the normal metric [Wa-Zi]. The methodology we used there was the reduction of the Einstein equation to an algebraic system of equations through a Lie theoretic description of the Ricci curvature $Ric(g)$ and the G-invariant metric g. Let $\mathbf{g}$ and $\mathbf{k}$ be the Lie algebras of G and K respectively and $\mathbf{h}$ be a fixed Cartan subalgebra of $\mathbf{k}$. In this paper we show that the Einstein equation for a generalized flag manifold is invariant under the group of inner automorphisms of $\mathbf{g}$ that preserve $\mathbf{h}$ and $\mathbf{k}$.

2 Generalized flag manifolds: Lie theoretic description

Let G be a compact, connected and semisimple Lie group. A *generalized flag manifold* is a homogeneous space $M = G/K$ whose isotropy group K is the centralizer of a

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torus in G. Equivalently, M is the adjoint orbit $Ad(G)w$ (w an element in $\mathbf{g}$) of w under the action of the adjoint representation Ad of G in $\mathbf{g}$ [B-F-R, Wa]. Since G is semisimple and compact, the Killing form

$$B(X,Y) = tr\, ad\,(X)\, ad\,(Y)$$

of $\mathbf{g}$ is nondegenerate and negative definite on $\mathbf{g}$, thus giving rise to an orthogonal decomposition of $\mathbf{g}$ as the direct sum $\mathbf{g} = \mathbf{k} \oplus \mathbf{m}$. Here $\mathbf{m} = \mathbf{k}^{\perp}$.

Moreover the tangent space T_pM can be identified with $\mathbf{m}$. This identification is given by

$$X \to X^*\,(p) = \frac{d}{dt}\,(\exp tX \cdot p)\Big|_{t=0}, \quad X \in \mathbf{m},\ p \in M.$$

We fix a Cartan subalgebra $\mathbf{h}^c$ of the complexified Lie algebra $\mathbf{k}^c$; then the Cartan decompositions of $\mathbf{g}^c$ and $\mathbf{k}^c$ are given as follows:

$$\mathbf{g}^c = \mathbf{h}^c + \sum_{\alpha\in R} \mathbf{g}^{(\alpha)},\ \mathbf{k}^c = \mathbf{h}^c + \sum_{\alpha\in R_K} \mathbf{g}^{(\alpha)},\ \mathbf{m}^c = \sum_{\alpha\in R_M} \mathbf{g}^{(\alpha)},$$

where R and R_K are the root systems of the pairs $(\mathbf{g}^c, \mathbf{h}^c)$ and $(\mathbf{k}^c, \mathbf{h}^c)$ respectively. The root system R is decomposed as $R = R_K \cup R_M$ where R_M is the set of *complementary roots*. The spaces $\mathbf{g}^{(\alpha)}$ are the 1-dimensional root spaces whose elements X_α are characterized by the equation $[H, X_\alpha] = \alpha(H)X_\alpha$, $H \in \mathbf{h}^c$. We a also recall that for any root α we can choose elements $E_\alpha \in \mathbf{g}^{(\alpha)}$ ($\alpha \in R$) which have the properties $B(E_\alpha, E_{-\alpha}) = -1$, $[E_\alpha, E_{-\alpha}] = -H_\alpha$, where H_α is determined by the equation $B(H_\alpha, H) = \alpha(H)$ for all $H \in \mathbf{h}^c$, as well as $[E_\alpha, E_\beta] = N_{\alpha,\beta}E_{\alpha+\beta}$ for α, $\beta \in I\!R$, $\alpha + \beta \in I\!R$ with coefficients $N_{\alpha,\beta}$. (*structural constants*). The set $\{E_\alpha : \alpha \in R_M\}$ constitutes a basis for $\mathbf{m}^c$.

3 t-roots and invariant metrics

Having the decomposition $\mathbf{g}^c = \mathbf{k}^c \oplus \mathbf{m}^c$ associated with the generalized flag manifold $M = G/K$ and the decomposition $R = R_K \cup R_M$, of the root system R, we set $\mathbf{h} = \mathbf{g} \cap \mathbf{h}^c$ and define

$$\mathbf{t} = Z\,(\mathbf{k}^c) \cap \mathbf{h} = \{X \in \mathbf{h} : \varphi\,(X) = 0,\ \forall\varphi \in R_X\}\,.$$

If $\mathbf{h}^*$ and $\mathbf{t}^*$ are the dual spaces of $\mathbf{h}$ and $\mathbf{t}$, consider the restriction map

$$\begin{aligned} \kappa : \mathbf{h}^* &\longrightarrow \mathbf{t}^* \\ \alpha &\longmapsto \kappa\,(\alpha) = \alpha\,|_{\mathbf{t}} \end{aligned}$$

and set $R_t = \kappa(R) = \kappa(R_M)$ (note that $\kappa(R_X) = 0$). The elements of R_t are called $\mathbf{t}$-*roots*. The benefit from these is that there is a one-to-one correspondence between $\mathbf{t}$-roots τ and irreducible $ad_{\mathbf{g}}(\mathbf{k}^C)$-invariant submodules $M_\tau^C = \sum_{\kappa(\alpha)=\tau} \mathcal{C} E_\alpha$ of $\mathbf{m}^c$ #

Now, a G-invariant metric on $M=G/K$ can be described by an $ad_{\mathbf{g}}(\mathbf{k})$-invariant scalar product g on $\mathbf{m}$, and we extend $\mathbf{g}$ without any change in notation to $\mathbf{m}^c$. Let $\{\omega^\alpha\colon \alpha\in R_M\}$ be the vector space basis in $(\mathbf{m}^c)^*$ which is dual to the basis $\{E_\alpha : \alpha\in R_M\}$. We fix a system of positive roots $R^+ = R_K^+\cup R_M^+$ where $R_K^+ = R_K\cap R^+$, $R_M^+ = R_M\cap R^+$ and set $R_t^+ = \kappa(R^+)$. The following proposition gives a description of the invariant metrics on M.

Proposition 1 *[Alek] Any real $ad_g(\mathbf{k}^c)$-invariant scalar product g on $\mathbf{m}^c$ has the form*

$$g=\sum_{\alpha\in R_M^+} g_\alpha \omega^\alpha\vee\omega^{-\alpha} = \sum_{\tau\in R_t^*} g_\tau \sum_{\alpha\in\kappa^{-1}(\tau)} \omega^\alpha\vee\omega^{-\alpha},$$

where $\omega\vee\omega' = \frac{1}{2}(\omega\otimes\omega' + \omega'\otimes\omega)$, $g_\alpha\in\mathbf{R}^+$, and $g_\alpha = g_\beta$ if $\alpha\mid_{\mathbf{t}} = \beta\mid_{\mathbf{t}}$ so the invariant Riemannian metrics on a generalized flag manifold $M=G/K$ depend (modulo the scale factor) on $\mid R_t^+\mid$ parameters.

4 The Ricci tensor and the Einstein equation

The Ricci tensor can now be determined by its value on the basis $\{E_\alpha : \alpha\in R_M\}$. We have the following:

Proposition 2 *[A] The Ricci tensor for an invariant metric g described in proposition 1 is given by*

$$Ric(E_\alpha,E_\beta)=0,\ \alpha,\beta\in R_M,\ \alpha+\beta\notin R_M,$$

$$Ric(E_\alpha,E_{-\alpha}) = (\alpha,\alpha) + \sum_{\substack{\varphi\in R_K\\ \alpha+\varphi\in R}} N_{\alpha\beta}^2 + \frac{1}{4}\sum_{\beta\in R_M^*} \frac{N_{\alpha,\beta}^2}{g_{\alpha+\beta}g_\beta}\left(g_\alpha^2 - (g_{\alpha+\beta}-g_\beta)^2\right),$$

where $R_M^* = R_M - \kappa^{-1}(\kappa(\alpha))$. Thus the Einstein equation $Ric(g)=cg$ reduces (after normalizing either one of the g_α or c to 1) to an algebraic system of $\mid R_t^+\mid$ equations with $\mid R_t^+\mid$ unknowns.

5 Inner automorphisms

We recall that $Ad\, z$, $z\in G$ is the derivative of the conjugation $C_z : g\to zgz^{-1}$ in G. The group of inner automorphisms of a complex Lie algebra $\mathbf{g}$ consists of finite products of the form $Ad\, z$, $z\in G$ and it is a subgroup of the group of all automorphisms of $\mathbf{g}$. Further, the Weyl group of R is the set of all linear transformations on $\mathbf{h}_R$ $(=\sum_{\alpha\in R} RH_\alpha)$ induced by inner automorphisms of $\mathbf{g}$ that preserves $\mathbf{h}$. We can now state the main theorem.

Theorem 1 *The set of equations that determine the Einstein condition for $M = G/K$ as given in Proposition 2 is invariant under the group of inner automorphisms of* $\mathbf{g}$ *that preserve* $\mathbf{h}$ *and* $\mathbf{k}$*. Equivalently, it is invariant under those elements in the Weyl group of R that preserve R_K.*

Proof. Without loss of generality we need to examine the effect of $w = Ad\, z$ on the root elements E_α, the structural constants $N_{\alpha,\beta}$, the components g_α of the G-invariant metric g, and finally on the set of equations that determine the Einstein equation.

STEP 1 Action of $Ad\, z$ on E_α.

Since $Ad\, z$ preserves $\mathbf{h}$ the equation $a^*(H) = B(H, Ad\, z(H_\alpha))$ defines a root α^* so that $Ad\, z(H_\alpha) = H_{\alpha^*}$.

Claim: $Ad\, z(E_\alpha) = E_{\alpha^*}$.

We apply $Ad\, z$ to the equation $[H, E_\alpha] = \alpha(H)E_\alpha$ and we obtain

$$[Ad\, z(H), Ad\, z(E_\alpha)) = \alpha(H) Ad\, z(E_\alpha). \tag{1}$$

By the invariance of the Killing form under $Ad\, z$ we have that

$$\begin{aligned} a(H) = B(H, H_\alpha) &= B(Ad\, z(H), Ad\, z(H_\alpha)) \\ &= B(Ad\, z(H), H_{\alpha^*}) = \alpha^*(Ad\, z(H)). \end{aligned} \tag{2}$$

From (1) and (2) we obtain

$$[Ad\, z(H), Adz(E_\alpha)] = \alpha^*(Ad\, z(H)) Ad\, z(E_\alpha)$$

which implies that $Ad\, z(E_\alpha)$ is the root vector E_{α^*} corresponding to the root α^* up to a constant. However the E_α's have been chosen so that this constant is normalized to 1. Notice that α^* also satisfies the equation

$$\alpha^*(H) = B(Ad\, z^{-1}(H), H_\alpha) = \alpha(Ad\, z^{-1}(H)) = w \cdot \alpha(H)$$

which is the usual definition of the action of an element w in the Weyl group on the roots.

STEP 2 Transformation of $N_{\alpha,\beta}$.

The numbers $N_{\alpha,\beta}$ are determined by the equation $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ ($\alpha + \beta \in R$). Applying $Ad\, z$ to this equation and using step 1 we get $[E_{\alpha^*}, E_{\beta^*}] = N_{\alpha,\beta} E_{(\alpha+\beta)^*}$, ($\alpha^* + \beta^* \in R$), or $N_{\alpha^*,\beta^*} E_{\alpha^*+\beta^*} = N_{\alpha,\beta} E_{(\alpha+\beta)^*}$. The last equation determines the action of $Ad\, z$ on $N_{\alpha,\beta}$ implicitly.

STEP 3 Transformation of g_α.

Let w be an element in the Weyl group of R that preserves R_K. Then the diffeomorphism C_z preserves K thus it induces a map $\bar{C}_z$ on G/K. Then $Ad\, z$ restricts to a map $d\bar{C}_z$ on $\mathbf{m} = T_0(G/K)$, and consequently it takes an invariant metric g to a new invariant metric $Ad\, z \cdot g$ defined by

$$Ad\, z \cdot g(X, Y) = g(Ad\, z(X), Ad\, z(Y)).$$

For $X = E_\alpha$ and $Y = E_{-\alpha}$ this gives

$$Ad\,z \cdot g_\alpha = g(E_{\alpha^*}, E_{(-\alpha)^*}) = g_{\alpha^*} \ .$$

STEP 4 Transformation of the system of equations.

We apply $w = Ad\,z$ to the system of equations in Proposition 2 and we obtain

$$(\alpha^*, \alpha^*) + \sum_{\substack{\varphi^* \in R_K \\ \alpha^* + \varphi^* \in R}} N^2_{\alpha^*, \varphi^*} + \frac{1}{4} \sum_{\beta^* \in R^*_M} \frac{N^2_{\alpha^*, \beta^*}}{g_{\alpha^*+\beta^*} g_{\beta^*}} \left(g^2_{\alpha^*} - (g_{\alpha^*+\beta^*} - g_{\beta^*})^2 \right) = g_{\alpha^*}. \tag{3}$$

We need to show that (3) is equivalent to

$$(\alpha^*, \alpha^*) + \sum_{\substack{\varphi \in R_K \\ \alpha^* + \varphi \in R}} N^2_{\alpha^*, \varphi} + \frac{1}{4} \sum_{\beta \in R^*_M} \frac{N^2_{\alpha^*, \beta}}{g_{\alpha^*+\beta} g_{\beta}} \left(g^2_{\alpha^*} - (g_{\alpha^*+\beta} - g_{\beta})^2 \right) = g_{\alpha^*}. \tag{4}$$

Since $\alpha^* = w \cdot a$ we can replace φ and β in (4) by ψ^* and γ^* (for some ψ and γ) respectively. Then we can use the invariance of R_K, under w to obtain equation (3).

Example. Let $G/K = SU(n)/S(U(n_1) \times \ldots \times U(n_s))$, $n = \sum_{i=1}^{s} n_i$.

According to [Alek] and [A] the Einstein equation reduces to the following system

$$n_i + n_j + \frac{1}{2} \sum_{l \neq i,j} \frac{n_l}{g_{il} g_{jl}} \left(g^2_{ij} - (g_{il} - g_{jl})^2 \right) = g_{ij}$$

of $\frac{1}{2}s(s-1)$ equations with $\frac{1}{2}s(s-1)$ unknowns g_{ij}, the components of the $SU(n)$-invariant metric g. The Weyl group of $SU(n)$ is the group of permutations w of the set $\{1, ..., n\}$ which acts on the set $R = \{\varepsilon_i - \varepsilon_j : i \neq j\}$ of roots of $SU(n)$ according to $w(\varepsilon_i - \varepsilon_j) = \varepsilon_{w(i)} - \varepsilon_{w(j)}$, and on g_{ij}, by $wg_{ij} = g_{w(i),w(j)}$.

The integers n_i are transformed by $wn_i = n_{w(i)}$. Since w preserves R_K the set of equivalence classes $\{[g_{ij}] : g_{ij} \approx g_{kl} \iff \kappa(\varepsilon_i - \varepsilon_j) = \kappa(\varepsilon_k - \varepsilon_l)\}$ is closed under w. Thus the equations are preserved.

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