APPLICATIONS OF THE VARIATIONAL PRINCIPLE IN THE FIBERED FINSLERIAN APPROACH

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Abstract

The paper describes the procedure used in the fibered Finslerian approach, for obtaining the generalized Einstein-Yang Mills equations and the equation of stationary curves on vector bundles. As applications, in §2 and §3 are presented these equations for a certain generalized Lagrange space GL^n, which provides a convenient relativistic model, having the E.P.S. conditions satisfied [13].

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1 Preliminaries

Let $\xi = (E, p, M)$ be a vector bundle having the local coordinates (x^i, y^a), $i = \overline{1,n}$, $a = \overline{1,m}$, $(n = \dim M,\ m = \dim E - n)$, endowed with a non-linear connection $N = \{N_i^a(x,y)\}$, a (h,v)-metric given by the d-tensor fields $\{g_{ij}(x,y)\}$ and $\{h_{ab}(x,y)\}$, and the covariant derivations ([14, 6])

$$(1) \qquad \begin{cases} D_i T_j^b = \delta_i T_j^b + L_{ci}^b T_j^c - L_{ji}^k T_k^b = T_{j|i}^b \\ D_a T_j^b = \dot{\partial}_a T_j^b + C_{ca}^b T_j^c - C_{ja}^k T_k^b = T_j^b\ |_a \end{cases},$$

where $\delta_i = \partial_i - N_i^a \dot{\partial}_a$, $\quad \partial_i = \frac{\partial}{\partial x^i}$, $\quad \dot{\partial}_a = \frac{\partial}{\partial y^a}$, $\quad i = \overline{1,n},\ a = \overline{1,m}$ and

$$(2) \qquad D\Gamma(N) = \{L_{jk}^i,\ L_{bk}^a,\ C_{ja}^i,\ C_{bc}^a\}$$

are the coefficients of a d-connection ([6, 4, 5, 11]).

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Let the *torsion d-tensor fields* of $D\Gamma(N)$ be given by

$$T^i_{jk} = L^i_{(jk)},\ R^a_{jk} = \delta_{(k}N^a_{j)},\ P^a_{kb} = \dot{\partial}_b N^a_k - L^a_{bk},\ P^i_{ja} = C^i_{ja},\ S^a_{bc} = C^a_{(bc)}, \tag{3}$$

where we denote $\mu_{(ij)} = \mu_{ij} - \mu_{ji},\ \mu_{\{ij\}} = \mu_{ij} + \mu_{ji}$, and let the *curvature d-tensor fields* described in [11, 15, 6]

$$\{R_j{}^i{}_{kl}, R_b{}^a{}_{kl},\ P_j{}^i{}_{kc},\ P_b{}^a{}_{kc},\ S_j{}^i{}_{cd},\ S_b{}^a{}_{cd}\}. \tag{4}$$

By *generalized gauge transformation* on ξ we shall understand an automorphism of a fixed subgroup $\mathcal{H}$ of $\mathcal{A}ut(\xi)$ ([6]).

A d-tensor field ([11, 14]) whose components in the local adapted basis of $\mathcal{X}(M)$ obey tensorial transformation rules relative to $\mathcal{H}$ ([4]), will be called *generalized gauge tensor field* (g.t.f.), and a function of $\mathcal{F}(E)$ which is invariant under $\mathcal{H}$ will be called *gauge scalar field* (g.s.f.) on ξ ([2, 3, 6]).

We shall assume that the non-linear connection N and the linear d-connection $D\Gamma(N)$ satisfy certain transformation laws ([4]) with respect to the action of $\mathcal{H}$ on ξ, which are provided by the requests that δ_i applied to gauge scalar fields produces a g.t.f and that the associated h-and v-covariant derivatives of $D\Gamma(N)$ preserve the gauge tensorial character of the g.t.f. . In this case, the associated h- and v-covariant derivation laws are called *generalized gauge covariant derivations* (g.c.d.) and the non-linear connection N is called *generalized gauge non-linear connection* (g.n.c.).

Let denote by $\{X_\alpha\} = \{\delta_i, \dot{\partial}_a\},\ \alpha = \overline{1, n+m},\ i = \overline{1, n},\ a = \overline{1, m}$, the local basis of $\mathcal{X}(E)$ adapted to N, and the corresponding dual basis $\{\delta\xi^\alpha\} = \{dx^i, \delta y^a\}$ ([11, 15]). A linear connection D on ξ has in the adapted basis the coefficients given by $D_{X_\gamma} X_\beta = \Gamma^\alpha_{\beta\gamma} X_\alpha,\ \beta, \gamma = \overline{1, n+m}$, has the torsion given by $T(X_\gamma, X_\beta) = \mathcal{T}^\alpha_{\beta\gamma} X_\alpha$, where $\mathcal{T}^\alpha_{\beta\gamma} = \Gamma^\alpha_{(\beta\gamma)} + w^\alpha_{\beta\gamma}$, and the *coefficients of non-holonomy* $w^\alpha_{\beta\gamma}$ are given by $[X_\beta, X_\gamma] = w^\alpha_{\beta\gamma} X_\alpha$. The associated curvature is described by

$$R(X_\delta, X_\gamma)X_\beta = \mathcal{R}^\alpha_{\beta\gamma\delta} X_\alpha, \quad \text{with} \quad \mathcal{R}^\alpha_{\beta\gamma\delta} = X_{(\delta}\Gamma^\alpha_{\beta\gamma)} + \Gamma^\alpha_{\phi(\delta}\Gamma^\alpha_{\beta\gamma)} + \Gamma^\alpha_{\beta\phi} w^\alpha_{\beta\delta}$$

The coefficients of the covariant derivation (1) provide the coefficients (2) of a linear connection D having

$$\Gamma^i_{jk} = L^i_{jk},\ \Gamma^a_{bk} = L^a_{bk},\ \Gamma^i_{ja} = C^i_{ja},\ \Gamma^a_{bc} = C^a_{bc};\ \Gamma^i_{a\alpha} = 0,\ \Gamma^a_{i\alpha} = 0.$$

In this case the non-holonomy coefficients are given by

$$w^i_{jk} = 0,\ w^i_{bk} = 0,\ w^i_{bc} = 0,\ w^i_{jb} = 0,\ w^a_{bc} = 0,\ w^a_{jk} = R^a_{jk},\ w^a_{kb} = -w^a_{bk} = \dot{\partial}_b N^a_k,$$

the torsion coefficients $\mathcal{T}^\alpha_{\beta\gamma}$ are described by the d-tensor fields (3)

$$\mathcal{T}^i_{jk} = T^i{}_{jk},\ \mathcal{T}^i_{ja} = -\mathcal{T}^i_{ja} = C^i_{bc},\ \mathcal{T}^i_{bc} = 0,\ \mathcal{T}^a_{kl} = R^a_{kl},\ \mathcal{T}^a_{kb} = -\mathcal{T}^a_{bk} = P^a_{kb},\ \mathcal{T}^a_{bc} = S^a_{bc}$$

and the curvature $\mathcal{R}^\alpha_{\beta\ \gamma\delta}$ is represented by the d-tensor fields (4)

$$\mathcal{R}_j{}^i{}_{kl} = R_j{}^i{}_{kl},\ R_b{}^a{}_{kl} = R_b{}^a{}_{kl},\ \mathcal{R}_j{}^i{}_{kc} = P_j{}^i{}_{kc},\ \mathcal{R}_b{}^a{}_{kc} = P_b{}^a{}_{kc},$$

$$R_j{}^i{}_{cd} = S_j{}^i{}_{cd},\ \mathcal{R}_b{}^a{}_{cd} = S_b{}^a{}_{cd},\ \mathcal{R}_b{}^i{}_{\alpha\beta} = \mathcal{R}_i{}^b{}_{\alpha\beta} = 0.$$

The Ricci tensor field of D has the components $\mathcal{R}_{\alpha\beta} = \mathcal{R}_\alpha{}^\delta{}_{\beta\delta}$, given by

$$\begin{aligned} \mathcal{R}_{ij} &= R_{ij} = R_i{}^h{}_{jh},\ \mathcal{R}_{ia} = -\overset{2}{P}_{ja} = -P_i{}^h{}_{ha}, \\ \mathcal{R}_{ai} &= \overset{1}{P}_{ai} = P_a{}^b{}_{ib},\ \mathcal{R}_{ab} = S_{ab} = S_a{}^c{}_{bc} \end{aligned}$$

and the scalar curvature is $\mathcal{R} = \mathcal{G}^{\alpha\beta}\mathcal{R}_{\alpha\beta} = R + S,\ R = g^{ij}R_{ij},\ S = h^{ab}S_{ab}$, where $\mathcal{G}^{\alpha\beta}$ is the inverse of the matrix associated to the metric

$$\mathcal{G} = g_{ij}dx^i \otimes dx^j + h_{ab}\delta y^a \otimes \delta y^b = \mathcal{G}_{\alpha\beta}\delta\xi^\alpha \otimes \delta\xi^\beta. \tag{5}$$

Then the Einstein equations on ξ are ([11])

$$\mathcal{R}_{\alpha\beta} - \frac{1}{2}\mathcal{R}\mathcal{G}_{\alpha\beta} = \kappa\mathcal{T}_{\alpha\beta}, \tag{6}$$

where κ is the gravitational constant and $\mathcal{T}_{\alpha\beta}$ are the components of the energy-momentum tensor field in the adapted basis $\{X_\alpha\}$.

I. Let L be a Lagrangian g.s.f. depending functionally on

$$\Phi \in \{\Gamma^\alpha_{\beta\gamma}, \mathcal{G}_{\alpha\beta}, N^a_i\} = \mathcal{F},$$

and let $\mathcal{L} = L(\det(\mathcal{G}))^{1/2}$ be the associated density. Then the variational problem

$$\delta \int \mathcal{L}dxdy = 0$$

provides the Euler-Lagrange equations

$$\frac{\delta\mathcal{L}}{\delta\Phi} = \partial_k\left(\frac{\partial\mathcal{L}}{\partial(\partial_k\Phi)}\right) + \dot{\partial}_a\left(\frac{\partial\mathcal{L}}{\partial(\partial\Phi/\partial y^a)}\right) - \frac{\partial\mathcal{L}}{\partial\Phi} = 0. \tag{7}$$

The general case was developed in [4, 6, 7], where the explicit form of the Einstein - Yang Mills (E.Y.M.) equations were obtained, and they were shown to have a gauge covariant character. We remark that if the g.c.d. $D\Gamma(N)$ and N depend only on $\mathcal{G}$, and $\mathcal{L}$ depends on the family of gauge fields $\mathcal{F}$, then the associated E.Y.M. equations become

$$\frac{\delta\mathcal{L}}{\delta\Phi} = 0, \qquad \Phi \in \mathcal{F}.$$

We shall exemplify this case in the following paragraph.

II. The geodesics of the fibration ξ can be defined considering the variational problem

$$\delta \int_C L\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}\right) dt = 0 \tag{8}$$

for the Lagrangian L given by the length of a smooth curve in E parametrized by $t, C = (x^i(t), y^a(t))$. Denoting the h- and v-velocities by

$$A^i = \dot{x}^i = \frac{dx^i}{dt}, \quad A^a = \frac{dy^a}{dt} + N^a_i \frac{dx^i}{dt},$$

then the Euler-Lagrange equations associated to L with respect to $\{x^i, y^a\}$ yield the gauge covariant equations of stationary curves. Their explicit expressions, for ξ and GL^n will be presented in §3.

2 The generalized Einstein-Yang Mills equations for the space $GL^n = (M, g_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x))$

Let $M^n = (M, g_{ij}(x,y))$ be a generalized Lagrange space (GLS) endowed with the nonlinear connection N and a linear connection $\Gamma(N) = (L^i_{jk}, C^a_{bc})$, ([15, 11, 14]). For $\xi = (TM, p, M)$ and $h_{ab} = \delta^i_a \delta^j_b g_{ij}$, the normal lift of $\Gamma(N)$ to ξ produces the d-connection $D\Gamma(N)$ having

$$L^a_{bk} = \delta^a_i \delta^j_b L^i_{jk}, \quad C^i_{ja} = \delta^i_b \delta^c_j C^b_{ca}. \tag{9}$$

Let $\{\gamma_{ij}(x)\}$ be a Riemannian metric tensor field on M, with γ^i_{jk} and $r_j{}^i{}_{kl}$ given by

$$r^i_{jkl} = \partial_{(l}\gamma^i_{jk)} + \gamma^i_{h(l}\gamma^h_{jk)}, \quad \gamma^i_{jk} = g^{ih}(\partial_{\{k}\gamma_{hj\}} - \partial_h\gamma_{jk})/2,$$

having the Ricci tensor field r_{ij}, and the scalar curvature r; let also the non-linear connection be given by $N^a_i = \gamma^a_{0i}$, where the null index denotes contraction by y. Let the metric tensor field be defined on $\tilde{TM}$ by

$$g(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x),$$

where $\sigma \in \mathcal{F}(\tilde{TM})$ is continuous on TM and C^2- differentiable on $\tilde{TM} = TM \setminus \{0\}$. This metric was studied in [14], and provides a nontrivial example of GLS, non-reducible in general to a Lagrange or Finsler space, and still obeying the Ehlers-Pirani-Schild (E.P.S.) postulates.

Assuming further that σ and $\gamma_{ij}(x)$ are g.s.f. and g.t.f., respectively, we can state
Proposition 1. The canonical metrical connection $\Gamma(N)$ for GL^n ([14, 12]) provides via (9) a gauge linear d-connection $D\Gamma(N)$; it has the coefficients

$$L^i_{jk} = \gamma^i_{jk} + \Lambda^i_{jk}, \quad C^i_{jk} = \delta^i{}_{\{j}\dot{\sigma}_{k\}} - \gamma_{jk}\gamma^{is}\dot{\sigma}_s,$$

where

$$\Lambda^i_{jk} = \delta^i{}_{\{j}\sigma_{k\}} - \gamma_{jk}\gamma^{is}\sigma_s, \quad \sigma_k = \delta_k\sigma, \ \dot{\sigma}_k = \dot{\partial}_k\sigma.$$

Proposition 2. a) The torsions of the metrical canonical d-connection $D\Gamma(N)$ are the g.t.f. given by

$$\{P^a_{kb} = -\Lambda^a_{kb}, \ R^a_{kl} = \delta_{(l}N^a_{k)} = r_0{}^a{}_{kl}, \ C^i_{jk}, \ T^i_{jk} = S^i_{jk} = 0\}.$$

b) The curvature tensor fields of $D\Gamma(N)$ are the g.t.f. given by

$$\begin{aligned}
R_b{}^a{}_{kl} &= r_b{}^a{}_{kl} + \delta^a{}_{(k}\sigma_{bl)} - \gamma^{as}\gamma_{b(k}\sigma_{sl} + \gamma_{bc}\dot{\sigma}^{(c}r_0{}^{a)}{}_{kl},\\
P_b{}^a{}_{kc} &= \delta^a_k\overset{1}{\sigma}_{bc} - \delta^a_c\overset{1}{\sigma}_{kb} - \gamma^{as}(\gamma_{bk}\overset{1}{\sigma}_{sc} - \gamma_{bc}\overset{1}{\sigma}_{ks}) + \gamma^{as}\gamma_{ck}\sigma_{(s}\dot{\sigma}_{b)},\\
S^a_{bcd} &= \delta^a{}_{(c}\dot{\sigma}_{bd)} - \gamma^{as}\gamma_{b(c}\dot{\sigma}_{sd)},
\end{aligned}$$

where we use the notations

$$\begin{aligned}
\sigma_{sl} &= \sigma_{s|l} + \sigma_s\sigma_l - \frac{1}{2}\gamma_{sl}\overset{H}{\sigma},\\
\overset{1}{\sigma}_{sl} &= \sigma_s\mid_l + \sigma_s\dot{\sigma}_l - \frac{1}{2}\gamma_{sl}\overset{M}{\sigma},\\
\dot{\sigma}_{sl} &= \sigma_s\mid_l + \dot{\sigma}_s\dot{\sigma}_l - \frac{1}{2}\gamma_{sl}\overset{V}{\sigma}
\end{aligned}$$

and

$$\begin{aligned}
\overset{H}{\sigma} &= \sigma_k\sigma^k, \qquad \overset{M}{\sigma} = \sigma_k\dot{\sigma}^k, \qquad \overset{V}{\sigma} = \dot{\sigma}_k\dot{\sigma}^k,\\
\sigma_{k|l} &= \delta_l\sigma_k - L^s_{kl}\sigma_s, \qquad \sigma_k\mid_l = \dot{\partial}_l\sigma_k - C^s_{kl}\sigma_s.
\end{aligned}$$

c) The following Lagrangian functions are g.s.f. :

$$\begin{aligned}
L_1 &= R_k{}^a{}_l R^k{}_a{}^l,\ L_2 = C_j{}^i{}_a C^j{}_i{}^a = (3n-2)\overset{V}{\sigma}e^{-2\sigma},\\
L_3 &= P_b{}^a{}_k P^b{}_a{}^k = (3n-2)\overset{H}{\sigma}e^{-2\sigma},\\
L_4 &= R = \left[r - 2(n-1)\gamma^{ij}\sigma_{ij} + 2r_{sc}y^s\dot{\sigma}^c\right]e^{-2\sigma},\\
L_5 &= S = 2(1-n)\gamma^{ij}\dot{\sigma}_{ij}e^{-2\sigma},\ L_0 = \sum_{i=1}^{5} n_i L_i,\ n\in\mathbf{R},\quad L = L_0 + \Lambda,
\end{aligned}\tag{10}$$

where Λ is a g.s.f. on $T\tilde{M}$, R and S are the horizontal and vertical scalar curvatures of $D\Gamma(N)$ respectively, and the raising/lowering of the indices is performed using the metric g.t.f. ([2, 6]).

The proof is computational ([12]). We remark that the Lagrangians considered above depend essentially on the gauge fields

$$\Phi \in \{\gamma_{ij}(x),\ \sigma(x,y)\}$$

and their derivatives. From relation (6) we can infer the following result.

Theorem 1 ([11]). *The Einstein equations of the space GL^n are given by*

$$R_{ij} - \frac{1}{2}g_{ij}(R+S) = \kappa\overset{H}{T}_{ij}, \quad S_{ij} - \frac{1}{2}g_{ij}(S+R) = \kappa\overset{V}{T}_{ij},$$

where $\overset{H}{T}_{ij}$ and $\overset{V}{T}_{ij}$ are the h- and v-components of the energy-momentum g.t.f., and R_{ij}, S_{ij} are the h- and v-Ricci g.t.f. of $D\Gamma(N)$.

Applying the variational Hilbert-Palatini method to the Lagrangian density $\mathcal{L} = GL$, where $G = |\det(g_{ij})|$, we obtain the following

Theorem 2. *If L is a Lagrangian of the form (10), then the following generalized Einstein - Yang Mills equations for $\mathcal{L} = GL$ take place*

$$R_{ij} - \frac{1}{2} g_{ij}(R+S) = \kappa \overset{1}{T}_{ij}, \quad S_{ij} - \frac{1}{2} g_{ij}(S+R) = \kappa \overset{2}{T}_{ij}$$

where

$$\overset{1}{T}_{ij} = g_{ij}L^* - \gamma_{ia}g_{jb}\left[\frac{1}{G}\partial_k\left(\frac{\partial(L^*G)}{\partial B_{kab}}\right) - g^{rs}\frac{\partial R_{rs}}{\partial\gamma_{ab}} - \frac{1}{2}\frac{\partial L}{\partial\gamma_{ab}}\right],$$

$$\overset{2}{T}_{ij} = g_{ij}L^* - \gamma_{ia}g_{jb}\left[\frac{1}{G}\partial_k\left(\frac{\partial(L^*G)}{\partial B_{kab}}\right) - g^{rs}\frac{\partial S_{rs}}{\partial\gamma_{ab}} - \frac{1}{2}\frac{\partial L}{\partial\gamma_{ab}}\right],$$

$L^h = L - R,\ L^v = L - S,\ L^* = (L + L^h + L^v)/2$ and $B_{kab} = \partial_k\gamma_{ab}$.

Corollary 1. For $L = R$, the first E.Y.M. set of equations described above have the equivalent expressions

$$r_{ij} - \frac{1}{2}\gamma_{ij} = \overset{1}{T}_{ij},$$

where

$$\left\{\begin{array}{ll} t_{ij} = u_{ij} + \gamma_{ij}[u + (1-n)\gamma^{rs}\dot{\sigma}_{rs}], & u = (1-n)\gamma^{rs}\sigma_{rs} + r_{oc}\dot{\sigma}^c, \\ u_{ij} = \gamma^{rs}\gamma_{i(j}\sigma_{sr)} - \gamma_{is}\dot{\sigma}^{(s}{r_0}^{t)}{}_{jt} + (n-1)\sigma_{ij}, & r_{ij} = {r_i}^s{}_{js} \end{array}\right. .$$

Proposition 3. The E.Y.M. equations for the Lagrangian $L = L_6 = \overset{1}{P}/[2(1-n)]$ with respect to $\gamma_{ij}(x)$ are given by

$$\overset{1}{P}_{ab} = \overset{M}{T}_{ab},$$

where

$$\overset{M}{T}_{ab} = \overset{1}{P}g_{ab} - \gamma_{ia}g_{jb}\left[\frac{1}{G}\partial_k\left(\frac{\partial LG}{\partial B_{kij}}\right) - \gamma^{rs}\frac{\partial\overset{1}{P}_{rs}}{\partial\gamma_{ij}}\right], \quad \overset{1}{P} = \overset{1}{P}_{ij}g^{ij},\ \overset{1}{P}_{ij} = {P_i}^s{}_{js}.$$

Theorem 3. *a) The E.Y.M equations for the Lagrangian*

$$L = n_1L_1 + n_2L_2 + n_3L_3$$

with respect to γ_{ij} have the form

$$\frac{\delta\mathcal{L}}{\delta\gamma_{lm}} = n_1\left\{\frac{1}{G}\partial_k\left(G\frac{\partial L_1}{\partial B_{klm}}\right) - \left(\gamma^{lm}L_1 + \frac{\partial L_1}{\partial\gamma_{lm}}\right)\right\} + n_2(3n-2)\mu(\gamma^{lm}\overset{V}{\sigma} - \dot{\sigma}^l\dot{\sigma}^m) +$$
$$+ n_3(3n-2)\left\{\frac{2}{G}\partial_k\left(\mu\sigma^s\frac{\partial\sigma_s}{\partial B_{klm}}\right) + \mu\left(\sigma^l\sigma^m - \sigma^s\frac{\partial\sigma_s}{\partial\gamma_{lm}}\right) - g^{lm}\overset{H}{\sigma}\right\} = 0,$$

where $\mu = e^{-2\sigma}$.

b) The generalized Einstein-Yang Mills equation for the Lagrangian L (10) corresponding to σ is given by $\frac{\delta\mathcal{L}}{\delta\sigma} = 0$.

3 Stationary curves in the fibered Finslerian approach

Applying the gauge variational principle for the classical Lagrangian

$$L = (\mathcal{G}_{\alpha\beta}(x,y)\mathcal{V}^\alpha\mathcal{V}^\beta)^{1/2}, \tag{11}$$

where $\{\mathcal{V}^\alpha\} = \{A^i, A^a\}$, we can infer the subsequent result.

Theorem 4. *The equations of geodesics for the Lagrangian (11) are*

$$\frac{DA^i}{dt} = F^i, \quad \frac{DA^a}{dt} = F^a, \quad i = \overline{1,n}, \ a = \overline{1,m}\,, \tag{12}$$

where

$$\frac{DA^i}{dt} = \frac{dA^i}{dt} + L^i_{jk}A^jA^k + C^i_{ja}A^jA^a, \quad \frac{DA^a}{dt} = \frac{dA^a}{dt} + L^a_{bk}A^bA^k + C^a_{bc}A^bA^c$$

and the occuring h- and v-forces have the expressions

$$F^i = \left[-A^jA^kg^{il}(g_{lj|k} - \frac{1}{2}g_{jk|l} + T_{ljk}) - A^bA^jg^{ik}(g_{kj}\mid_b + P_{kjb} - R_{kjb})\right] +$$

$$+A^bA^c\left(\frac{1}{2}g^{ij}h_{bc|j} - P^i{}_{bc}\right) + A^i\frac{dlnL}{dt}\,,$$

$$F^a = -A^bA^ch^{ad}\left(h_{dc}\mid_b - \frac{1}{2}h_{bc}\mid_d + S_{dbc}\right) - A^bA^jh^{ad}\left(h_{db|j} + P_{jdb}\right) +$$

$$+A^jA^kh^{ad}\left(\frac{1}{2}g_{jk}\mid_d + P_{jkd}\right) + A^a\frac{dlnL}{dt}.$$

The same procedure, applied for the Lagrangian L considered by G.S.Asanov in the fibered Finslerian ansatz, is given by ([2, 3])

$$L = \alpha L_H + \beta L_v, \quad \alpha, \beta \in \mathbf{R}, \tag{13}$$

where

$$L_H = (g_{ij}(x,y)A^iA^j)^{1/2}, \quad L_V = (h_{ab}(x,y)A^aA^b)^{1/2},$$

and provides the following result.

Theorem 5. ([4, 6]) *The equations of geodesics for the gauge connection $D\Gamma(N)$ and G.S.Asanov's Lagrangian (13) and (12), provide the expressions of the forces are*

$$F^i = \left[A^i\frac{d\ln L_H}{dt} - A^jA^kg^{il}(g_{lj|k} - \frac{1}{2}g_{jk|l} + T_{ljk}) - A^bA^jg^{ik}(g_{kj}\mid_b + P_{kjb})\right] +$$

$$\frac{\beta L_H}{\alpha L_V}\left[\frac{1}{2}A^bA^cg^{ij}h_{bc|j} - A^bA^jR^i{}_{bj} - A^bA^cP^i{}_{bc}\right]\,,$$

$$F^a = \left[A^a \frac{d \ln L_V}{dt} - A^b A^c h^{ad} (h_{dc} \mid_b - \frac{1}{2} h_{bc} \mid_d + S_{dbc}) - A^b A^j h^{ad} (h_{db|j} + P_{jdb}) \right] +$$

$$+ \frac{\alpha L_V}{\beta L_H} A^j A^k h^{ad} \left(\frac{1}{2} g_{jk} \mid_d + P_{jkd} \right).$$

Remark. The equations (12) represent the horizontal and the vertical part of the equations of geodesics, respectively; F^i and F^a are forces produced by the geometry of the fibre. The proof of the theorem is computational. For the particular case considered in §2, these forces get simpler expressions, as follows

Proposition 4. The forces which occur in the equations of geodesics ([12]) for the canonical connection $\Gamma(N)$ described in Proposition 1 are given by

$F^i = A^i \frac{d \ln L_h}{dt} - A^j A^k P^i{}_{jb} + \frac{\beta L_h}{\alpha L_v} (A^b A^c \Lambda^i{}_{bc} - A^b A^j g^{ik} r_{obkj})$,

$F^a = A^a \frac{d \ln L_v}{dt} - A^b A^j g^{ad} \Lambda_{jdb}$.

Conclusions. Using the gauge variational principle, the generalized Einstein-Yang Mills equations (7) for a vector bundle endowed with a (h, v)-metric g.t.f., a nonlinear gauge connection N and a g.c.d. $D\Gamma(N)$, are infered. In particular, in §2, for the canonical connection on a generalized Lagrange space we derive the explicit form of the generalized Einstein equations. Also, in §3, using the same principle, the equations of stationary curves are obtained for the classical and for G.S.Asanov's extended approach.

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References

[1] A.K.Aringazin, G.S.Asanov, *Problems of Finslerian Theory of Gauge Fields and Gravitation*, Rep. Math. Phys., 25, 2(1988), 35-93.

[2] G.S.Asanov, S.F.Ponomarenko, *Finsler Bundles and Space-Time, Associated Gauge Fields and Connections*, Nauka Eds., Chishinew, 1989.

[3] G.S.Asanov, P.C.Stavrinos, *Finslerian Deviations of Geodesics over Tangent Bundle*, Rep. Math. Phys., 30, 1(1991), 63-69.

[4] V.Balan, *Generalized Einstein - Yang Mills Equations in Vector Bundles*, Bull. Soc. Sci. Mat. Rom., v3(84), n.1(1992), 17-22.

[5] V.Balan, *On Geodesics and Deviations of Geodesics in the Fibered Finslerian Approach*, St. Cerc. Mat., Academy Eds., 46, 4(1994), 415-422.

[6] V.Balan, *On the Generalized Einstein - Yang Mills Equations*, Publ. Math. Debrecen, 43, 1-2(1993), 1-11.

[7] V.Balan, *Palatini Methods for Vector Bundles*, Proc. of the XXII-nd Nat. Conf. of Geometry and Topology, Bucharest, 1991, Sci. Bull. of the Polytechnic Inst. of Bucharest, v.53, n.3-4(1991), 19-22.

[8] V.Balan, *Tidal Forces and Deviations of Geodesics in Fiber Bundles*, The Proc. of the Workshop on Diff. Geometry and Its Applications, 1993, Sci. Bull. of the Politehnica Univ. of Bucharest, Ser. A, 55, 3-4(1993), 47-53.

[9] M.Matsumoto, *Foundation of Finsler Geometry and Special Finsler Spaces*, Kaiseisha Press, Tokyo, 1986.

[10] R.Miron, *Introduction to the Theory of Finsler Spaces*, Proc. of the Nat. Sem. of Finsler Spaces, Braşov, feb. 1980, Timişoara 1981.

[11] R.Miron, A.Anastasiei, *The Geometry of Lagrange Spaces. Theory and Applications*, FTPH 59, Kluwer Academic Publishers, 1994.

[12] R.Miron, V.Balan, R.K.Tavakol, *Einstein and Maxwell Equations for the Space* $GL^n = (M, g(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x))$, Proc. of the National Seminar of Finsler Spaces, Braşov 1992, to appear.

[13] R.Miron, R.K.Tavakol, *Geometry of Space-Time and Generalized Lagrange Spaces*, Publ. Math. Debrecen, 4(1994).

[14] R.Miron, R.K.Tavakol, V.Balan, I.Roxburgh, *Geometry of Space-Time and Generalized Lagrange Gauge Theory*, Publ. Math. Debrecen, 3-4, 42(1993), 215-224.

[15] R.Miron, S.Watanabe, S.Ikeda, *Some Connections on Tangent Bundle and Their Applications to the General Relativity*, Tensor N.S., 46(1987),8-22.

[16] H.Rund, *The Geometry of Finsler Spaces*, Springer, Berlin, 1959.

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