LIE ALGEBRAS METHODS IN UNIFIED THEORIES - GAUGE THEORY OF SPIN $\frac{3}{2}$ FIELD IN $S^3 \times R$ SPACETIME

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Abstract

Using the symmetry properties of S^3, which is the manifold of $SU(2)$, we develope the $SO(3,1) \times SU(N)$ gauge - covariant formulation of the spin 3/2 field, in $S^3 \times R$ spacetime. The explicit form of the $SU(N)$ gauge - invariant Rarita - Schwinger Lagrangian density, as well as the corresponding field equation are derived, pointing out additional terms, due to the gravitational spin effects and to the choice of the pseudo - orthonormal basis.

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1 INTRODUCTION

The $S^3 \times R$ spacetime has always been of great interest for Friedman - Robertson - Walker cosmological models. The analysis of both statical and dynamical $S^3 \times R$ Universe has an important contribution in generalizing the Standard Cosmological Model based on the conformal - flat Robertson - Walker metric. The maximal symmetry of S^3 and its compactness offer the possibility of using group theory techniques in formulating gauge theories, enlarging the Minkowskian background case. Preserving the temporal dimension flat allows us to work out the quantization of the complex scalar field on $\Sigma^3 \times R$ manifolds. M. Carmeli and S. Malin have developed field theories on $S^3 \times R$ topology, exploiting the high symmetry of S^3 spacelike hypersurface, whose Lie algebra is $SU(2) \times SU(2)$. They derived the Klein - Gordon, Weyl and Dirac - type equations, by simply going from the momentum representation to the angular momentum one [1-3]. In our work, we are using a modified method, in order to obtain the Rarita - Schwinger equation on $S^3 \times R$ spacetime. Our results generalize those of D. Sen, who provides an excellent formulation of fermionic fields, in $R \times S^3$ [7].

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The goal of our works is to develope gauge field theories on $S^3 \times R$ spacetime, in order to obtain a good description of physical processes in $S^3 \times R$ Universe, by considering, in a gauge framework, the matter sources [4, 5, 6, 8]. The complete set of the Dirac - Klein - Gordon - Maxwell - Yang - Mills equations, generalizing those of the unified electroweak theory, are presented in [5].

2 REVIEW OF $S^3 \times R$ GEOMETRY

We shall use the same parametrization for the S^3 sphere with the unit radius, as in our previous works [4, 5, 8], namely:

$$\begin{aligned} x^1 &= \cos\alpha\,\cos\theta\ ; \\ x^2 &= \sin\alpha\,\cos\theta\ ; \\ x^3 &= \cos\beta\,\sin\theta\ ; \qquad 0 \le \theta \le \frac{\pi}{2}\ ;\ 0 \le \alpha,\beta \le 2\pi\ ; \\ x^4 &= \sin\beta\,\sin\theta\ . \end{aligned} \tag{1}$$

with the corresponding metric

$$d\sigma^2 = \cos^2\theta(d\alpha)^2 + \sin^2\theta(d\beta)^2 + (d\theta)^2. \tag{2}$$

Exploiting the maximal symmetry of S^3, which is the manifold of $SU(2)$, we define the right - handed triad

$$\begin{aligned} L_1 &= \cos(\alpha+\beta)\left(\tan\theta\frac{\partial}{\partial\alpha} - \cot\theta\frac{\partial}{\partial\beta}\right) - \sin(\alpha+\beta)\frac{\partial}{\partial\theta}\ , \\ L_2 &= \sin(\alpha+\beta)\left(\tan\theta\frac{\partial}{\partial\alpha} - \cot\theta\frac{\partial}{\partial\beta}\right) + \cos(\alpha+\beta)\frac{\partial}{\partial\theta}\ , \\ L_3 &= -\left(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}\right)\ , \end{aligned} \tag{3}$$

satisfying the commutation relations of the angular momentum operators:

$$[L_i,\, L_j] = 2\,\varepsilon_{ij}{}^{k} L_k\ ,\ i,j,k = 1,2,3. \tag{4}$$

By adding $L_4 = \frac{\partial}{\partial t}$, we introduce the pseudo - orthonormal tetradic frame

$$\{L_\alpha\} = \{L_i,\ L_4 = \partial_t\}\ ,\ \alpha = \overline{1,\,4}\ , \tag{5}$$

whose Lie algebra is

$$[L_\mu,\, L_\nu] = -2\,\varepsilon^{\alpha}_{\ \mu\nu 4} L_\alpha = C^{\alpha}_{\mu\nu} L_\alpha\ ,\ \text{with } \varepsilon_{1234} = -1, \tag{6}$$

that means the $SU(2) \times T$ algebra (T being the time translations group). This allows the use of the group theory techniques in describing field theories on $S^3 \times R$ manifold.

In these assumptions, the metric has only diagonal constant terms [4]

$$g_{11} \;=\; g_{22} \;=\; g_{33} \;=\; -g_{44} \;=\; 1. \tag{7}$$

The Levi - Civita coefficients are given by [5]

$$\Gamma_{\alpha\mu\nu} \;=\; \varepsilon_{\alpha\mu\nu 4} \tag{8}$$

and the spin - connection 1-forms read:

$$w_{\mu\nu} \;=\; \varepsilon_{\mu\nu\alpha 4}\omega^{\alpha}, \;\; \text{with} \;\; \begin{array}{rcl} w_{ij} &=& -\varepsilon_{ijk}\omega^{k} \;, \\ w_{i4} &=& 0 \;, \end{array} \tag{9}$$

where ω^{μ} is the dual basis:

$$\begin{array}{rcl} \omega^{1} &=& \cos(\alpha+\beta)\sin(2\theta)(d\alpha - d\beta) \;-\; \sin(\alpha+\beta)d\theta \;, \\ \omega^{2} &=& \sin(\alpha+\beta)\sin(2\theta)(d\alpha - d\beta) \;+\; \cos(\alpha+\beta)d\theta \;, \\ \omega^{3} &=& -\left(\cos^{2}\theta d\alpha \;+\; \sin^{2}\theta d\beta\right) . \end{array} \tag{10}$$

3 YANG - MILLS LAGRANGIAN

We consider G as some simple Lie group, with the associated Lie algebra

$$[T_a,\, T_b] \;=\; i\, f_{ab}^{c}\, T_c \;,\;\; a,b,c = \overline{1,n} \tag{11}$$

and we put the Yang - Mills theory in the $SO(3,1)$ - gauge framework. So, the G gauge - tensor is [8]

$$\begin{array}{rcl} F_{\mu\nu}^{a} &=& L_{\mu}A_{\nu}^{a} \;-\; L_{\nu}A_{\mu}^{a} \;+\; g\; f_{bc}^{a}A_{\mu}^{b}A_{\nu}^{c} \;+\; 2\,\varepsilon_{\alpha\mu\nu 4}A^{\alpha a} \;= \\ &=& \tilde{F}_{\mu\nu}^{a} \;+\; 2\,\varepsilon_{\alpha\mu\nu 4}A^{\alpha a}, \end{array} \tag{12}$$

where the last term comes from the non - commutativity of $\{L_i\}_{i=\overline{1,3}}$.

Hence, the Yang - Mills Lagrangian becomes

$$\begin{array}{rcl} \mathcal{L}_{YM} \;=\; -\dfrac{1}{4}F_{\mu\nu}^{a}F_{a}^{\mu\nu} &=& -\dfrac{1}{4}\left(\tilde{F}_{\mu\nu}^{a} \;+\; 2\,\varepsilon_{\alpha\mu\nu 4}A^{a\alpha}\right)\left(\tilde{F}_{a}^{\mu\nu} \;+\; 2\,\varepsilon^{\beta\mu\nu 4}A_{a\beta}\right) \;= \\ &=& \tilde{\mathcal{L}}_{YM} \;-\; \varepsilon_{\alpha\mu\nu 4}\tilde{F}_{a}^{\mu\nu}A^{a\alpha} \;-\; 2\left(A_{\alpha}^{a}A_{a}^{\alpha} \;-\; A_{4}^{\alpha}A_{\alpha}^{4}\right), \end{array} \tag{13}$$

pointing out some additional terms, namely

$$-\,\varepsilon_{\alpha\mu\nu 4}\tilde{F}_{a}^{\mu\nu}A^{a\alpha} \;-\; 2\left(A_{\alpha}^{a}A_{a}^{\alpha} \;-\; A_{4}^{\alpha}A_{\alpha}^{4}\right) \;,$$

absent in the usual Minkowskian case, described by $\tilde{\mathcal{L}}_{YM}$.

4 RARITA-SCHWINGER-TYPE EQUATION

In flat space, spin $\frac{3}{2}$ field is described by the Lagrangian density

$$\tilde{\mathcal{L}}_{RS} \;=\; \frac{1}{2}\,\varepsilon^{\eta\rho\mu\nu}\bar{\psi}_\eta\gamma_5\gamma_\rho\partial_\mu\psi_\nu \tag{14}$$

In curved space, its coupling to gravity can be written using the idea introduced by M. Carmeli in [1], namely to go from the momentum representation to the angular momentum one. In addition, we shall introduce also the gravitational spin effects, by replacing the usual derivatives ∂_μ, not by $\{L_\mu\}$ as in [1], but by the operator

$$J_\mu \;=\; L_\mu \,+\, S_\mu\,, \tag{15}$$

where $\{L_\mu\}_{\mu=\overline{1,4}}$ are given by (5) and $\{S_\mu\}_{\mu=\overline{1,4}}$ are the spin operators

$$S_\mu \;=\; \frac{i}{2}\varepsilon_{\mu\alpha\beta 4}\Sigma^{\alpha\beta}\;;\;\Sigma^{\alpha\beta} \;=\; \frac{i}{4}\left[\gamma^\alpha,\,\gamma^\beta\right]\,, \tag{16}$$

$\{\gamma^\mu\}$ being the usual Dirac matrices.

In these assumptions, the $SU(N)$ gauge - invariant Rarita - Schwinger Lagrangian density on $S^3\times R$ spacetime takes the form

$$\mathcal{L}_{RS} \;=\; \frac{1}{2}\,\varepsilon^{\eta\rho\mu\nu}\bar{\psi}_\eta\gamma_5\gamma_\rho D_\mu\psi_\nu \;-\; \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a\;. \tag{17}$$

Here, the gauge - covariant derivative

$$D_\mu\psi_\nu \;=\; \nabla_\mu\psi_\nu \,+\, ig\,B_\mu\psi_\nu\,, \tag{18}$$

where

$$B_\mu \;=\; A^a_\mu\,T_a\;,\; a=\overline{1,n}\,, \tag{19}$$

contains the Levi - Civita covariant derivative written in the form:

$$\nabla_\mu\psi_\nu \;=\; J_\mu\psi_\nu \,+\, \psi^\rho\Gamma_{\nu\rho\mu}\;. \tag{20}$$

Using in (18) the expressions (20), (16) and (8), we obtain the explicit form of the Lagrangian density (17) as:

$$\begin{aligned}\mathcal{L}_{RS} \;&=\; \frac{1}{2}\varepsilon^{\eta\rho\mu\nu}\overline{\psi}_\eta\gamma_5\gamma_\rho\left[L_\mu\psi_\nu \,+\, \varepsilon_{\nu\sigma\mu 4}\psi^\sigma \,-\, \frac{1}{4}\varepsilon_{\mu\alpha\beta 4}\gamma^\alpha\gamma^\beta\psi_\nu\right] \,+\\ &+\; ig\varepsilon^{\eta\rho\mu\nu}\overline{\psi}_\eta\gamma_5\gamma_\rho\psi_\nu B_\mu \,-\, \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a\;.\end{aligned} \tag{21}$$

Starting from (21), we can derive now the $SO(3,1)\times G$ gauge - covariant form of the Rarita - Schwinger equation on $S^3\times R$ manifold as being:

$$\begin{aligned}\varepsilon^{\eta\rho\mu\nu}\gamma_5\gamma_\rho(L_\mu+igB_\mu)\psi_\nu \;&-\; \frac{7}{2}\gamma_5\gamma^\mu\psi_\mu\delta^\eta_4 \,+\, \frac{3}{2}\gamma_5\gamma_4\psi^\eta \,+\\ &+\; \frac{3}{2}\gamma_5\gamma^\eta\psi_4 \,+\, \frac{1}{2}\gamma_5\gamma_4\gamma^\eta\gamma^\mu\psi_\mu \;=\; 0\,.\end{aligned} \tag{22}$$

If someone applies mechanically Carmeli's method, described in [1], and replaces $\vec{\nabla}$ by $\vec{L}$, he should obtain the following $SU(N)$ gauge - covariant Rarita - Schwinger - type equation

$$\varepsilon^{\eta\rho\mu\nu}\gamma_5\gamma_\rho(L_\mu + igB_\mu)\psi_\nu \;=\; 0 \;, \tag{23}$$

missing the additional term

$$T \;=\; -\frac{7}{2}\gamma_5\gamma^\mu\psi_\mu\delta_4^\eta \;+\; \frac{3}{2}\gamma_5\gamma_4\psi^\eta \;+\; \frac{3}{2}\gamma_5\gamma^\eta\psi_4 \;+\; \frac{1}{2}\gamma_5\gamma_4\gamma^\eta\gamma^\mu\psi_\mu \;. \tag{24}$$

This comes in the theory as a result of the following two expressions:

- the first one, namely

$$T_S \;=\; -\frac{3}{2}\gamma_5\gamma^\mu\psi_\mu\delta_4^\eta \;-\; \frac{1}{2}\gamma_5\gamma_4\psi^\eta \;+\; \frac{3}{2}\gamma_5\gamma^\eta\psi_4 \;+\; \frac{1}{2}\gamma_5\gamma_4\gamma^\eta\gamma^\mu\psi_\mu \;, \tag{25}$$

  which in the case of the Dirac field becomes $-(3/2)\gamma_5\gamma_4\psi$, being absent in Carmeli's paper [3], expresses the spin effects;

- the second one, namely

$$T_\Gamma \;=\; -2\gamma_5\gamma^\mu\psi_\mu\delta_4^\eta \;+\; 2\gamma_5\gamma_4\psi^\eta \;, \tag{26}$$

  expresses the gravitational effects.

For $\eta = i$, we have

$$\varepsilon^{i\rho\mu\nu}\gamma_5\gamma_\rho(L_\mu + igB_\mu)\psi_\nu \;+\; \frac{3}{2}\gamma_5\gamma_4\psi^i \;+\; \frac{3}{2}\gamma_5\gamma^i\psi_4 \;+\; \frac{1}{2}\gamma_5\gamma_4\gamma^i\gamma^\mu\psi_\mu \;=\; 0 \tag{27}$$

and for $\eta = 4$, we obtain

$$\varepsilon^{4ijk}\gamma_5\gamma_i(L_j \;+\; igB_j)\psi_k \;-\; 3\gamma_5\gamma^i\psi_i \;=\; 0 \;. \tag{28}$$

These results can be now materialized in some important physical cases, like the one corresponding to the gauge group $G = SU(2)\times U(1)$ on which is based the unified electroweak theory.

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