

REDUCTIONS ASSOCIATED WITH VECTOR SUBBUNDLES

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Abstract

The aim of this paper is to study the reductions of the extended and restricted principal bundle associated with an almost complex vector subbundle. These principal bundles are defined earlier for real vector bundles, where the authors studied the induced geometrical objects. The results are similar as in the real case, where the extended reduction is determined by a Finsler splitting, while for the restricted reduction, the Finsler splitting is not sufficient.

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Let $\xi = (E, \pi, M)$ be a real vector bundle, where E is the total space, π is the canonical projection and M is the base. We use [1, 2] as general references on vector bundles. A *nonlinear connection* on ξ is a left splitting C of the canonical inclusion $i : V\xi \rightarrow \tau E$ (i.e. $C \circ i = id_{V\xi}$), where $\tau E = (TE, p, E)$ is the tangent bundle of E , $V\xi = (VE, p|_{VE}, E) = \ker \pi_*$ is the *vertical bundle* of ξ and π_* is the differential map of π .

We consider a vector subbundle $\xi' = (E', \pi_1, M)$ of ξ . According to [3], a ξ' -*F-splitting* of the inclusion $I : \xi' \rightarrow \xi$, is a left splitting of the canonical inclusion $I' = \pi_1^* I : \pi_1^* \xi' \rightarrow \pi_1^* \xi$. This definition is valid if we take open subfibred manifolds of ξ' and ξ instead of the vector bundles ξ' and ξ (as an example the open subfibred manifolds ξ'_0 and ξ_0 of the non-null vectors on ξ' and ξ respectively).

In the sequel we suppose that an almost complex structure J is given on ξ , which induces an almost complex structure J' on ξ' . In this case we say that ξ' is an *almost complex vector subbundle*. If open fibred submanifolds are considered instead of ξ and ξ' , we suppose that they are invariant by J and J' respectively. For example, ξ'_0 and ξ_0 , considered above, enjoy this property.

We give the following example of an almost complex vector subbundle. On the vector bundle τM (which has the total space TM), every nonlinear connection C defines canonically an almost complex structure on the vector bundle τTM . It is integrable iff it is weak flat [1, 5]. Let M' be a submanifold of M . It can be shown that every $\tau M'$ -*F-splitting* of the inclusion $\tau M' \subset \tau M|_{M'}$ induces a nonlinear connection C' on $\tau M'$, such that $\tau M'$ is a total geodesic manifold relating to the nonlinear connection C iff $\tau TM'$ is an almost complex vector subbundle of $\tau TM|_{TM'}$.

In the general case, we consider local coordinates, adapted to the vector bundle structures, which are real on the base M and complex on the fibres of ξ' and ξ respectively: (x^i) on M , (x^i, z^α) on E' , (x^i, z^α, z^u) on E , $i = \overline{1, m}$, $\alpha = \overline{1, k}$, $u = \overline{k+1, n}$, where

$x^i \in \mathbf{R}$, $z^\alpha, z^u \in \mathbf{C}$. The change law of the local coordinates is: $x^{i'} = x^i(x^i)$, $z^{\alpha'} = g_\alpha^{\alpha'}(x^i)z^\alpha + g_u^{\alpha'}(x^i)z^u$, $z^{u'} = g_u^{u'}(x^i)z^u$, where $(g_\alpha^{\alpha'}) \in GL_k(\mathbf{C})$, $(g_u^{u'}(x^i)) \in GL_{n-k}(\mathbf{C})$ and $(g_u^{\alpha'}(x^i)) \in M_{k, n-k}(\mathbf{C})$ are complex matrices. Let $\{s_a\}_{\alpha=1, \bar{k}}$ be a local base of $\Gamma(\xi')$ and $\{s_a\}_{\alpha=1, \bar{k}} \cup \{t_u\}_{u=\bar{k}+1, \bar{n}}$ be a local base of $\Gamma(\xi)$. There are the following local bases: $\{\bar{s}_a\}_{\alpha=1, \bar{k}}$ of $\Gamma(\xi'(\xi'))$, where $\pi_1^* \xi' \stackrel{not}{=} \xi'(\xi')$, and $\{\bar{s}_a\}_{\alpha=1, \bar{k}} \cup \{\bar{t}_u\}_{u=\bar{k}+1, \bar{n}}$ of $\Gamma(\xi'(\xi))$, where $\pi_1^* \xi \stackrel{not}{=} \xi'(\xi)$. The local expression of a ξ' -F-splitting S of the inclusion i is $(x^i, z^\alpha, Z^\alpha, Z^u) \rightarrow (x^i, z^\alpha, Z^\alpha + S_u^\alpha(x^i, z^\alpha)Z^u)$, where $Z^\alpha, Z^u \in \mathbf{C}$. On the fibres it has the matrix form expression: $\begin{pmatrix} Z^\alpha \\ Z^u \end{pmatrix} \rightarrow \begin{pmatrix} \delta_\alpha^\beta & S_u^\alpha \\ 0 & g_u^{u'} \end{pmatrix} \cdot \begin{pmatrix} Z^\alpha \\ Z^u \end{pmatrix}$. Since the splitting S is a morphism of vector bundles, it follows that: $\begin{pmatrix} \delta_{\alpha'}^{\beta'} & S_{u'}^{\alpha'} \\ 0 & g_u^{u'} \end{pmatrix} \cdot \begin{pmatrix} g_\alpha^{\alpha'} & g_u^{\alpha'} \\ 0 & g_u^{u'} \end{pmatrix} = \begin{pmatrix} g_\beta^{\beta'} \\ \delta_\alpha^\beta & S_u^\alpha \end{pmatrix}$, therefore:

$$S_{u'}^{\alpha'}(x^{i'}, z^{\alpha'})g_u^{u'}(x^i) = S_u^\alpha g_\alpha^{\alpha'}(x^i) - g_u^{\alpha'}(x^i). \quad (1)$$

Using the splitting S , we can consider the local sections $\tilde{t}_u = \bar{t}_u - S_u^\alpha \bar{s}_\alpha$ on $\Gamma(\eta)$, where $\eta = \ker S$, such that $\{s_a\}_{\alpha=1, \bar{k}} \cup \{\tilde{t}_u\}_{u=\bar{k}+1, \bar{n}}$ are local bases on $\Gamma(\xi'(\xi))$, adapted as well to $\Gamma(\xi'(\xi'))$ and $\Gamma(\eta)$. A straightforward calculus shows that $\tilde{t}_u = g_u^{u'}(x^i)\tilde{t}_{u'}$.

Proposition 1 *Let ξ' be an almost complex vector subbundle of the vector bundle ξ .*

Then every ξ' -F-splitting of the inclusion $i : \xi' \rightarrow \xi$ induces a reduction of $\xi'(\xi)$ as $\xi'(\xi') \oplus \eta$. Conversely, every reduction of $\xi'(\xi)$ as the Whitney sum $\xi'(\xi') \oplus \eta$ defines a ξ' -F-splitting of the inclusion i . In the both cases η is a vector bundle isomorphic with $\xi'(\xi'')$, where $\xi'' = \xi/\xi'$.

Let us consider the following subgroups of $GL_n(\mathbf{C})$:

$$G_0 = \left\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}; A \in GL_k(\mathbf{C}), B \in GL_{n-k}(\mathbf{C}), C \in M_{k, n-k}(\mathbf{C}) \right\}, \quad (2)$$

$$H_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}; A \in GL_k(\mathbf{C}), B \in GL_{n-k}(\mathbf{C}) \right\}.$$

Proposition 2 *There is a canonical identification*

$$G_0/H_0 \cong M_k = \left\{ \begin{pmatrix} 0 & P \\ 0 & I_{n-k} \end{pmatrix}; P \in M_{k, n-k}(\mathbf{C}) \right\}, \quad (3)$$

of the left classes G_0/H_0 , such that the left action, denoted as \odot , of the group G_0 on M_k is expressed by the adjunction $\begin{pmatrix} E & G \\ 0 & F \end{pmatrix} \odot \begin{pmatrix} 0 & P \\ 0 & I_{n-k} \end{pmatrix} = \begin{pmatrix} E & G \\ 0 & F \end{pmatrix} \cdot \begin{pmatrix} 0 & P \\ 0 & I_{n-k} \end{pmatrix} \cdot \begin{pmatrix} E & G \\ 0 & F \end{pmatrix}^{-1}$, i.e.,

$$\begin{pmatrix} E & G \\ 0 & F \end{pmatrix} \odot \begin{pmatrix} 0 & P \\ 0 & I_{n-k} \end{pmatrix} = \begin{pmatrix} 0 & (E \cdot P + G) \cdot F^{-1} \\ 0 & I_{n-k} \end{pmatrix}. \quad (4)$$

Another canonical identification of the left classes G_0/H_0 can be made with the elements of the subgroup $G_{01} \subset G_0$, where

$$G_{01} = \left\{ \begin{pmatrix} I_k & P \\ 0 & I_{n-k} \end{pmatrix}; P \in M_{k, n-k}(\mathbf{C}) \right\},$$

$$\text{by } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \cdot H_0 \xrightarrow{\Psi} \begin{pmatrix} I_k & C \cdot B^{-1} \\ 0 & I_{n-k} \end{pmatrix}.$$

Let us consider the Lie subgroup $G_{m,n}^1$ of $GL(m+n, \mathbf{C})$ which consists of matrices which have the form $\begin{pmatrix} A_j^i & 0 \\ 0 & B_b^a \end{pmatrix}$, $(A_j^i) \in GL_m(\mathbf{C})$, $(B_b^a) \in G$.

In the sequel, we define the *extended prolongation* (or *e-prolongation*) of order $r \in \mathbb{N}^*$ of the group G , denoted by $G_{m,n}^r$. The elements of $G_{m,n}^r$ have the form

$$a = (A_{j_1}^i, A_{j_1 j_2}^i, \dots, A_{j_1 j_2 \dots j_r}^i; B_b^a, B_{b j_1}^a, \dots, B_{b j_1 \dots j_{r-1}}^a), \quad (5)$$

where all the components in the parentheses are symmetric in the low indices j_k , when $j_k = \overline{1, m}$, $(\forall) k = \overline{1, r}$, and $(A_j^i) \in GL_m(\mathbf{C})$, $(B_b^a) \in G$. For two elements which have the form (5) the composition rule follows considering them as linear maps. More exactly, for $a = (A, B) = ((A_1(\cdot), A_2(\cdot, \cdot), \dots, A_r(\cdot, \dots, \cdot)), B_0(\cdot), B_1(\cdot, \cdot), \dots, B_{r-1}(\cdot, \cdot, \dots, \cdot)))$ and $b = (C, D)$, we have $b \cdot a = (A', B')$, where

$$\begin{aligned} A'_1(u_1) &= C_1 A_1(u_1), \\ A'_2(u_1, u_2) &= C_1 A_2(u_1, u_2) + C_2(A_1(u_1), A_1(u_2)), \\ A'_3(u_1, u_2, u_3) &= C_1 A_3(u_1, u_2, u_3) + C_2(A_2(u_1, u_2), A_1(u_3)) + \\ &+ C_2(A_1(u_1), A_2(u_2, u_3)) + C_2(A_2(u_1, u_3), A_1(u_2)) + C_3(A_1(u_1), A_1(u_2), A_1(u_3)), \\ &\dots \dots \dots \\ B'_0(v) &= D_0 B_0(v), \\ B'_1(v, u_1) &= D_0 B_1(v, u_1) + D_1(B_0(v), A_1(u_1)), \\ B'_2(v, u_1, u_2) &= D_0 B_2(v, u_1, u_2) + D_1(B_1(v, u_1), A_1(u_2)) + \\ &+ D_1(B_0(v), A_2(u_1, u_2)) + D_1(B_1(v, u_2), A_1(u_1)) + D_2(B_0(v), A_1(u_1), A_1(u_2)), \\ &\dots \dots \dots \end{aligned} \quad (6)$$

It is easy to see that if H is a subgroup of G , then $H_{m,n}^r$ is also a subgroup of $G_{m,n}^r$. We can consider the principal bundle $\mathcal{O}G\xi(\xi)^r$ on the base space E and the structural group $G_{m,n}^r$. The structural functions of this bundle are:

$$\varphi_{UU'}(u) = \left(\frac{\partial x^{i'}}{\partial x^i}(x), \dots, \frac{\partial^r x^{i'}}{\partial x^{i_1} \dots \partial x^{i_r}}(x), g_a^{a'}(x), \dots, \frac{\partial^{r-1} g_a^{a'}}{\partial x^{j_1} \dots \partial x^{j_{r-1}}}(x) \right), \quad (7)$$

where $\pi(u) = x$, and $\{g_a^{a'}(x)\}$ are the structural functions of the vertical bundle $V\xi$, constant on the fibres of ξ , which corresponds to an atlas on E of the form $\{U = \pi^{-1}(V), V \subset M\}$.

Proposition 3 *There is a canonical identification $G_{0m,n}^r/H_{0m,n}^r \cong G_0/H_0$ and a surjective morphism of Lie groups $\varphi : G_{0m,n}^r \rightarrow G_0$, which is compatible, via the identification, with the natural left actions: $g \cdot [a] \cong \varphi(g) [\varphi(a)]$, $g \in G_{0m,n}^r$, $[a] \in G_{0m,n}^r/H_{0m,n}^r$.*

Theorem 1 *Let ξ' be an almost complex vector subbundle of the vector bundle ξ and $r \geq 1$. The following properties hold:*

1. *Every ξ' -F-splitting of the inclusion $i : \xi' \rightarrow \xi$ defines canonically a reduction of the structural group $G_{0m,n}^r$ of the principal bundle $\mathcal{O}G_0\xi'(\xi)^r$ to $H_{0m,n}^r$, the reduced principal bundle being $\mathcal{O}H_0\xi'(\xi)^r$.*
2. *Every reduction of the structural group $G_{0m,n}^r$ of the principal bundle $\mathcal{O}G_0\xi'(\xi)^r$ to $H_{0m,n}^r$, coincides with $\mathcal{O}H_0\xi'(\xi)^r$ and it is induced by a ξ' -F-splitting, as above.*

If G is a subgroup of $GL_n(\mathbf{C})$, we denote by $\mathcal{A}(G)$ the subalgebra of $M_n(\mathbf{C})$, generated by G . The algebra $\mathcal{A}(G)$ consists of \mathbf{C} -linear combination of finite products of matrices from G . If $A, B \in \mathcal{A}(G)$, then $[A, B] = A \cdot B - B \cdot A \in \mathcal{A}(G)$, thus $\mathcal{A}(G)$ is a Lie subalgebra of $M_n(\mathbf{C})$. As an example, the matrix subgroups G_0 and H_0 , given by (2), have the following matrix subalgebras:

$$\mathcal{A}(G_0) = \left\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}; A \in \mathcal{M}_k(\mathbf{C}), B \in \mathcal{M}_{n-k}(\mathbf{C}), C \in \mathcal{M}_{k, n-k}(\mathbf{C}) \right\}$$

and

$$\mathcal{A}(H_0) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}; A \in \mathcal{M}_k(\mathbf{C}), B \in \mathcal{M}_{n-k}(\mathbf{C}) \right\}.$$

Let $G \subset GL_n(\mathbf{C})$ be a matrice subgroup.

We define the *restricted prolongation* (or *r-prolongation*) of order $r \in \mathbb{N}^*$ of the group $G_{m,n}^1$, denoted by $\bar{G}_{m,n}^r$, similarly for the e-prolongation, but asking additionally that the matrices $\{B_{b_{j_1 \dots j_{p-1}}}^a\}_{a,b \in \overline{1,n}}$ (considering the indices j fixed) belong to the algebra $\mathcal{A}(G)$. More exactly, considering the element $a = (A, B) = ((A_1(\cdot), A_2(\cdot, \cdot), \dots, A_r(\cdot, \dots, \cdot)), B_0(\cdot), B_1(\cdot, \cdot), \dots, B_{r-1}(\cdot, \cdot, \dots, \cdot)))$ as linear map, then, for every fixed $v_1, \dots, v_p \in \mathbf{C}^m$, the map $\mathbf{C}^n \ni u \rightarrow B_p(u, v_1, \dots, v_p) \in \mathbf{C}^n$ is an endomorphism which belongs to $\mathcal{A}(G)$, using the canonical base in \mathbf{C}^n . It is easy to see that the composition law of $\bar{G}_{m,n}^r$ defined by (6) is well defined and defines on $\bar{G}_{m,n}^r$ a Lie group structure. In fact, $\bar{G}_{m,n}^r$ is a Lie subgroup of the Lie group $G_{m,n}^r$.

If the structural functions of the principal bundle $\mathcal{O}G\xi(\xi)^r$, given by the formula (7) belong to the group $\bar{G}_{m,n}^r$, they define a reduction of the structural group $G_{m,n}^r$ to the subgroup $\bar{G}_{m,n}^r$. The reduced principal bundle is denoted as $\mathcal{O}\bar{G}\xi(\xi)^r$. The reduction is possible when the structural functions $(g_a^{a'}) \in G_0$ or $(g_a^{a'}) \in H_0$. Notice that in the general case, the relation $(g_a^{a'}(x)) \in G$ does not imply that $(\frac{\partial x^{i'}}{\partial x^i}, \dots, \frac{\partial^r x^{i'}}{\partial x^{i_1} \dots \partial x^{i_r}}, g_a^{a'}, \dots, \frac{\partial^{r-1} g_a^{a'}}{\partial x^{j_1} \dots \partial x^{j_{r-1}}}) \in \bar{G}_{m,n}^r$, since for any fixed j_1, \dots, j_p , it does not follow automatically that the matrix $\left(\frac{\partial^p g_a^{a'}}{\partial x^{j_1} \dots \partial x^{j_p}} \right)_{a,a' \in \overline{1,n}}$ belongs to $\mathcal{A}(G)$.

In order to study the reductions of the group $\bar{G}_{0m,n}^r$ of the principal bundle $\mathcal{O}\bar{G}_0\xi'(\xi)^r$ to the group $\bar{H}_{0m,n}^r$, we need a simple form of $\bar{G}_{0m,n}^r/\bar{H}_{0m,n}^r$. In proposition 3 we proved that there is a canonical isomorphism $G_{0m,n}^r/H_{0m,n}^r \cong G_0/H_0$, which allowed to prove in theorem 1 the equivalence between the reductions of the structural group $G_{0m,n}^r$ of the principal bundle $\mathcal{O}G_0\xi'(\xi)^r$ to the subgroup $H_{0m,n}^r$ and the ξ' -F-splitting of the inclusion $i : \xi' \rightarrow \xi$. It can be proved that in the case of the r-prolongations there is no equivalence between $\bar{G}_{0m,n}^r/\bar{H}_{0m,n}^r$ and G_0/H_0 , which allows to conclude that the reductions of the structural group $\bar{G}_{0m,n}^r$ of the principal bundle $\mathcal{O}\bar{G}_0\xi'(\xi)^r$ to the subgroup $\bar{H}_{0m,n}^r$ are not equivalent with ξ' -F-splittings. We state the result in the case $r = 2$, the general situation being analogous, but the proof is more complicated.

Theorem 2 *There is a canonical identification of the left classes $\bar{G}_{0m,n}^2/\bar{H}_{0m,n}^2$ with the subgroup $\bar{G}_{01m,n}^2 \subset \bar{G}_{0m,n}^2$ of the elements which have the form:*

$$\left(I_m, 0, \begin{pmatrix} I_k & B_u^\alpha \\ 0 & I_{n-k} \end{pmatrix}, \begin{pmatrix} 0 & B_{ui}^\alpha \\ 0 & 0 \end{pmatrix} \right).$$

Notice that the proofs of the propositions and theorems are similiary with those of [4], with the corresponding adjustments. The complex case considered in this paper gives new results, essentially different from the real case considered in [4].

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