ISOTOPIES OF DIFFERENTIAL CALCULUS AND ITS APPLICATION TO MECHANICS AND GEOMETRIES

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Abstract

We present a simple, axiom-preserving, isotopic generalization of the ordinary differential calculus, here called *isodifferential calculus*, which is based on the generalization of the basic unit with compatible generalizations of fields, vector spaces and manifolds. The new calculus is applied to the isotopic lifting of Newton's equations with a number of novel possibilities, such as: the representation of the actual nonspherical and deformable shape of particles (which is absent in Newtonian mechanics); the admission of nonlocal-integral forces (which is not possible for the topology of Newton's equations); and the capability to turn Newtonian systems which are non-Hamiltonian in the frame of the observer into a form in the same frame which is Hamiltonian in isospaces. We then introduce the isotopies of the Lagrangian and Hamiltonian mechanics and show that the most general possible isotopic Newton's equations are derivable from a first-order variational principle in isospace. The *calculus of isovariations* and related isotopies of optimization methods are indicated. We also show that the construction of the isoanalytic representations from the given equations of motion (here called *inverse isotopic Newtonian problem*) is considerably easier than that of the conventional inverse Newtonian problem. We finally apply the isodifferential calculus to the construction of novel isotopies of the symplectic and Riemannian geometries which are nonlinear in the velocities and integro-differential, thus being particularly significant for interior dynamical problems. The paper is written by a physicist to stimulate mathematical studies on nonlinear-integral dynamical systems which have recently emerged in particle physics, astrophysics, superconductivity and other disciplines.

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1 Background notions on isotopies

The basic notion of this paper, that of *isotopies*, is rather old. As Bruck [5] recalls, the notion can be traced back to the early stages of set theory where two Latin squares were said to be *isotopically related* when they can be made to coincide via permutations. Since Latin square can be interpreted as the multiplication table of quasigroups, the isotopies propagated to quasigroups, then to algebras and more recently to most of mathematics. As an illustration, the isotopies of Jordan algebras were studied by McCrimmon [19], those of Lie algebras by Santilli [25], and subsequently extended to fields, vector spaces, manifolds, groups, functional analysis, etc. A comprehensive literature on isotopies up to 1984 can be found in Tomber's bibliography [2] while subsequent references can be found in the recent monograph by Lõhmus, Paal and Sorgsepp [17].

In this paper we study the isotopies of differential calculus, here called *isodifferential calculus*, and identify the consequential isotopies of mechanics and geometries. The isocalculus is presented here for the first time, although it is implicit in other studies by this author [32, 33], as we shall indicated later on. In this section we recall only those aspects of the isotopies which are essential for the understanding of this paper. The topics are also selected on the basis of their applicability to specific problems in physics and other disciplines. Due to the emphasis on applications, our treatment is local, while abstract, realization-free profiles are merely indicated.

Let $F = F(n, +, \times)$ be a field (hereon assumed to have characteristic zero) with elements $n, m, \ldots$ (hereon assumed to be real $\mathbf{R}$, complex $\mathbf{C}$ or quaternionic numbers $\mathbf{Q}$), sum $n + m$, multiplication $n \times m$, additive unit 0, multiplicative unit 1, and familiar properties $n + 0 = 0 + n = n$, $n \times 1 = 1 \times n = n$, $\forall n \in F$, and others. An *isofield* [31] is the image $\hat{F} = \hat{F}(\hat{n}, +, \hat{\times})$ of $F(n, +, X)$ under the lifting

$$1 \to \hat{I}, \tag{1}$$

where the quantity $\hat{I}$ is sufficiently smooth, everywhere invertible, symmetric, real-valued and positive-definite but otherwise arbitrary (conditions which are hereon assumed), thus generally being outside the original set F (e.g., for $F = \mathbf{R}$, $\hat{I}$ can be a well behaved integral or an $N \times N$ matrix), while preserving unchanged the additive unit 0. The set F is then reconstructed in such a way to admit $\hat{I}$ as the correct left and right unit. This requires the lifting of: the numbers $n \in F$ into the *isonumbers* $\hat{n} = n \times \hat{I}$, the sum $n + m$ into the *isosum* $\hat{n} + \hat{m} = (n + m) \times \hat{I}$, and the multiplication into the *isomultiplication*

$$n \times m \to \hat{n} \hat{\times} \hat{m} = \hat{n} \times \hat{T} \times \hat{m},\ \hat{T} = \hat{I}^{-1}. \tag{2}$$

Under these conditions it is easy to see that: $\hat{1}$ is the correct left and right unit of $\hat{F}$, $\hat{I} \hat{\times} \hat{n} = \hat{n} \hat{\times} \hat{I} = \hat{n}$, $\forall\ \hat{n} \in \hat{F}(\hat{n}, +, \hat{\times})$; $\hat{F}(\hat{n}, +, \hat{\times})$ preserves all axioms of $F(n, +, \times)$ thus being a field; and $\hat{F}(\hat{n}, +, \hat{\times})$ is isomorphic to $F(n, +, \times)$. Due to the preservation of the original axioms, the lifting $F(n, +, \times) \to \hat{F}(\hat{n}, +, \hat{\times})$ is called an *isotopy.* In this case the new unit $\hat{I}$ is called *isounit* and its inverse $\hat{T}$ is called the *isotopic element.* All conventional operations dependent on the multiplication on $F(n, +, \times)$

are generalized on $\hat{F}(\hat{n},+,\hat{\times})$ in such a way that $\hat{I}$ preserves all the original axiomatic properties of I, i.e. $\hat{I}^{\hat{n}}$ $=\hat{I}*\hat{I}*\ldots*\hat{I}$($n$-times) $=\hat{I}$,$\hat{I}=\hat{I}^{\frac{\hat{I}}{2}}=\hat{I}$, $\hat{I}/\hat{I}=\hat{I}$, etc. Despite its simplicity, the lifting $F(n,+,\times)\rightarrow\hat{F}(\hat{n},+,\hat{\times})$ has significant implications. For instance, real numbers which are conventionally prime (under the tacit assumption of the unit 1) are not necessarily prime with respect to a different unit [31]. As a result, most of the properties and theorems of the contemporary number theory are dependent on the assumed unit and, as such, admit intriguing isotopies.

The notion of isonumbers was presented, apparently for the first time, by this author at the conference *Differential Geometric Methods in Mathematical Physics*, held at the University of Clausthal, Germany, in 1980. The first mathematical treatment appeared in ref. [21] of 1982. A systematic mathematical study is available in ref. [31], while additional studies and applications are presented in monographs [32, 33].

Let $E(x,\delta,R)$ be an N-dimensional Euclidean space, with local chart $x=\{x^k\}$, $k=l,2,\ldots,N$, N-dimensional metric $\delta=diag.(1,1,\ldots 1)$ and line element $x^2=x^i\delta_{ij}x^j$ over the reals $R(n,+,\times)$, where the convention on the sum of repeated indices is assumed hereon . The lifting of the fields $R(n,+,\times)$ evidently requires for necessary compatibility a corresponding lifting of the space $E(x,\delta,R)$. The isotopies of the Euclidean, Minkowskian and Riemannian spaces were introduced by this author in paper [28] of 1983, subjected to a deeper study in memoirs [29, 30] of 1988 and systematically studied with various applications in the recent monographs [32, 33]. Isospaces can be introduced as follows.

The *isoeuclidean* spaces $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ are the image of $E(x,\delta,R)$ constructed over the isofields $\hat{R}(\hat{n},+,\hat{\times})$ and are characterized by: the lifting of the original N-dimensional unit $I=diag.(l,1,...,1)$ into $N\times N$-dimensional isounits $\hat{I}=(\hat{I}^i_j)=\hat{T}^{-1}$ (verifying the above assumed conditions); the joint deformation of the metric δ into the *isometric* $\hat{\delta}$, $\delta\rightarrow\hat{\delta}=\hat{T}\delta$; the assumption that the isounit of the underlying isofield coincides with that of the isospace; and use of the original local coordinates in contravariant form, $\hat{x}^k\equiv x^k$, although different coordinates in their covariant form, $\hat{x}_k=\hat{\delta}_{kj}\hat{x}^j=\hat{T}^i_kx_i$. Because of the latter occurrence, the symbol x will be used for the coordinates of conventional spaces, while the symbol $\hat{x}$ will be used for the coordinates of isospaces. When writing $\hat{\delta}(x,\dot{x},\ddot{x},\ldots)$ we refer to the *projection* of the isometric $\hat{\delta}$ in the original space. Since the deformation of the metric $\delta\rightarrow\hat{\delta}=\hat{T}\delta$ is compensated by the inverse deformation of the unit, $I\rightarrow\hat{I}=T^{-1}$, it is easy to see that $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ preserves all geometric axioms of $E(x,\delta,R)$. Therefore, the map $E(x,\delta,R)\rightarrow\hat{E}(\hat{x},\hat{\delta},\hat{R})$ is also an isotopy and $\hat{E}(\hat{x},\hat{\delta},\hat{R})\approx E(x,\delta,R)$. Similar results occur for the isotopies of the Minkowski, Riemannian and other spaces [28, 32].

Despite its simplicity, the lifting $E(x,\delta,R)\rightarrow\hat{E}(\hat{x},\hat{\delta},\hat{R})$ also has significant implications. ln fact, the functional dependence of the matrix elements $\hat{I}^i_j$ of the isounit $\hat{I}$ is completely unrestricted. The isometric $\hat{\delta}$ can therefore depend on the local coordinates x as well as their derivatives with respect to an independent variable t of arbitrary order, and we have the liftings

$$I=diag(1,1,\ldots,1)\rightarrow\hat{I}=\left(\hat{I}^i_j\right)=\hat{I}\left(t,x,\dot{x},\ddot{x},\ldots\right)=\hat{T}^{-1}, \qquad (3)$$

$$x^2 = x^i\delta_{ij}x^j \in \mathbf{R} \rightarrow \hat{\mathbf{x}}^{\hat{2}} = \left[x^i\hat{\delta}\left(t, x, \dot{x}, \ddot{x}, \dots\right)x^j\right]\hat{I} \in \hat{R}(\hat{n}, +, \hat{\times}),\ \hat{\delta} = \hat{T}\delta. \quad (4)$$

Despite the above generalizations, the original space is flat and therefore its image under isotopies is *isoflat*, that is, the axioms of flatness are verified in isospace (i.e. when the basic unit is $\hat{I}$). The understanding is that the projection of $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ into the original space $E(x, \delta, R)$ is curved (when the unit is the conventional 1). Note that Riemannian metrics $g(x)$ are a *particular* case of the broader isometric $\hat{\delta}(x, \dot{x}, \ddot{x}, \dots)$. This indicates that the N-dimensional Riemannian space $\mathcal{R}(x, g, R)$ over the reals can be reinterpreted as the isospace $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$, $\hat{\delta} = g(x)$, over the isoreals via the factorization $g(x) = \hat{T}(x)\delta$. The assumption of the isounit $\hat{I} = T^{-1}$ then eliminates curvature in isospace [32].

Isospaces have intriguing and novel properties which do not appear to have propagated into the mathematical literature. For instance, the conventional trigonometry on the two- dimensional Euclidean space $E(x, \delta, R)$, $\delta = diag.(1, 1)$ (Gauss plane) is lost under lifting to a two-dimensional Riemannian space $\mathcal{R}(x, g(x), R)$, but it can be reformulated in the two-dimensional isospace $\hat{E}(\hat{x}, \hat{\delta}(x, \dot{x}, \ddot{x}, \dots), \hat{R})$ resulting in the so-called *isotrigonometry* (see [32], App. 6.A, for brevity). An intriguing application is the formulation of the Pythagorean theorem for a triangle with *curved* sides (because it can be mapped via the isotopies into an ordinary Pythagorean configuration with straight sides in isospace).

Similarly, all infinitely possible spheroidal ellipsoids in three-dimensional Euclidean space $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \in R(n, +, \times)$, $a, b, c \neq 0$, are unified by the perfect sphere in isospace called *isosphere*

$$\left(x^2/a^2 + y^2/b^2 + z^2/c^2\right)\hat{I} = \hat{I} \in \hat{R}(\hat{n}, +, \hat{\times}), \quad (5)$$

$$\hat{I} = diag(a^2, b^2, c^2),\ \hat{T} = diag(a^{-2}, b^{-2}, c^{-2}) \quad (6)$$

In fact, under isotopies the semiaxes $(1, 1, 1)$ of the original perfect sphere are deformed into the values (a^2, b^2, c^2), but the corresponding units are deformed of the *inverse* amount (a^{-2}, b^{-2}, c^{-2}) thus preserving the perfect sphericity in isospace. When the conditions of positive-definiteness and non-singularity of the isounit are relaxed, the isosphere unifies all possible compact and noncompact quadrics and cones in three-dimension. The use of yet more general isounits then yields new notions, such as an isosphere whose isounit is singular or a distribution. For corresponding isotopies of the Minkowski space see [33].

The isotopies of the various branches of Lie's theory (enveloping algebra, Lie algebra, Lie groups, representation theory, etc.) were introduced by this author in memoir [25] of l978 under the name of *Lie-isotopic theory*, where systematically studied in monographs [26, 27] and [32, 33] and today called *Lie-Santilli isotheory* (see independent monographs [12, 17, 34] or review paper [14] and literature quoted therein). In essence, Lie's theory in its contemporary formulation (on conventional spaces over conventional fields) is linear, local and canonical and, as such, it possesses limitations in its applications. The isotopies of Lie's theory are the most general possible nonlinear, nonlocal and noncanonical maps which are however capable of reconstructing linearity, locality and canonicity when formulated in isospaces over isofields. As such, the isotopies imply a considerable broadening of the applications of the conventional Lie theory while preserving its axioms at the abstract level.

The isotopies of functional analysis, called *isofunctional analysis*, were introduced by Kadeisvili [13], including the notions of *isofunction* $\hat{f}(\hat{x})$ on isospaces, *isocontinuity*, *isolimits*, etc. They are simple isotopies of the conventional notions and will be tacitly assumed hereon. We merely recall that the notion of isocontinuity implies that of continuity, but the inverse statement is not necessarily true.

Ref. [13] also introduced a classification of isounits into five topologically different classes, which is called *Kadeisvili's classification* and generally used in current literature. This paper is devoted to the isotopies of Kadeisvili's Class I, i.e., those with isounits verifying the assumed conditions. The isotopies of Class II occur when the isounits satisfy the same conditions except that they are *negative-definite*. The isotopies of Class III are the union of those of classes I and II; those of Class IV include all preceding ones plus *singular* isounits; and those of Class V include all preceding ones plus isounits of unrestricted characteristics, such as step-functions, distributions, lattices, etc.

Kadeisvili's classification is significant because it illustrates the broad character of the isotopies. For instance, Lie's theory is unique (because referred to the single unit I), while the Lie- Santilli isotheory admits five topologically distinct classes (because based on five distinct isounits). It should be stressed that, despite all the studies conducted to date, the isotopies remain vastly unexplored at this writing. In fact, only the isotopies of Class I, II and III have been preliminarily studied until now [32, 33], while those of Classes IV and V are unknown.

The isotopies of manifolds, called *isomanifolds*, have been systematically studied by the mathematicians G. Tsagas and D. S. Sourlas [34, 35, 36]. They here called *Tsagas-Sourlas* isomanifolds and can be indicated as follows. Let M be a manifold. An *isochart* is the pair (U_α, Φ_α) [34], where $U_\alpha \subseteq M$ and Φ_α is a homeomorphism of U_α onto an open subset $\hat{V}_\alpha$ of $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$:

$$\Phi_\alpha :\to \Phi_\alpha(U_\alpha) = \hat{V}_\alpha \subseteq \hat{E}(\hat{x}, \hat{\delta}, \hat{R}). \tag{7}$$

A Tsagas -Sourlas isomanifold is an N-dimensional real isomanifold reducible to the study of $M[\hat{E}(\hat{x}, \hat{\delta}, \hat{R})]$. For the basic properties of isomanifolds we refer to [35] for brevity. We only mention that, conventional manifolds have a topology which is everywhere local-differential, while the Tsagas-Sourlas isomanifold have an *integro-differential topology*, i.e., a topology which is everywhere local-differential except *in the isounit*. Nonlocal-integral terms can therefore be treated via isomanifolds provided that they are embedded in the isounit.

2 Isodifferential calculus on Tsagas-Sourlas isomanifolds

Let $E(x, \delta, R)$ be the ordinary N-dimensional Euclidean space with local coordinates $x = \{x^k\}$, $k = 1, 2..., N$, and metric $\delta = diag.(1, 1, 1)$ over the reals $R(n, +, \times)$.

Let $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ be its isotopic image with local coordinates $\hat{x} = \{\hat{x}^k\}$ and isometric $\hat{\delta}{=}\hat{T}\delta$ over the isoreals $\hat{R}(\hat{n}, +, \hat{\times})$. Let the isounit be given by the $N \times N$ nowhere singular, symmetric, real-valued and positive-definite matrix $\hat{I} = (\hat{I}_i^j) = (\hat{I}_i^j) = \hat{T}^{-1} = (T_i^j)^{-1} = (T_i^j)^{-1}$ whose elements have a smooth but otherwise arbitrary functional dependence on the local coordinates, their derivatives with respect to an independent variable and any needed additional quantity, $\hat{I} = \hat{I}(\hat{x}, \ldots)$. The following properties then hold

$$\hat{x}^k \equiv x^k,\ \hat{x}_k = \hat{\delta}_{ki}\hat{x}^i = \hat{T}_k^i \delta_{ij}\hat{x}^j = \hat{T}_k^i \delta_{ij} x^j = \hat{T}_k^i x_i,\ x_i = \delta_{ij} x^j, \tag{8}$$

$$\hat{x}^i \delta_{ij} \hat{x}^j = \hat{x}^i \hat{T}_i^j \delta_{jm} \hat{x}^m = \hat{x}_i \delta^{ij} \hat{x}_j \equiv \hat{x}^k \hat{x}_k = \hat{x}_k \hat{x}^k,\ \hat{\delta}^{ij} = \left[\left(\hat{\delta}_{mn}\right)^{-1}\right]^{ij}, \tag{9}$$

$$x^i \delta_{ij} x^j = x_i \delta^{ij} x_j = x^i x_j = x_i x^j,\ \delta^{ij} = \left[(\delta_{mn})^{-1}\right]^{ij}. \tag{10}$$

Let $M\left[\hat{E}(\hat{x}, \hat{\delta}, \hat{R})\right]$ be the Tsagas-Sourlas isomanifold on $\hat{E}$ hereon refered as $M(\hat{E})$. The *isodifferential calculus* on $M(\hat{E})$ can be defined as an isotopic lifting of the conventional differential calculus on $M(E)$, that is, a lifting based on the generalization $I \to \hat{I}$ of the unit I of E, under the condition of preserving the original axioms, including the condition of the invariance of the isounit (see below).

The *first-order isodifferentials* of the contravariant and covariant variables $\hat{x}^k$ and $\hat{x}_k$, respectively, are here defined as the quantities

$$\hat{d}\hat{x}^k = \hat{I}_k^i(x, \ldots)dx^i,\ \hat{d}\hat{x}_k = \hat{T}_k^i(x, \ldots)dx_i\ , \tag{11}$$

where the expressions $\hat{d}\hat{x}^k$ and $\hat{d}\hat{x}_k$ are defined on $M(\hat{E})$ while the corresponding expressions $\hat{I}_i^k dx^i$ and $\hat{T}_k^i dx_i$ are the *projections* on $M(E)$.

Let $\hat{f}(\hat{x})$ be a sufficiently smooth isofunction on a closed domain $\hat{D}(\hat{x})$ of contravariant coordinates $\hat{x}^k$ on $M(\hat{E})$. By using Kadeisvili's [13] notions of isocontinuity, isolimits and isoconvergence, we shall say that $\hat{f}(\hat{x})$ admits the *isoderivative* at a point $\hat{a} \in \hat{D}$ when the following isolimits exist

$$\hat{f}'(\hat{a}^k) = \frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^k}_{|\hat{x}^k = \hat{a}^k} = \hat{T}_k^i \frac{\partial f(x)}{\partial x^i}_{|\hat{x}^k = \hat{a}^k} = Lim_{d\hat{x}^k \to \hat{0}^k} \frac{\hat{f}\left(\hat{a}^k + \hat{d}\hat{x}^k\right) - \hat{f}\left(\hat{a}^k\right)}{\hat{d}x^k} \tag{12}$$

where, again, $\hat{\partial}\hat{f}(\hat{x})/\hat{\partial}\hat{x}^k$ is computed in $M(\hat{E})$ and $\hat{T}_k^i \partial f(x)/\partial x^i$ is the projection in $M(E)$. When $\hat{f}(\hat{x})$ is a isofunction of a covariant coordinate $\hat{x}_k$ on a closed domain $\hat{D}(\hat{x}^k) \in M(\hat{E})$, we have the *isoderivative* at $\hat{a}_k$ when the following limit exists:

$$\hat{f}'(\hat{a}_k) = \frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}_k}_{|\hat{x}_k = \hat{a}_k} = \hat{I}_k^i \frac{\partial f(x)}{\partial x_i}_{|\hat{x}_k = \hat{a}_k} = Lim_{d\hat{x}_k \to \hat{0}_k} \frac{\hat{f}\left(\hat{a}_k + \hat{d}\hat{x}_k\right) - \hat{f}\left(\hat{a}_k\right)}{\hat{d}x_k} \tag{13}$$

The *isodifferentials of an isofunction* of contravariant (covariant) coordinates $\hat{x}^k(\hat{x}_k)$ on $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ are defined via the isoderivatives according to the respective rules

$$\hat{d}\hat{f}(\hat{x})_{|contrav.} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}^k}\hat{d}\hat{x}^k = \hat{T}^i_k \frac{\partial f}{\partial x^i}\hat{I}^k_j dx^j = df(x), \tag{14}$$

$$\hat{d}f(x)_{|covar.} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}_k}\hat{d}\hat{x}_k = \hat{I}^k_i \frac{\partial f}{\partial x_i}\hat{T}^j_k dx_j = df(x). \tag{15}$$

An iteration of the notion of isoderivative leads to the *second-order isoderivative*

$$\frac{\hat{\partial}^2\hat{f}(\hat{x})}{\hat{\partial}x^{k\,2}} = \hat{T}^i_k\hat{T}^j_k \frac{\partial^2 f(x)}{\partial x^i \partial x^j}, \ \frac{\hat{\partial}^2\hat{f}(\hat{x})}{\hat{\partial}x^2_k} = \hat{I}^k_i\hat{I}^k_j \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \ (\textit{no sums on } k) \tag{16}$$

and similarly for isoderivatives of higher order.

The *isolaplacian* on $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ is given by

$$\hat{\Delta} = \hat{\partial}_k\hat{\partial}^k = \hat{\partial}^i\hat{\delta}_{ij}\hat{\partial}^j = \hat{\partial}^i\delta_{ij}\delta^j = \hat{I}^i_k\delta^k\delta_{ij}\partial^j, \ \hat{\partial}_k = \hat{\partial}/\hat{\partial}\hat{x}^k, \ \partial_k = \partial/\partial x^k, \ \textit{etc.}, \tag{17}$$

and differs from the corresponding expression on a Riemannian space $\mathcal{R}(x, g, R)$ with metric $g(x) = \hat{\delta}$, $\Delta = \hat{\delta}^{-1/2}\partial_i\hat{\delta}^{1/2}\hat{\delta}^{ij}\partial_j$.

A few examples are in order. First note the following properties derived from definitions (12) and (13),

$$\hat{\partial}\hat{x}^i/\hat{\partial}\hat{x}^j = \delta^i_j, \ \hat{\partial}\hat{x}_i/\hat{\partial}\hat{x}_j = \delta^j_i, \ \hat{\partial}\hat{x}_i/\hat{\partial}\hat{x}^j = \hat{T}^j_i, \ \hat{\partial}\hat{x}^i/\hat{\partial}\hat{x}_j = \hat{I}^i_j \tag{18}$$

Next, we have the simple isoderivatives:

$$\frac{\hat{\partial}\left(\hat{x}_k\hat{x}^k\right)}{\hat{\partial}\hat{x}^r} = \frac{\hat{\partial}\left(\hat{x}^i\hat{\delta}_{ij}\hat{x}^j\right)}{\partial x^r} = \hat{T}^i_r \frac{\partial\left(x^i\delta_{ij}x^j\right)}{\partial x^i} = \hat{T}^i_r 2x^i = 2\hat{x}_r \ , \tag{19}$$

$$\frac{\hat{\partial}\ln\hat{\psi}(\hat{x})}{\hat{\partial}\hat{x}^k} = \hat{T}^i_k \frac{\partial \ln\psi(x)}{\partial x^i} = \frac{1}{\hat{\psi}(\hat{x})}\frac{\hat{\partial}\hat{\psi}(\hat{x})}{\hat{\partial}\hat{x}^k}, \tag{20}$$

and similarly for other cases.

For completeness we merely mention the (indefinite) *isointegration* which, when defined as the inverse of the isodifferential, is given by

$$\int \hat{d}\hat{x} = \int \hat{T}\hat{I}dx = \int dx = x, \tag{21}$$

namely, $\hat{\int} = \int \hat{T}$. Definite isointegrals are formulated accordingly.

The above basic notions are sufficient for our needs at this time. Isodifferentiable isofunctions of order m will be indicated $\hat{C}^m$. Systematic studies on the isotopies of the various theorems of the conventional calculus (see, e.g., [38]) will be studied elsewhere.

Remark: The isodifferential, isoderivative and isodifferentiation verify the condition of preserving the basic isounit $\hat{I}$. Mathematically, this condition is *necessary* to prevent that a set of isofunctions $\hat{f}(\hat{x}), \hat{g}(\hat{x})$, ..., on $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ over the isofield $\hat{R}(\hat{n}, +, \hat{\times})$ with isounit $\hat{I}$ are mapped under isoderivative into a set of functions $\hat{f}'(\hat{x}), \hat{g}'(\hat{x})$, ... defined over a *different* field because of the alteration of the isounit. Physically, the condition is also necessary because the unit is a pre-requisite for measurements. Lack of conservation of the unit therefore implies lack of consistent physical applications.

As an example, the following alternative definition of the isodifferential

$$\hat{d}\hat{x}^k = d\left(\hat{I}_i^k x^i\right) = \left[\left(\partial_i \hat{I}_r^k\right) x^r + \hat{I}_i^k\right] dx^i = \hat{W}_i^k dx^i, \tag{22}$$

would imply the alteration of the isounit,$\hat{I} \rightarrow \hat{W} \neq \hat{I}$, thus being mathematically and physically unacceptable.

Nevertheless, when using isoderivatives of independent variables, say, coordinates and time, the above rule does not apply and we have:

$$\hat{\partial}_t \hat{\partial}_k \hat{f}\left(t, \hat{x}\right) = \hat{\partial}_t \left[\hat{\partial}_k \hat{f}\left(t, \hat{x}\right)\right] = \hat{\partial}_t \left[T_k^i\left(t, x, \ldots\right) \partial_i f\left(t, x\right)\right]. \tag{23}$$

Additional properties of the isodifferential calculus will be identified during the course of our analysis.

3 Isotopic lifting of Newtonian mechanics

Newton's equations have remained essentially unchanged since their formulation in 1687 [20]. Their re-inspection is now warranted because classical Hamiltonian mechanics has been constructed to represent Newton's equations and, in turn, quantum mechanics has been constructed as an operator image of Hamiltonian mechanics. The applicability of these mechanics is essentially restricted to local-differential and potential systems, while the advancement of knowledge in various disciplines is requesting the treatment of nonlocal-integral and nonpotential systems. It then follows that a possible broadening of contemporary dynamics must originate from its foundations, Newton's equations.

In this section we introduce, apparently for the first time, the isotopies of Newton's equations characterized by the isodifferential calculus as one (not necessarily unique) way of broadening their original conception [20]. The isotopies have been selected over a variety of other possibilities because of their axiom-preserving character as well as of the consequential broadening of classical and quantum mechanics outlined in subsequent sections.

The contemporary formulation of Newton's equations requires the tensorial space $S(t, x, v) = E(t) \times E(x, \delta, R) \times E(v, \hat{\delta}, R)$, where $E(t)$ is the one-dimensional space representing time t, $E(x, \delta, R)$ is the conventional three-dimensional Euclidean space with local trajectories $x(t) = [x^k] = \{x, y, z\}$ and $E(v, \delta, R)$ is the tangent pace TE (see Sect. 6) which, at this Newtonian level, can be considered as an independent space representing the contravariant velocities $v = \{v^k\} = dx^k/dt$. Newton's equations for a test body of mass $m = const.$ ($\neq 0$) moving within a resistive medium (a.e., our atmosphere) can then be written

$$mdv_k/dt - F_k^{SA}(t,x,v) - F_k^{NSA}(t,x,v) = 0, \ k = 1,2,3(= x,y,z), \tag{24}$$

where SA (NSA) stands for *variational self-adjointness* (*variational non-self-adjointness*), i.e. the verification (violation) of the necessary and sufficient conditions for the existence of a potential $U(t,x,v)$ originally due to Helmholtz [11] (see monograph [26] for historical notes and systematic studies). It should be recalled that in Newtonian mechanics the potential $U(y,x,v)$ must be linear in the velocities (to avoid a redefinition of the mass),

$$U(t,x,v) = U_k(t,x)v^k + U_0(t,x). \tag{25}$$

Eq.s (24) can then be written

$$\left\{m\frac{dv_k}{dt} - \frac{d}{dt}\frac{\partial U(t,x,v)}{\partial v^k} + \frac{\partial U(t,x,v)}{\partial x^k} - F_k^{NSA}(t,x,v)\right\}^{NSA} =$$

$$= \left\{m\frac{dv_k}{dt} - \frac{\partial U_k(t,x)}{\partial x^s}\frac{dv^s}{dt} + \frac{\partial U_0(t,x)}{\partial x^k} - F_k^{NSA}(t,x,v)\right\}^{NSA} = 0, \tag{26}$$

namely, they are not in general derivable from Lagrange's [10] or Hamilton's [10] equations in the local chart $\{t,x,v\}$, as well known [26, 27] (see later on for coordinate transforms). The extension to systems of n particles with masses $m_k (\neq 0)$ is straighforward and will be ignored for brevity.

The representation space of the desired isotopic image of Newton's equations is given by the Kronecker product of isospaces $\hat{S}(\hat{t},\hat{x},\hat{v}) = \hat{E}(\hat{t}) \times \hat{E}(\hat{x},\hat{\delta},\hat{R}) \times \hat{E}(\hat{v},\hat{\delta},\hat{R})$ characterized by the (one- dimensional) *time isounit* $\hat{I}_0^0 = (\hat{T}_0^0)^{-1}$ and the (three-dimensional) *space isounit* $\hat{I} = (\hat{I}_i^k) = (\hat{T}_k^i)^{-1}$ where, for clarity, we have differentiated the *isotime* $\hat{t}$, *isocoordinates* $\hat{x}^k(\hat{t})$ and *isovelocities* $\hat{v}^k(t)$ from the original respective quantities t, x^k and v^k, with the following relationships in addition to (8-10):

$$\hat{t} = t, \ \hat{v}^k \equiv v^k, \ \hat{v}_k = \hat{\delta}_{kj}\hat{v}^j = \hat{T}_k^i\delta_{ij}\hat{v}^j = \hat{T}_k^i v_i \neq v_k = \delta_{ki}v^i. \tag{27}$$

The desired isotopic lifting of Newton's equations (26) in isospace $\hat{S}(\hat{t},\hat{x},\hat{v})$, here called *isotopic Newton equations* and submitted apparently for the first time, are given by

$$\hat{\Gamma}_k(\hat{t},\hat{x},\hat{v}) = \hat{m}\frac{\hat{d}\hat{v}_k}{\hat{d}\hat{t}} - \frac{\hat{d}}{\hat{d}\hat{t}}\frac{\hat{\partial}\hat{U}(\hat{t},\hat{x},\hat{v})}{\hat{\partial}\hat{v}^k} + \frac{\hat{\partial}\hat{U}(\hat{t},\hat{x},\hat{v})}{\hat{\partial}\hat{x}^k} =$$

$$= \hat{m}\frac{\hat{d}\hat{v}_k}{\hat{d}\hat{t}} - \frac{\hat{\partial}\hat{U}_k(\hat{t},\hat{x})}{\hat{\partial}\hat{x}^i}\frac{\hat{d}\hat{x}^i}{\hat{d}\hat{t}} + \frac{\hat{\partial}\hat{U}_0(\hat{t},\hat{x})}{\hat{\partial}\hat{x}^k} = 0, \tag{28}$$

$$\hat{U}(\hat{t},\hat{x},\hat{v}) = \hat{U}_k(\hat{t},\hat{x})\hat{v}^k + \hat{U}_0(\hat{t},\hat{x}), \tag{29}$$

where we have used properties (17), $\hat{m} = const$ ($\neq 0$) is the *isotopic mass*, that is, the image of the Newtonian mass in isospace and one should note the preservation of the linearity of isopotential (29) in $\hat{v}^k$.

Theorem 1 *All possible sufficiently smooth, regular and variationally non-self-adjoint Newton's equations (26) always admit in a neighborhood of a point (t, x, v) the representation in terms of the isotopic equations (28 and 29)*

$$\hat{m}\frac{\hat{d}\hat{v}_k}{\hat{d}\hat{t}} - \frac{\hat{d}}{\hat{d}\hat{t}}\frac{\partial\hat{U}(\hat{t},\hat{x},\hat{v})}{\hat{\partial}\hat{x}^k} + \frac{\partial\hat{U}(\hat{t},\hat{x},\hat{v})}{\hat{\partial}\hat{x}^k} =$$

$$= \hat{T}^i_k\left\{m\frac{dv_i}{dt} - \frac{\partial U_i(t,x)}{\partial x^s}\frac{dx^s}{dt} + \frac{\partial U_0(t,x)}{dx^i} - F^{NSA}_i(t,x,v)\right\} = 0. \tag{30}$$

Proof. When projected in the original space $S(t, x, v)$, Eq.s (28-29) can be written

$$\hat{m}\hat{T}^0_0\frac{d\left(\hat{T}^i_k v_i\right)}{dt} - \hat{T}^0_0\frac{d}{dt}\hat{T}^i_k\frac{\partial\hat{U}\left(\hat{t},\hat{x},\hat{v}\right)}{\partial\hat{v}^i} + \hat{T}^i_k\frac{\partial\hat{U}\left(\hat{t},\hat{x},\hat{v}\right)}{\partial x^i} =$$

$$\hat{m}\hat{T}^0_0\hat{T}^i_k\frac{dv_i}{dt} - \hat{T}^0_0\hat{T}^i_k\frac{\partial\hat{U}_i\left(t,x\right)}{\partial x^s}v^s + \hat{T}^i_k\frac{\partial\hat{U}_0\left(t,x\right)}{\partial x^i} + \hat{m}\hat{T}^0_0\frac{d\hat{T}^i_k}{dt}v_i = 0. \tag{31}$$

Sufficient conditions for identities (30) are then given by

$$\hat{m}T^0_0 dv_i/dt = m dv_i/dt, \tag{32}$$

$$\hat{T}^0_0\frac{\partial\hat{U}_i(t,x)}{\partial x^s}v^s = \frac{\partial U_i(t,x)}{\partial x^s}v^s, \tag{33}$$

$$\frac{\partial\hat{U}_0(t,x)}{\partial x^i} = \frac{\partial U_0(t,x)}{\partial x^i}, \tag{34}$$

$$\hat{m}\hat{T}^0_0\frac{d\hat{T}^i_k\left(t,x,\ldots\right)}{dt}v_i = -\hat{T}^i_k F^{NSA}_i\left(t,x,v\right), \tag{35}$$

which, under the assumed continuity and regularity conditions (see [26] for details) always admits a solution in the unknown quantities $\hat{m}$, $\hat{T}^0_0$, $\hat{T}^i_k$, $\hat{U}_k$ and $\hat{U}_0$ for given equations (26). In fact, system (32-33) is overdetermined and a solution exists for *diagonal space isounit* and *constant time isounit*,

$$\hat{I}^i_k = \delta^i_k e^{f_k(t,x,v)},\ \hat{I}^0_0 = constant > 0, \tag{36}$$

for which

$$\hat{m}T^0_0 \equiv m,\ \hat{U}_k\left(t,x\right) = \hat{I}^0_0 U_k\left(t,x\right),\ \hat{U}_0\left(t,x\right) = U_0\left(t,x\right), \tag{37}$$

$$f_k\left(t,x,v\right) = -m^{-1}\int_0^1 dt F^{NSA}_k\left(t,x,v\right)/v_k, \tag{38}$$

where there are no repeated indices $\hat{m}$ is constant and the functions f_k are computed from Eq.s (38). □

The primary motivations for the submission of the isotopic Newton's equations are expressed by the following properties with self-evident proofs which will be only illustrated.

Corollary 1 *The isotopic Newton equations permit a representation of the actual nonspherical shape of the body considered and of its possible deformations via the generalized unit (or isotopic element) of the theory.*

Recall that Newton's equations can only approximate the body considered as a massive point, as well known since Newton's time [20]. The point-like representation of particles then persists under analytic representations via Hamilton's equations as well as under symplectic map to quantum mechanical formulations. A representation of the extended character of particles is reached in *second* quantization via the form factors. However, this representation is restricted to spherical shapes from the fundamental symmetry of quantum mechanics, the rotational symmetry. The latter symmetry is known to be a symmetry of *rigid* bodies. Form factors cannot therefore represent the *deformations* of particles under sufficiently intense external interactions which is studied via other rather complicated procedures.

A first motivation for the studies presented in this paper is to introduce a representation of actual *nonspherical and deformable* shapes of particles at the primitive *Newtonian level*, which then persists under *classical* analytic representations and maps to *first* quantization. The isotopic Newton equations do indeed achieve these objectives by setting the foundations for possible new advances in classical and quantum physics. The objective is achieved via the new degrees of freedom of the generalized unit of the theory which are evidently absent in the conventional Newtonian, classical and quantum formulations.

As a simple case, suppose that the body considered is a rigid spheroidal ellipsoid with semiaxes n_1^2, n_2^2, n_3^2= constants. Such a shape is directly represented by the isotopic element of the theory in the simple diagonal form

$$\hat{T} = diag\left(n_1^{-2}, n_2^{-2}, n_3^{-2}\right),\ n_k = const > 0,\ k = 1, 2, 3,\ \hat{T}_0^0 = 1. \tag{39}$$

The representation of the shape in isospace $\hat{S}(\hat{t}, \hat{x}, \hat{v})$ is then embedded in the *isoderivatives* of the isotopic Newton equations and, when projected in the conventional space $S(t, x, v)$ can be written

$$m\hat{T}_k^i \frac{dv_i}{dt} - \hat{T}_k^i \frac{\partial U_i\,(t,x)}{\partial x^s} v^s + \hat{T}_k^i \frac{\partial U_0\,(t,x)}{\partial x^i} = 0, \tag{40}$$

namely, the shape terms $\hat{T}_k^i$ are admitted as factors.

Note that *the representation of shape occurs only in isospace because, when projected in the conventional Euclidean space,. the shape terms cancel out by recovering the conventional point-like character of Newton's equations.* This illustrates the necessity of the isotopy for the representation of shape. Moreover, *the nonspherical character of the shape emerges only in the projection in ordinary spaces, because all deformed spheres in ordinary spaces are mapped into the perfect sphere in isospace, the isosphere of Sect. 1,*

$$\hat{x}^2 = \left(x^1 n_1^{-2} x^1 + x^2 n_2^{-2} x^2 + x^3 n_3^{-2} x^3\right) \hat{I} \in \hat{R}(\hat{n}, +, \hat{\times}). \tag{41}$$

The representation of shapes more complex than the spheroidal ellipsoids is possible with non-diagonal isounits. The representation of the original shape due to motion within resistive media or other reasons, can be achieved via a suitable functional dependence of the $\hat{T}_k^i$ terms in velocities, pressure, etc. [32, 33].

A simple application discussed in detail in [33] is given by a charged, spinning and spherical metallic shell which is subjected to a sufficiently intense external electromagnetic field represented by the known Lorentz force with potential $U = eA_k v^k + e\phi$, where e is the charge and (A_k, ϕ) are the familiar electromagnetic potentials. It is evident that the original spherical shape is deformed by the Lorentz force, with consequential alteration of its magnetic moment. Such a deformation is not representable by Newton's equations as well as by its Hamiltonian representation, but it is easily representable via our isotopic equations (28-29).

The operator image of this classical setting illustrates the relevance of the theory herein submitted. In first quantization, the constituents of a nuclear structure (protons and neutrons) are represented as point-like particles. As such, they maintain in the nuclear structure their intrinsic magnetic moments when in vacuum. However, this approach has not permitted an *exact* representation of the total magnetic moments of few-body nuclei (such as deuteron, tritium, etc.). The isotopic representation of protons and neutrons as they are in the physical reality (*extended and therefore elastic, spinning*, charge distributions) has instead permitted the achievement of an exact representation of said total magnetic moments because each particle experiences a (generally small) deformation of its shape when under the short-range strong forces of a nuclear structure, resulting in an alteration of the intrinsic magnetic moment in vacuum which is missing in conventional quantum treatments. In turn, such alteration permits the *exact* representation of the total magnetic moments of few-body nuclei as well as other intriguing implications and novel predictions [32, 33].

Corollary 2 *The isotopic Newton equations permit a novel representation of variationally non-self-adjoint forces via the isometric of the underlying geometry, according to the rules*

$$m dv_k/dt - F_k^{NSA}(t, x, v) = \hat{I}_k^i m d\hat{T}_i^j v_j/dt, \tag{42}$$

while leaving unchanged the representation of conventional self-adjoint forces (except for the constant factor $\hat{T}_0^0$ of U_k).

In fact, the non-self-adjoint forces are embedded in the *covariant* coordinates in isospace $\hat{v}_i = T_i^j v_j$, where the v_j are the covariant coordinates in conventional space. The novelty therefore lies on the fact that non-self-adjoint forces are represented by the isogeometry itself, thus providing another motivation for the istopies.

The simplicity of representation (42) should be kept in mind and compared to the complexity of the conventional solution of the *inverse problem of Newtonian mechanics* ([26], i.e., the representation of non-self-adjoint systems via a Lagrangian or a Hamiltonian. Moreover, under the assumed conditions, the latter exists in the fixed coordinates (t, x, v) of the observer only for a restricted class called *nonessentially nonselfadjoint* [loc. cit.], while isorepresentation (30) always exist in the given coordinates (t, x, v) under the same conditions.

When coordinate transformations are admitted, an *indirect analytic representation* (i.e., a representation in transformed coordinates (t', x', v')) always exists for all local-differential, analytic and regular, nonselfadjoint Newtonian systems in a star-shaped region of the variables (this is the Lie-Koening theorem [27] as the analytic counterpart of the geometric *Darboux's theorem* of Sect. 6). However, the latter representation has a number of *physical* drawbacks. First, the transformations $(t, x, v) \rightarrow (t', x', v')$ are nonlinear and, as such, the new coordinates are not realizable in laboratory. Also, their nonlinearity implies the loss of the original inertial character of the reference frame with consequential loss of conventional relativities (in fact, the Galilei and Einstein relativities are solely applicable to inertial systems as well known).

These are the reasons why, after completing the studies of ref.s [26, 27], this author continued the search for a representation of nonselfadjoint systems which occurs in the given *inertial* reference frame of the observer, and it is universal, i.e., applicable to all systems occurring in the physical reality.

The following examples illustrate isorepresentation (30). The equation of the linearly damped particle in one dimension

$$m\, dv/dt + \gamma\, v = 0,\ \gamma \in R(n, +, \times)\ , \gamma > 0, \tag{43}$$

admits isorepresentation (30) with values

$$\hat{T} = \hat{S}_0 e^{\gamma t/2m},\ \hat{T}^0_0 = 1,\ U_k = U_0 = 0, \tag{44}$$

where $\hat{S}_0$ is a *shape factor*, e.g., of the spheroidal type (39) which is prolate in the direction of motion. In this way, the isotopic Newton equations represent:1) the nonselfadjoint force $F^{NSA} = -\gamma v$ experienced by an object moving within a physical medium; 2) its extended character (which is necessary for the existence of the resistive force); and 3) the deformation of the original shape (in the case considered a perfect sphere) caused by the medium.

The equation for the linearly damped harmonic oscillator in one dimension

$$m\ \ddot{x} + \gamma\ \dot{x} + kx = 0,\ k \in R(n, +, \times),\ k > 0, \tag{45}$$

admits isorepresentation (30) with the values

$$\hat{T} = \hat{S}_0 e^{\gamma t/2m},\ U_0 = -\frac{1}{2}kx^2,\ U_k = 0,\ \hat{T}^0_0 = 1, \tag{46}$$

where $\hat{S}_0$ represents the shape of the body oscillating within a resistive medium. The interested reader can construct a virtually endless variety of isorepresentations of non-self-adjoint forces. A systematic study will be conducted elsewhere.

Corollary 3 *The isotopic Newton equations permit the representation of nonlocal-integral forces when completely embedded in the isounit of the theory.*

The above occurrence is permitted by the integro-differential topology of the Tsagas- Sourlas isomanifolds recalled in Sect. 1. Consider as an example the integro-differential equation

$$m\, dv/dt + \gamma v^2 \int_\sigma d\sigma \mathcal{F}(\sigma, \ldots) = 0,\ \gamma > 0, \tag{47}$$

representing an extended object (such as a space-ship during re-entry in our atmosphere) with local-differential center-of-mass trajectory $x(t)$ and corrective terms of integral type due to the shape (surface) σ of the body moving within a resistive medium, where $\mathcal{F}$ is a suitable kernel depending on σ as well as on other variables such as pressure, temperature, density, etc. The above equation admits isorepresentation (30) with the values

$$\hat{T} = \hat{S}_\sigma e^{\gamma m^{-1} \times \int_\sigma d\sigma \mathcal{F}(\sigma, \dots)}, \; \hat{T}_0^0 = 1, \; U_k = U_0 = 0, \tag{48}$$

where $\hat{S}_\sigma$ is the shape factor, which is admitted by the integro-differential topology of the isomanifold $M(\hat{E})$ because all integral terms are embedded in the isounit. Similar isorepresentations can be easily constructed by the interested reader.

It should be recalled that the representation of nonlocal-integral terms is prohibited in Hamiltonian mechanics because the underlying geometry and topology are local-differential. In fact, the Lie-Koening Theorem requires a *local-differential approximation* of systems and it is inapplicable to integral systems of type (47).

In the author's opinion, the generalization of Newton's equations into a form admitting nonlocal-integral forces has the most important epistemological, mathematical and physical implications. Recall that contemporary mathematical and physical knowledge is generally restricted to point-like/local formulations. The isotopies therefore permit the study of more general nonlocal-integral systems begining at the primitive Newtonian level. Mathematically, the representation of nonlocal-integral forces requires the study of new methods, such as new topologies, geometries and mechanics. Physically, the implications are equally important and they deal with the historical legacy, due to Blockint'sev, Fermi and others, that the strong interactions have a nonlocal-integral component. In fact, all strongly interacting particles (hadrons) have a charge radius which is of the same order of the range of the strong interactions (about 10^{-13} cm). A necessary condition to activate the strong interactions is therefore that hadrons enter into mutual penetration of their charge distributions. But hadrons are some of the densest objects measured in laboratory until now. The historical legacy on the nonlocality of the strong interactions then follows.

A quantitative treatment of the historical legacy of the nonlocality of strong interactions has been the primary motivation for this author to conduct his studies on the isotopies, with evident need to initiate the studies at the primitive Newtonian level, then passing to classical analytic representations and finally to operator treatment.

The isotopic Newton equation on a curved space are submitted in Sect. 7.

4 Variational iso-self adjointness

The fundamental methods of the Inverse Newtonian Problem are the conditions of variational self-adjointness in $E(t) \times E(x, \delta, R) \times E(v, \delta, R)$ [11, 26]. In this section we shall identify, apparently for the first time, their image in isospace here called *conditions of variational iso-self-adjointness*.

Theorem 2 *A necessary and sufficient condition for a system of ordinary second-order isodifferential equations in* $E(t) \times \hat{E}(\hat{x}, \hat{\delta}, \hat{R}) \times \hat{E}(\hat{v}, \hat{\delta}, \hat{R})$

$$\hat{\Gamma}_k \left(\hat{t}, \hat{x}, \hat{v}, \hat{a}\right) = 0,\ k = 1, 2, \ldots, N,\ \hat{v} = \hat{d}\hat{x}/\hat{d}\hat{t},\ \hat{a} = \hat{d}\hat{v}/\hat{d}\hat{t} \tag{49}$$

which are isodifferentiable at least up to the third order and regular in a region $\widehat{\mathcal{R}}$ *of points* $(\hat{t}, \hat{x}, \hat{v}, \hat{a}, \hat{d}\hat{a}/\hat{d}\hat{t})$ *(i.e.,* $\det(\hat{\partial}\hat{\Gamma}_i/\hat{\partial}\hat{a}^j) \neq 0$*) to be variationally iso-self-adjoint (ISOSA) in* $\widehat{\mathcal{R}}$ *is that all the following conditions:*

$$\frac{\hat{\partial}\hat{\Gamma}_i}{\hat{\partial}\hat{a}^k} = \frac{\hat{\partial}\hat{\Gamma}_k}{\hat{\partial}\hat{a}^i}, \tag{50}$$

$$\frac{\hat{\partial}\hat{\Gamma}_i}{\hat{\partial}\hat{v}^k} + \frac{\hat{\partial}\hat{\Gamma}_k}{\hat{\partial}\hat{v}^i} = 2\frac{\hat{d}}{\hat{d}\tilde{t}}\frac{\hat{\partial}\hat{\Gamma}_i}{\hat{\partial}\hat{a}^k} = \frac{\hat{d}}{\hat{d}\tilde{t}}\left(\frac{\hat{\partial}\hat{\Gamma}_i}{\hat{\partial}\hat{a}^k} + \frac{\hat{\partial}\hat{\Gamma}_k}{\hat{\partial}\hat{a}^i}\right), \tag{51}$$

$$\frac{\hat{\partial}\hat{\Gamma}_i}{\hat{\partial}\hat{x}^k} - \frac{\hat{\partial}\hat{\Gamma}_k}{\hat{\partial}\hat{x}^i} = \frac{\hat{d}}{\hat{d}\tilde{t}}\left[\frac{\hat{d}}{\hat{d}\tilde{t}}\left(\frac{\hat{\partial}\hat{\Gamma}_k}{\hat{\partial}\hat{v}^i}\right) - \frac{\hat{\partial}\hat{\Gamma}_k}{\hat{\partial}\hat{v}^i}\right] = = \frac{1}{2}\frac{\hat{d}}{\hat{d}\tilde{t}}\left(\frac{\hat{\partial}\hat{\Gamma}_i}{\hat{\partial}\hat{a}^k} - \frac{\hat{\partial}\hat{\Gamma}_k}{\hat{\partial}\hat{a}^i}\right) \tag{52}$$

are identically verified in $\widehat{\mathcal{R}}$.

Proof. The proof is provided by an elementary isotopy of the conventional case, ref. [26], Theorem 2.1.2, p. 60, and consists in computing the isovariational forms of system (49), proving their uniqueness and showing that conditions (50)-(52) are necessary and sufficient for the isovariational forms to coincide with their adjoint. □

The novelty of conditions (50-52) is illustrated by the following:

Corollary 4 *Systems of ordinary isodifferential equations which are variationally iso-self-adjoint in isospace are generally variational non-self-adjoint when projected in ordinary spaces.*

Proof. Conditions (50-52) imply no restriction on the isotopic terms $\hat{T}^i_k$ in isospace while the same terms are restricted by the ordinary conditions of self-adjointness in ordinary spaces □

Theorem 3 *The isotopic Newton equations (28)-(29) are variationally iso-self-adjoint.*

Proof. The verification of the first set of conditions (50) reads

$$\frac{\hat{\partial}\hat{F}_i}{\hat{\partial}\hat{a}^j} - \frac{\hat{\partial}\hat{F}_j}{\hat{\partial}\hat{a}^i} = \hat{T}^m_j\frac{\hat{\partial}\hat{F}_i}{\hat{\partial}\hat{a}^m} - \hat{T}^m_i\frac{\hat{\partial}\hat{F}_j}{\hat{\partial}\hat{a}^m} = \hat{T}^m_j\hat{T}^m_i - \hat{T}^m_i\hat{T}^m_j \equiv 0, \tag{53}$$

and the same identities hold for all remaining conditions. □

It is an instructive exercise for the interested reader to work out the isotopies of the remaining theorems for second-order ordinary differential equations (see [26], Sections 2.2 and 2.3).

We now introduce the conditions of variational iso-self-adjointness for N-dimensional systems (49) in an equivalent $2N$-dimensional first-order form. Let $T^*\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ be the isocotangent space (see Sect. 6 for a geometric treatment), which in this section can be characterized via the independent space $\hat{E}(\hat{p}, \hat{\delta}, \hat{R})$ with new, independent, covariant coordinates $\hat{p}^k$ and let the total representation space be $\hat{T}(t) \times \hat{E}(\hat{x}, \hat{\delta}, \hat{R}) \times \hat{E}(\hat{p}, \hat{\delta}, \hat{R})$ with local chart $\hat{b} = \{b^\mu\} = \{\hat{x}^k, \hat{p}_k\}$, $\mu = 1, 2, \ldots, 2N$, $k = 1, 2, \ldots, N$. Assign sufficiently smooth and invertible prescriptions for the characterization of the new variables $\hat{p}_k$

$$\hat{p}_k = \hat{g}_k \left(\hat{t}, \hat{x}, \hat{v}\right), \tag{54}$$

with unique system of implicit functions $v^k = f^k(\hat{t}, \hat{x}, \hat{p})$ (see [26], Sect. 2.4, for the conventional case). By using the latter implicit functions, system (49) can be written in the equivalent $2N$- dimensional form

$$\hat{\Gamma}_\mu \left(\hat{t}, \hat{b}, \hat{c}\right) = \hat{C}_{\mu\nu} \left(\hat{t}, \hat{b}\right) \hat{c}^\nu + \hat{D}_\mu \left(\hat{t}, \hat{b}\right) = 0, \; \hat{c}^\nu = \hat{d}\, \hat{b}^\nu / \hat{d}\hat{t}. \tag{55}$$

Theorem 4 *A necessary and sufficient condition for system (55) which is at least twice isodifferentiable and regular (*$\det(\hat{C}_{\mu\nu})(\widehat{\mathcal{R}}) \neq 0$*) in a (*$6N+1$*)- dimensional region* $\widehat{\mathcal{R}}$ *of points (*$\hat{t}$, $\hat{b}$, $\hat{c}, \hat{d}\hat{c}/\hat{d}\hat{t}$*) to be iso-self-adjoint in* $\widehat{\mathcal{R}}$ *is that all the following conditions:*

$$\hat{C}_{\mu\nu} + \hat{C}\nu\mu = 0, \tag{56}$$

$$\frac{\hat{\partial}\hat{C}_{\mu\nu}}{\hat{\partial}\hat{b}^\rho} + \frac{\hat{\partial}\hat{C}_{\nu\rho}}{\hat{\partial}\hat{b}^\mu} + \frac{\hat{\partial}\hat{C}_{\rho\mu}}{\hat{\partial}\hat{b}^\nu} = 0, \tag{57}$$

$$\frac{\hat{\partial}\hat{D}_\mu}{\hat{\partial}\hat{b}^\nu} + \frac{\hat{\partial}\hat{D}_\nu}{\hat{\partial}\hat{b}^\mu} = \frac{\hat{\partial}\hat{C}_{\mu\nu}}{\hat{\partial}\hat{t}} \tag{58}$$

are identically satisfying in $\widehat{\mathcal{R}}$.

Proof. The proof is also a simple isotopy of the proof of Theorem 2.7.2, p. 87, ref. [26]. Also, conditions (56)-(58) are uniquely derivable from conditions (50)-(52) when systems (49) are assumed to be $2N$-dimensional and of first-order. □

The following property is self-evident,

Corollary 5 *When systems (55) assume the isocanonical form*

$$\hat{\Gamma}_\mu \left(\hat{t}, \hat{b}, \hat{c}\right) = \omega_{\mu\nu}\hat{c}^\nu - \hat{\Xi}_\mu \left(\hat{t}, \hat{b}\right) = 0, \tag{59}$$

where $\omega_{\mu\nu}$ *is the conventional canonical symplectic tensor*

$$(\omega_{\mu\nu}) = \left(\begin{array}{cc} 0_{N\times N} & -I_{N\times N} \\ I_{N\times N} & 0_{N\times N} \end{array} \right), \tag{60}$$

the conditions of variational iso-self-adjointness (56)-(58) reduce to

$$\frac{\hat{\partial}\hat{\Xi}_\mu}{\hat{\partial}\hat{b}^\nu} - \frac{\hat{\partial}\hat{\Xi}_\nu}{\hat{\partial}\hat{b}^\mu} = 0. \tag{61}$$

Note that a conventional canonical system which is self-adjoint is also iso-self-adjoint. Additional and this illustrates the reason why a potential representation of a selfadjoint forces persists at the isotopic level. Additional properties of variational iso-self-adjointness will be identified later on.

Let us recall the following meanings of the conditions of variational self-adjointness for 2N-dimensional systems of ordinary first-order differential equations

$$\Gamma_\mu\,(t,b,c) = C_{\mu\nu}\,(t,b)\,c^\nu + D_\mu\,(t,b) = 0,\; b = \left\{x^k, p_k\right\},\; c^\nu = db^\nu/dt\,, \tag{62}$$

on a conventional space (see [26, 27] for detailed studies):

1) *Analytic meaning.* The conditions imply the direct derivability (i.e., derivability without change of local variables or integrating factors) of the equations from a first-order variational principle

$$\delta\, A = \delta \int_{t_1}^{t_2} dt\; [R_\mu\,(t,b)\,db^\mu - H\,(t,b)] = 0, \tag{63}$$

$$C_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu\,,\; D_\mu = \partial_\mu H - \partial_t R\,,\; \partial_\mu = \partial/\partial b^\mu\,,\; \partial_t = \partial/\partial t\,; \tag{64}$$

2) *Geometric meaning.* The two form

$$C = C_{\mu\nu} db^\mu \wedge db^\nu, \tag{65}$$

characterized by the covariant symplectic tensor $C_{\mu\nu}(b)$; and

3) *Algebraic meaning.* The brackets among two smooth functions $A(b)$ and $B(b)$

$$[A,B] = (\partial_\mu A)\,C^{\mu\nu}(b)\,(\partial_\nu B)\,, \tag{66}$$

characterized by the contravariant version of $C_{\mu\nu}$

$$C^{\mu\nu} = \left[(C_{\alpha\beta})^{-1}\right]^{\mu\nu},$$

are Lie.

In the next sections we show that the above properties persist in their entirety when formulated under isotopies in isospaces.

5 Isolagrangian and isohamiltonian mechanics

We now show the derivability of the isotopic Newton equations from a first-order iso-variational principle and then study the isotopies of Lagrange's [15] and Hamilton's [10] mechanics.

Proposition 1 *All Newtonian action functionals of second or higher order in Euclidean space $E(t) \times E(x, \delta, R) \times E(v, \delta, R)$ whose integrand is sufficiently smooth and regular in a region $\mathcal{R}$ of their variables can always be identically rewritten as first-order action isofunctionals in isospace $\hat{E}(\hat{t}) \times \hat{E}(\hat{x}, \hat{\delta}, \hat{R}) \times \hat{E}(\hat{v}, \hat{\delta}, \hat{R})$ which are bilinear in the velocities,*

$$\hat{A} = \int_{t_1}^{t_2} dt\, \mathcal{L}\left(t, x, v, a, \ldots\right) = \widehat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, \widehat{\mathcal{L}}\left(\hat{t}, \hat{x}, \hat{v}\right), \tag{67}$$

$$\widehat{\mathcal{L}} = \frac{1}{2}\hat{m}\hat{v}^i \hat{\delta}_{ij}\hat{v}^j - \hat{U}^i\left(\hat{t}, \hat{x}\right)\hat{\delta}_{ij} v^j - U_0\left(\hat{t}, \hat{x}\right) = \frac{1}{2}\hat{m}\hat{v}_k\hat{v}^k - \hat{U}_k\left(\hat{t}, \hat{x}\right) v^k - \hat{U}_0\left(\hat{t}, \hat{x}\right). \tag{68}$$

In fact, identities (67) are overdetermined because, for each given $\mathcal{L}$, there exist infinitely many choices of $\hat{m}$, $\hat{T}_0^0$, $\hat{T}_j^i$, $\hat{U}_k$ and $\hat{U}_0$. We shall assume that integral terms are admitted in the integrand provided that they are all embedded in the isometric.

The *isovariational calculus* is a simple extension of the isodifferential calculus. In fact, we can write the following isovariation along an admissible isodifferentiable path $\hat{P}$:

$$\hat{\delta}\,\hat{A}\left(\hat{P}\right) = \widehat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t} \left(\hat{\delta}\hat{x}^k \frac{\hat{\partial}}{\hat{\partial}\hat{x}^k} + \hat{\delta}\hat{v}^k \frac{\hat{\partial}}{\hat{\partial}\hat{v}^k}\right) \hat{L}\left(\hat{P}\right) = \widehat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t} \left(\frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\hat{x}^k} - \frac{\hat{d}}{\hat{d}\hat{t}}\frac{\hat{\partial}\mathcal{L}}{\hat{\partial}\hat{v}^k}\right)\left(\hat{P}\right)\hat{\delta}\hat{x}^k, \tag{69}$$

where we have used isointegration by parts. The isotopy of the celebrated Euler [8] necessary condition can be formulated as follows.

Theorem 5 *(***Isoeuler Necessary Condition***): A necessary condition for an isodifferentiable path $\hat{P}$ in isospace $\hat{E}(t) \times \hat{E}(\hat{x}, \hat{\delta}, \hat{R}) \times \hat{E}(\hat{v}, \hat{\delta}, \hat{R})$ to be an extremal of action isofunctional $\hat{A}$ is that all the following isotopic equations*

$$\hat{L}_k\left(\hat{P}_0\right) = \left\{\frac{\hat{d}}{\hat{d}\hat{t}}\frac{\hat{\partial}\hat{L}\left(\hat{t}, \hat{x}, \hat{v}\right)}{\hat{\partial}\hat{v}^k} - \frac{\hat{\partial}\hat{L}\left(\hat{t}, \hat{x}, \hat{v}\right)}{\hat{\partial}\hat{x}^k}\right\}\left(\hat{P}_0\right) = 0 \tag{70}$$

are identically verified along $\hat{P}_0$.

It is an instructive exercise for the interested reader to prove the following:

Corollary 6 *The isotopic equations (70) are variationally iso-self-adjoint.*

The isotopies of the remaining aspects of the calculus of variations (see, e.g., Bliss[4]) with consequential isotopies of the optimal control theory are intriguing and significant, but they cannot be studied here for brevity. When dealing with the calculus of isovariations, Eq.s (70) will be referred to as *isoeuler equations*, and when dealing with analytic mechanics they will be referred to as *isolagrange equations*.

We shall say that the isotopic Newton equations (28)-(28) admit a *direct isoanalytic representation*, when there exists one isolagrangian $\widehat{\mathcal{L}}\left(\hat{t}, \hat{x}, \hat{v}\right)$ under which all the following identities occur

$$\left\{ \frac{\hat{d}}{\hat{d}\hat{t}} \frac{\hat{\partial}\hat{L}\left(\hat{t},\hat{x},\hat{v}\right)}{\hat{\partial}\hat{v}^k} - \frac{\hat{\partial}\hat{L}\left(\hat{t},\hat{x},\hat{v}\right)}{\hat{\partial}\hat{x}^k} \right\}^{ISOSA} =$$

$$= \left\{ \hat{m}\frac{\hat{d}v_k}{\hat{d}\hat{t}} - \frac{\hat{\partial}\hat{U}_k\left(\hat{t},\hat{x}\right)}{\hat{\partial}\hat{x}^i}\frac{\hat{d}\hat{x}^i}{\hat{d}\hat{t}} + \frac{\hat{\partial}\hat{U}_0\left(\hat{t},\hat{x}\right)}{\hat{\partial}\hat{x}^k} \right\}^{ISOSA} =$$

$$= \hat{T}^i_k \left\{ m\frac{dv_i}{dt} - \frac{\partial U_j\left(t,x\right)}{\partial x^S}\frac{dx^S}{dt} + \frac{\partial U_0\left(t,x\right)}{\partial x^i} - F_i^{NSA}\left(t,x,v\right) \right\}^{NSA} = 0, \tag{71}$$

$$\mathcal{L}\left(\hat{t},\hat{x},\hat{v}\right) = \frac{1}{2}\hat{m}\hat{v}^k\hat{v}_k - \hat{U},\ \hat{U}\left(\hat{t},\hat{x},\hat{v}\right) = \hat{U}_k\left(\hat{t},\hat{x}\right)\hat{v}^k + \hat{U}_0\left(\hat{t},\hat{x}\right). \tag{72}$$

Theorem 6 *(***Universality of isolagrangian mechanics***) All possible sufficiently smooth and regular dynamical systems in a star-shaped neighborhood of a point of their variables always admit a direct isorepresentation via the isolagrange equations in isospace.*

Proof. The universality of the isorepresentation follows from the fact that conditions (32)-(35) always admit solution (37)-(38) in the unknown functions. □

Remark: Newtonian systems are usually refered to systems with local-differential forces depending at most on velocities. Theorem 6 includes also non-Newtonian forces, e.g ., when they are of integral type or acceleration-dependent. Discontinuous Newtonian forces, such as those of impulsive type, have been removed from the theorem because of lack of current knowledge on the topology of isospaces with discontinuous isounits (isospaces of Kadeisvili's Class V [13]), although such an extension is expected to exist, and its study is left to interested readers.

Note the simplicity of the construction of an isolagrangian representation as compared to the complexity of the construction of a conventional Lagrangian representation [26, 27], when it exists.

We now introduce, apparently for the first time, the isotopies of the Legendre transform based on the isodifferential calculus (ref. [33] presents a different isotopies based on the isotopic degrees of freedom of the multiplication). For this purpose, we introduce the following isodifferentials in isospace $\hat{S}(\hat{t},\hat{x},\hat{p}) = \hat{E}(\hat{t}) \times \hat{E}(\hat{x},\hat{\delta},\hat{R}) \times \hat{E}(\hat{p},\hat{\delta},\hat{R})$:

$$\hat{d}\,\hat{t} = \hat{I}^0_0\,dt,\ \hat{d}\hat{x}^k = \hat{I}^k_i\,dx^i,\ \hat{\partial}\hat{x}^i/\hat{\partial}\hat{x}^j = \delta^i_j,\ etc., \tag{73}$$

$$\hat{d}\,\hat{p}_k = \hat{T}^i_j\,d\hat{p}_i,\ \hat{d}\,\hat{p}^k = \hat{I}^k_i\,d\hat{p}^i,\ \hat{\partial}\hat{p}_i/\hat{\partial}\hat{p}_j = \delta^j_i,\ etc. \tag{74}$$

The total isounits and isotopic elements of the isospace $\hat{E}(\hat{x},\hat{\delta},\hat{R}) \times \hat{E}(\hat{p},\hat{\delta},\hat{R})$ are therefore given by

$$\hat{I}_2 = diag.\left(\hat{I},\hat{T}\right),\ \hat{T}_2 = diag.\left(\hat{T},\hat{I}\right). \tag{75}$$

It should be indicated that, in view of the independence of the variables $\hat{p}_k$ from $\hat{x}^k$, we can introduce a new isounit $\hat{W} = \hat{Z}^{-1}$ for the isospace $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ which is different than the unit $\hat{I} = \hat{T}^{-1}$ of isospace $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$, in which case the total unit is $\hat{I}_2 = diag.\left(\hat{I}, \hat{W}\right)$. Selection (74) is based on the simplest possible case $\hat{W} = \hat{I}$ which is recommendable from the geometric isotopies studied in the next section. Other alternatives belong the problem of the degrees of freedom of the isotopic theories which is not studied in this paper for brevity.

We now introduce the *isocanonical momentum* via the following realization of prescriptions (54)

$$\hat{p}_k = \frac{\hat{\partial} \hat{L}(\hat{t}, \hat{x}, \hat{v})}{\hat{\partial} \hat{v}^k} = \hat{m}\hat{v}_k - \hat{U}_k\left(\hat{t}, \hat{x}\right), \tag{76}$$

under the condition of being regular in a $(2N+1)$-dimensional region $\widehat{\mathcal{R}}$ of points $(\hat{t}, \hat{x}, \hat{p})$

$$Det\left(\frac{\hat{\partial}^2 \hat{L}(\hat{t}, \hat{x}, \hat{v})}{\hat{\partial} \hat{v}^i \hat{\partial} \hat{v}^j}\right)\left(\widehat{\mathcal{R}}\right) \neq 0, \tag{77}$$

thus admitting a unique set of implicit functions $\hat{v}^k = f^k(t, \hat{x}, \hat{p})$. The *isolegendre transform* can then be defined by

$$\hat{L}\left(\hat{t}, \hat{x}, \hat{v}\left(\hat{t}, \hat{x}, \hat{p}\right)\right) = \hat{p}_k \hat{v}^k\left(\hat{t}, \hat{x}, \hat{p}\right) - \frac{1}{2}\hat{m}\hat{v}_i\left(\hat{t}, \hat{x}, \hat{p}\right)\hat{v}^i\left(\hat{t}, \hat{x}, \hat{p}\right) +$$
$$+\hat{U}_k\left(t, \hat{x}\right)\hat{v}^k\left(\hat{t}, \hat{x}, \hat{p}\right) + \hat{U}_0\left(\hat{t}, \hat{x}\right) = \hat{p}_k \hat{p}^k/2\hat{m} + \hat{V}^k\left(\hat{t}, \hat{x}\right)\hat{p}_k + \hat{V}^0\left(\hat{t}, \hat{x}\right) = \hat{H}\left(t, \hat{x}, \hat{p}\right). \tag{78}$$

We are now equipped to study the isotopies of Hamilton's principle [10]. By using the unified variables $\hat{b} = \{\hat{b}^\mu\} = \{\hat{x}^k, \hat{p}_k\}$, $\hat{c} = \hat{d}\,\hat{b}^\mu/\hat{d}\hat{t}$, and by introducing the notation

$$\hat{R}^\circ = \left\{\hat{R}^\circ_\mu\right\} = \left\{\hat{p}_k, \hat{0}\right\},\ \mu = 1, 2, \ldots, 2N,\ k = 1, 2, \ldots, N, \tag{79}$$

the *isocanonical principle* assumes the form along an actual path $\hat{P}_0$

$$\hat{\delta}\,\hat{A} = \hat{\delta}\widehat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\,\left(\hat{p}_k\,\hat{d}\hat{x}^k/\hat{d}\hat{t} - \hat{H}\right)\left(\hat{P}_0\right) = \hat{\delta}\widehat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\,\left(\hat{R}^\circ_\mu \hat{c}^\mu - \hat{H}\right)\left(\hat{P}_0\right) =$$
$$= \widehat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\,\left\{\hat{\delta}\hat{p}_i \frac{\hat{\partial}}{\hat{\partial}\hat{p}_i} + \hat{\delta}\hat{v}^i \frac{\hat{\partial}}{\hat{\partial}\hat{v}^i} + \hat{\delta}\hat{x}^i \frac{\hat{\partial}}{\hat{\partial}\hat{x}^i}\right\}\left(p_k v^k - H\right)\left(\hat{P}_0\right) =$$
$$\widehat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\left[\left(\frac{\hat{d}\hat{x}^k}{\hat{d}\hat{t}}\frac{\hat{\partial}\hat{p}_k}{\hat{\partial}\hat{p}_i} - \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_i}\right)\hat{\delta}\hat{p}_i - \left(\frac{\hat{d}}{\hat{d}\hat{t}}\left(\hat{p}_k \frac{\hat{\partial}\hat{v}^k}{\hat{\partial}\hat{v}^i}\right) + \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{x}^i}\right)\hat{\delta}\hat{x}^i\right]\left(\hat{P}_0\right) =$$
$$\widehat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\,\left\{\hat{\delta}\hat{b}^\mu \frac{\hat{\partial}}{\hat{\partial}\hat{b}^\mu} + \hat{\delta}\hat{c}^\mu \frac{\hat{\partial}}{\hat{\partial}\hat{c}^\mu}\right\}\left(\hat{R}^\circ_\mu\,\hat{d}\hat{b}^\mu - \hat{H}\,\hat{d}\hat{t}\right)\left(\hat{P}_0\right) =$$
$$\widehat{\int}_{\hat{t}_1}^{\hat{t}_2} \left\{\left(\frac{\hat{\partial}\hat{R}^\circ_\nu}{\hat{\partial}\hat{b}^\mu} - \frac{\hat{\partial}\hat{R}^\circ_\mu}{\hat{\partial}\hat{b}^\nu}\right)\frac{\hat{d}\hat{b}^\nu}{\hat{\partial}\hat{t}} - \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{b}^\mu}\right\}\left(\hat{P}_0\right) * \hat{\delta}\hat{b}^\mu = 0. \tag{80}$$

Theorem 7 (**Isohamilton Necessary Condition**): *A necessary condition for an isofunctional in isocanonical form whose integrand is sufficiently smooth and regular in a region $\mathcal{R}$ of points $(\hat{t}, \hat{b}, \hat{c})$ to have an extremum along a path $\hat{P}_0$ is that all the following isoequations in disjoint notation*

$$\frac{\hat{d}\hat{x}^k}{\hat{d}\hat{t}} = \frac{\hat{\partial}\hat{H}\left(\hat{t}, \hat{x}, \hat{p}\right)}{\hat{\partial}\hat{p}_k}, \ \frac{\hat{d}\hat{p}_k}{\hat{d}\hat{t}} = -\frac{\hat{\partial}\hat{H}\left(\hat{t}, \hat{x}, \hat{p}\right)}{\hat{\partial}\hat{x}^k}, \tag{81}$$

or in unified notation

$$\left(\frac{\hat{\partial}\hat{R}^\circ_\nu}{\hat{\partial}\hat{b}^\mu} - \frac{\hat{\partial}\hat{R}^\circ_\mu}{\hat{\partial}\hat{b}^\nu}\right)\frac{\hat{d}\hat{b}^\nu}{\hat{\partial}\hat{t}} - \frac{\hat{\partial}\hat{H}\left(\hat{t}, \hat{b}\right)\ A}{\hat{\partial}\hat{b}^\mu} = 0, \tag{82}$$

hold along an actual path $\hat{P}_0$.

It is also instructive for the interested reader to prove the following:

Corollary 7 *Isotopic equations (82) are variationally iso-self-adjoint.*

Eq.s (81) or (82) are called isohamilton equations and can be more simply written in the following respective covariant and contravariant forms

$$\omega_{\mu\nu}\frac{\hat{d}\hat{b}^\nu}{\hat{d}\hat{t}} = \frac{\hat{\partial}\hat{H}\left(\hat{t}, \hat{b}\right)}{\hat{\partial}\hat{b}^\mu}, \tag{83}$$

$$\frac{\hat{d}\hat{b}^\mu}{\hat{d}\hat{t}} = \omega^{\mu\nu}\frac{\hat{\partial}\hat{H}\left(\hat{t}, \hat{b}\right)}{\hat{\partial}\hat{b}^\nu}, \tag{84}$$

where the quantities

$$(\omega_{\mu\nu}) = \left(\frac{\hat{\partial}\hat{R}^\circ_\nu}{\hat{\partial}\hat{b}^\mu} - \frac{\hat{\partial}\hat{R}^\circ_\mu}{\hat{\partial}\hat{b}^\nu}\right) = \begin{pmatrix} 0_{N\times N} & -I_{N\times N} \\ I_{N\times N} & 0_{N\times N} \end{pmatrix}, \tag{85}$$

$$\left(\omega^{\alpha\beta}\right) = \left(\frac{\hat{\partial}\hat{R}^\circ_\nu}{\hat{\partial}\hat{b}^\mu} - \frac{\hat{\partial}\hat{R}^\circ_\mu}{\hat{\partial}\hat{b}^\nu}\right)^{-1} = \begin{pmatrix} 0_{N\times N} & \bar{I}_{N\times N} \\ -\bar{I}_{N\times N} & 0_{N\times N} \end{pmatrix}, \tag{86}$$

are the *conventional* covariant and contravariant canonical tensors, respectively, which hold in view of the identities originating from properties (73)-(74) and values (79)

$$\hat{\partial}\hat{R}^\circ_\nu/\hat{\partial}\hat{b}^\mu \equiv \partial R^\circ_\nu/\partial b^\mu. \tag{87}$$

The equivalence of the isolagrangian and isohamiltonian equations under the assummed regularity and invertibility of the isolegendre transform can be proved as in the conventional case (see, e.g., [26], Sect. 3.8).

We now study the following additional property of isohamiltonian mechanics which is important for operator maps. The *isotopic Hamilton-Jacobi problem* (see, e.g.,

[26], p. 201 and ff. for the conventional case) is the identification of an isocanonical transform under which the Hamiltonian becomes null. The generating function of such a transform is the isocanonical action itself, resulting in the end-point contributions

$$\hat{d}\hat{A} = \hat{d}\int_{t_0}^{t} \left(\hat{p}_k \, \hat{d}\hat{x}^k - \hat{H} \, \hat{d}\hat{t}\right) = \left|\hat{p}_k \, d\hat{x}^k - \hat{H} \, \hat{d}\hat{t}\right|_{t_0}^{t}, \tag{88}$$

with *isotopic Hamilton-Jacobi equations*

$$\frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{t}} + \hat{H}\left(t, \hat{x}, \hat{p}\right) = 0, \; \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{x}^k} - \hat{p}_k = 0, \tag{89}$$

plus initial conditions $\hat{\partial}\hat{A}/\hat{\partial}\hat{x}^{\circ k} = \hat{p}^{\circ}_k$, where $\hat{x}^{\circ}$ and $\hat{p}^{\circ}$ are constants.

Remark l: Note the *abstract identity between the conventional and isotopic mechanics.* Since the isounits are positive-definite, at the abstract level there is no distinction between dt and $\hat{d}\hat{t}$ or dx and $\hat{d}\hat{x}$, etc. The isolagrange and isohamilton equations therefore coincide at the abstract level with the conventional equations. This illustrates the axiom-preserving character of the isotopies.

Remark 2: The *direct universality* of the isohamiltonian mechanics in the fixed inertial frame of the observer should be compared with the corresponding *lack* of universality of the conventional Hamiltonian mechanics. A first direct universality was achieved by this author [27] via a step-by-step generalization of Hamiltonian mechanics called (for certain historical reasons) *Birkhoffian mechanics*. The latter mechanics is based on the most general possible *first-order* Pfaffian variational principle (63)-(64) in the unified variables $b = \{b^\mu\} = \{x^k, p_k\}$ in a conventional space $S(t, x, p)$, i.e.,

$$\delta \int_{t_1}^{t_2} \left[R_\mu \left(b\right) \, db^\mu - H\left(t, b\right) \, dt\right] = 0, \tag{90}$$

yielding Birkhoff's equations [3] in covariant form

$$\left\{\Omega_{\mu\nu}\left(b\right)\frac{\partial b^\nu}{dt} - \frac{\partial H\left(t,b\right)}{\partial b^\mu}\right\}^{SA} = 0, \quad \Omega_{\mu\nu}\left(b\right) = \frac{\partial R_\nu}{\partial b^\mu} - \frac{\partial R_\mu}{\partial b^\nu}, \tag{91}$$

with contravariant version

$$\frac{db^\mu}{dt} = \Omega^{\mu\nu}\left(b\right)\frac{\partial H\left(t,b\right)}{\partial b^\nu}, \quad \Omega^{\mu\nu} = \left[\left(\Omega_{\alpha\beta}\right)^{-1}\right]^{\mu\nu}. \tag{92}$$

The connection between the Birkhoffian and the isohamiltonian mechanics is intriguing. In fact, the Pfaffian action can always be identically rewritten as the isotopic action

$$\int_{t_1}^{t_2} \left[R_\mu \left(b\right) db^\mu - H\left(t,b\right) dt\right] \equiv \widehat{\int}_{\hat{t}_1}^{\hat{t}_2} \left[\hat{R}^0_\mu \left(b\right) \hat{d}\hat{b}^\mu - \hat{H}\left(\hat{t},\hat{b}\right) dt\right], \\ \hat{b}^\mu \equiv b^\mu \hat{H} \equiv H, \; \hat{d}\hat{t} = dt, \tag{93}$$

and the general, totally antisymmetric Lie tensor $\Omega^{\mu\nu}$ (see later on) always admits the factorization into the canonical Lie tensor $\omega^{\mu\nu}$ and a regular symmetric matrix $\hat{T}^{\nu}_{\mu}$

$$\Omega^{\mu\nu} \equiv \omega^{\mu\beta}\hat{T}^{\nu}_{\beta}\,, \tag{94}$$

under which Birkhoff's equations (92) coincides with the isohamilton's equations (84) for $\hat{I}^0_0 = 1$.

Despite these similarities, it should be indicated that the isohamiltonian mechanics is considerably broader than the Birkhoffian mechanics. In fact, the former is based on an action of arbitrary order, while the latter necessarily requires a first-order action. Also, the former can represent integral forces, while the latter cannot (because the underlying geometry, the symplectic geometry in its most general possible exact realization) only admits local-differential systems. Finally, the former is based on a broader mathematics, the isodifferential calculus, while the latter is based on conventional mathematics.

Remark 3: An important application of the isohamiltonian mechanics is to provide a novel classical realization of the Lie-Santilli isotheory [25, 12, 14, 17, 34]. Recall that the conventional classical realization of the Lie product is given by the familiar Poisson brackets among two functions $A(b)$ and $B(b)$ in the cotangent bundle (phase space)

$$[A,B]_{Poisson} = \frac{\partial A}{\partial x^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial x^k}\frac{\partial A}{\partial p_k} = \frac{\partial A}{\partial b^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial b^\nu}. \tag{95}$$

From the self-adjointness of Birkhoff's equations [27] and the algebraic meaning of the conditions of self-adjointness recalled in Sect. 4, the most general possible (regular, unconstrained) brackets in cotangent bundle verifying the Lie algebra axioms are given by the *Birkhoffian brackets* (also called generalized Poisson brackets) [27]

$$[A,B]_{Birkhoff} = \frac{\partial A}{\partial b^\mu}\Omega^{\mu\nu}(b)\,\frac{\partial B}{\partial b^\nu} \tag{96}$$

The novel brackets introduced in this paper are given by the following brackets among isofunctions $\hat{A}(\hat{b})$, $\hat{B}(\hat{b})$ on isocotangent bundle

$$[A,B]_{Isotopic} = \frac{\hat{\partial} A}{\hat{\partial}\hat{x}^k}\frac{\hat{\partial} B}{\hat{\partial}\hat{p}_k} - \frac{\hat{\partial} B}{\hat{\partial}\hat{x}^k}\frac{\hat{\partial} A}{\hat{\partial}\hat{p}_k} =$$

$$= \frac{\partial A}{\partial x^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial x^k}\frac{\partial A}{\partial p_k}\,, \tag{97}$$

and they *formally coincide* with the conventional brackets (95) when projected in the original space. This illustrates Bruck's [5] statement to the effect that "the isotopies are so natural to creep in unnoticed". However, one should remember that the underlying geometry is generalized. In fact, the isotopic brackets can be written

$$[A,B]_{Isotopic} = \frac{\partial A}{\hat{\partial}\hat{x}_i}\hat{T}^k_i\,(t,r,p,\ldots)\,\delta_{kj}\frac{\partial B}{\hat{\partial}\hat{p}_j} - \frac{\partial B}{\hat{\partial}\hat{x}_i}\hat{T}^k_i\,(t,r,p,\ldots)\,\delta_{kj}\frac{\partial A}{\hat{\partial}\hat{p}_j}, \tag{98}$$

thus showing their differences with the conventional brackets. Moreover, one should keep in mind from the comments following Eq.s (75) that we have selected the simplest possible isotopies for which the isounits of the independent variables $\hat{p}_k$ and x^k are the same. The use of different isounits for $\hat{p}^k$ and x^k evidently implies further differences between the isotopic and conventional brackets.

Note that the isotopic character of brackets (97) is assured by the iso-self-adjointness of the isohamilton equations. Note also that brackets (98) *do not verify* the Lie algebra axioms in conventional spaces, evidently because the isotopic elements $\hat{T}_i^j$ are unrestricted. This illustrates that *the isotopic theory of this paper verifies the Lie axioms only in isospace but not when projected in conventional spaces.* This occurrence should be compared to other realizations studied in ref.s [32, 33] in which the Lie axioms are verified in isospace as well as in their projection in conventional spaces.

In unified notation we evidently have the same occurrence. In fact, the isotopic brackets can be written

$$[A, B]_{Isotopic} = \frac{\hat{\partial} A}{\hat{\partial} \hat{b}^\mu} \omega^{\mu\nu} \frac{\hat{\partial} B}{\hat{\partial} \hat{b}^\nu} = \frac{\partial A}{\hat{\partial} \hat{b}^\alpha} \hat{T}_\mu^\alpha \omega^{\mu\nu} \hat{T}_\nu^\beta \frac{\partial B}{\partial \hat{b}^\beta} = \frac{\partial A}{\partial \hat{b}^\alpha} \omega^{\alpha\beta} \frac{\partial B}{\partial \hat{b}^\beta} , \tag{99}$$

where the last identity occurs in view of the properties

$$\hat{T}_\mu^\alpha \omega^{\mu\nu} \hat{T}_\nu^\beta = \omega^{\alpha\beta}. \tag{100}$$

It is also easy to see that the isohamiltonian mechanics provides a classical realization of the Lie-Santilli isogroups [25, 12, 14, 17, 34]. In fact, the integrated form of Eq. (84) yields the time evolution of a quantity $\hat{A}(\hat{t})$ in isospace here expressed in terms of expression (98)

$$\hat{A}\left(\hat{t}\right) = \exp\left\{t\left[\frac{\partial \hat{H}}{\partial \hat{x}} \hat{T} \frac{\partial}{\partial \hat{p}} - \frac{\partial \hat{H}}{\partial \hat{x}} \hat{T} \frac{\partial}{\partial \hat{p}}\right]\right\} \hat{A}\left(\hat{0}\right), \tag{101}$$

which is indeed a one-dimensional isogroup owing to the appearance of the isotopic matrix $\hat{T}$ in the exponent (see [32] for details). Realizations (97) and (101) are the classical counterpart of the operator isotopic realizations identified below.

Remark 4: One should note from isoprinciple (89) that

$$\hat{\partial} \hat{A} / \hat{\partial} \hat{p}_k \equiv 0, \; k = 1, 2, \ldots, N. \tag{102}$$

This occurrence is important for quantization in order to reach wavefunctions $\hat{\psi}(\hat{t}, \hat{x})$ independent from the momenta $\hat{p}$, as necessary for a correct isotopy of quantum mechanical wavefunctions $\psi(t, x)$.Note that property (98) occurs again because of the embedding of higher order terms in the geometry of the theory.

Remark 5: The significance of isohamiltonian mechanics can be also illustrated by the fact that its map under the conventional (or symplectic) quantization is not quantum mechanics, but instead a broader isotopic theory known under the name of *hadronic mechanics* [33]. Without entering into details, it is important for this paper to see that the isotopic operator theory preserves all the main features of the isotopic Newton equations, such as the representation of nonspherical-deformable shapes, nonselfadjoint forces and nonlocal-integral interactions.

Consider the map (called *naive isoquantization*)

$$\hat{A}\left(\hat{t},\hat{x}\right) \rightarrow -i\,\hat{I}\left(\hat{t},\hat{p}\right)\,Ln\,\hat{\psi}\left(\hat{t},\hat{x}\right),\ \hat{n}=1, \tag{103}$$

where the coordinates are in isospace, $\hat{I}$ is the isounit of the isotopic Newton equations which is here assumed to be independent from $\hat{x}$ for simplicity (see ref. [33] for the general case). The application of map (103) to Eq.s (89) yields the following *isotopic Schrödinger equations*

$$i\,\hat{\partial}\,\hat{\psi}\,/\,\hat{\partial}\hat{t} = \hat{H}\hat{T}\,\hat{\psi} = \hat{H} * \hat{\psi}, \quad \hat{p}_k\,\hat{T}\hat{\psi} = \hat{p}_k * \hat{\psi} = -i\,\hat{\partial}\,\hat{\psi}/\hat{\partial}\hat{x}^k \tag{104}$$

and are defined on a *isohilbert space* $\widehat{\mathcal{H}}$ with isostates $\hat{\psi}$, $\hat{\phi}$, ..., and *isoinner product* $<\hat{\psi}\hat{|}\hat{\phi}>=\hat{I}\int d\hat{x}^3\hat{\psi}\uparrow\hat{\phi}$ over the isocomplex field $\hat{C}(\hat{c},+,\hat{\times})$ originally submitted by Myung and Santilli [20] (see [33] for recent detailed studies). The equivalent *isoheisenberg equation* for an observable $\hat{O}$ are given by

$$i\,\hat{d}\hat{O}\,/\,\hat{d}\hat{t} = \left[\hat{O},\hat{H}\right] = \hat{O} * \hat{H} - \hat{H} * \hat{O} = \hat{O}\,\hat{T}\,\hat{H} - \hat{H}\,\hat{T}\,\hat{O} \tag{105}$$

and results to be defined on an enveloping algebra $\hat{\xi}$ of operators $\hat{A}$, $\hat{B}$,..., and isounit $\hat{I}=\hat{T}^{-1}$ on $\widehat{\mathcal{H}}$ equipped with the isoassociative product $\hat{A}*\hat{B}=\hat{A}\,\hat{T}\,\hat{B}$ over $\hat{C}(\hat{c},+,\hat{\times})$ originally submitted by Santilli [25]. The operator image of the isobrackets (98) is therefore given by

$$\left[\hat{A},\hat{B}\right] = \hat{A}\,\hat{T}\,\hat{B} - \hat{B}\,\hat{T}\,\hat{A}, \tag{106}$$

which constitute the operator realization of the Lie-Santilli isoproduct (see [25, 12, 14, 17, 25, 31, 33, 34] and references quoted therein).

The exponentiated form of Eq.s (106) yields the time evolution of isostates

$$\hat{\psi}' = \hat{U} * \hat{\psi} = \left\{\hat{e}^{i\hat{H}t}\right\} * \hat{\psi} = e^{iHTt}\,\hat{\psi}\ , \tag{107}$$

where $\hat{e}^{\alpha}$ is the *isoexponentiation* of an arbitrary (well behaved) quantity α, i.e., the exponentiation in $\hat{\xi}$ when it is turned into a universal enveloping isoassociative algebra via the isotopic Poincaré-Birkhoff-Witt theorem first formulated in [25] and then studied in [20]

$$\hat{e}^{\alpha} = \hat{I} + \alpha/1! + \alpha * \alpha/2! + \cdots = \left\{e^{\alpha T}\right\}\hat{I}\ , \tag{108}$$

$\hat{U}$ is a *isounitary transform*, i.e., a transform verifying the rules on $\widehat{\mathcal{H}}$

$$\hat{U} * \hat{U}^{\dagger} = \hat{U}^{\dagger} * \hat{U} = \hat{I}, \tag{109}$$

and the isotopic group laws can be written for an arbitrary isoparameters $\hat{w} \in \hat{R}(\hat{n},+,\hat{\times})$ [25]

$$\hat{U}\left(\hat{w}\right) * \hat{U}\left(\hat{w}'\right) = \hat{U}\left(\hat{w}+\hat{w}'\right), \quad \hat{U}\left(\hat{w}\right) * \hat{U}\left(-\hat{w}\right) = \hat{U}\left(\hat{0}\right) = \hat{I}. \tag{110}$$

As one can see, the matrix $\hat{T}$ of the isotopic Newton equations is preserved in its entirety at the operator level and this confirms the capability of the isotopic operator theory of representing nonspherical-deformable shapes, nonselfadjoint forces and nonlocal-integral interactions (see [33] for comprehensive studies).

The significance of the Lie-Santilli isotheory over the conventional formulation is illustrated by the appearance of the matrix $\hat{T}$ with arbitrary nonlinear-integral terms in the *exponent* of the isogroup, Eq. (107). This assures that the original linear, local and canonical Lie theory is mapped under isotopies into nonlinear, nonlocal and noncanonical forms, as desired. The reconstruction in isospace of linearity, locality and canonicity is illustrated in the next section.

Numerous applications of the above operator isotopic theories in various disciplines are studied in ref. [33]. An application illustrating the nonlinear-integral character occurs for the Cooper pair in superconductivity in which two *identical* electrons experience an *attractive* interaction under the intermediate action of cuprate ions. The use of quantum mechanical Coulomb interactions does not permit the achievement of the above attraction among identical electrons. The occurrence is preliminarily explained in superconductivity via the assumption of not fully known interactions between electrons and phonons. The occurrence is instead represented via the operator isotopic theory via the lifting studied in detail by Animalu [1] of the conventional Coulomb interactions characterized by the following isounit, today called *Animalu isounit*

$$\hat{I} = \exp\left\{\int dx\, \hat{\psi}_{\uparrow}^{\dagger}(x)\, \hat{\psi}_{\downarrow}(x)\, I \hat{\psi}_{\uparrow}(r) / \hat{\psi}_{\uparrow}(r)\right\} \tag{111}$$

where $_{\uparrow}$ and $_{\downarrow}$ represents spin up and down, respectively, $\hat{\psi}$ is the wavefunction of the isoelectron (i.e., the electron under isotopic treatment) and ψ is that of the conventional electron. In essence , the isotopy of the Coulomb interactions characterized by isounit (111) assumes a form at short distances which behaves likes the attractive Coulomb interactions but which is stronger than the conventional interactions, thus resulting in attraction irrespective of whether the charges are equal or opposite. The emerging theory, called *iso-super-conductivity* [1], results to be in remarkable agreement with experimental data and offers rather promising predictive capacities currently under study by various authors. Similar results occurs in the application of the isotopic theories in nuclear physics, particle physics. astrophysics and other disciplines [33].

Isounit (111) illustrates the type of interactions occurring in the isotopic operator theory, and it is therefore significant for the identification of the class of mathematical structures recommended for study. Isounit (111) contain the *integral term* $\int dx\, \hat{\psi}_{\uparrow}^{\dagger}(x)\, \hat{\psi}_{\downarrow}(x)$ which is representative of the *nonlocal* interactions occurring in the Cooper pair, which are given by the *volume integral over the region of overlapping of the two wavepackets* $\hat{\psi}_{\uparrow}$ and $\hat{\psi}_{\downarrow}$. These interactions are not permitted by quantum mechanics owing to its strict local-differential character, but they are permitted by isotopic theories when embedded in the isounit, as now familiar.

Moreover, the correct achievement of attraction among the identical electrons calls for the additional contribution which is *nonlinear in the wavefunctions* and which is represented by the term $\psi_\uparrow/\hat{\psi}_\uparrow$ in the exponent of (111). These contributions are today treated via nonlinear Schrödinger's equations of the type

$$H\left(t, r, p, \psi, \ldots\right)\psi\left(t, x\right) = E\,\psi\left(t, x\right). \tag{112}$$

Even though mathematically impeccable and actually intriguing, the latter theories are afflicted by problems of *physical* consistency, such as the loss of the superposition principle (with consequential inability to treat bound states as needed for the Cooper pair), the general nonunitary character of the time evolution which implies the general loss of invariance of the unit (with consequential impossibility of consistent measurements), the general loss of the original Hermiticity under the time evolution of the theory (with consequential loss of observables), and others.

The isotopic theories resolve all these problematic aspects via the decomposition

$$H\left(t, r, p, \psi, \ldots\right)\psi \equiv H_0\left(t, x, p\right)\hat{T}\left(\psi, \ldots\right)\psi = H * \psi = E\psi,$$

that is, via the embedding of all nonlinear terms in the isotopic element T, with consequential reconstruction in isospace of the superposition principle, the unitarity of the time evolution law, the invariance of the unit, the preservation of Hermiticity at all times, etc. (for brevity, see [33] for details).

With the understanding that quantum mechanics is exactly valid for all conditions in which nonlinear, nonlocal and nonhamiltonian effects are ignorable (such as the atomic structure as well as the electromagnetic and weak interactions at large), the advantage of isotopic theories for systematic studies of nonlinear, nonlocal, noncanonical interactions at short distances is then evident. At any rate, the conventional theory is recovered identically when the integral $\int dx\,\hat{\psi}^\dagger_\uparrow(x)\,\hat{\psi}_\downarrow(x)$ in Animalu's isounit (111) is identically null, i.e., when the nonlocal contributions due to overlapping of the wavepackets are ignorable, in which case $\hat{I} = I$.

6 Isosymplectic geometry

We identify in this section the isotopies of the symplectic geometry, called *isosymplectic geometry* for short, as the geometry underlying the isohamilton equations and related Lie-isotopic theory. These isotopies were first studied by this author in ref. [25], then subjected to deeper studies in memoir [29] and monograph [32] via the lifting of the units and of the conventional associative product. The formulation of the isosymplectic geometry based on the isodifferential calculus is presented in this section for the first time.

Unless otherwise stated, all quantities are assumed to satisfy the needed continuity conditions, e.g., of being of class C^∞ and all neighborhoods of a point are assumed to be star-shaped or have an equivalent topology. Owing to the emphasis on applications, the treatment of this section is restricted to local realizations, while coordinate-free treatments are left to the interested reader. In any case, as we shall see, all distinctions cease to exist at the abstract level between the symplectic and isosymplectic geometries. A comprehensive literature on the symplectic geometry is available in ref. [26] and it is omitted here for brevity.

Let $M(\hat{E}) = M(\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ be an N-dimensional Tsagas-Sourlas isomanifold [35] on the isoeuclidean space $\hat{E}(\hat{x}, \hat{\delta}, \hat{R})$ over the isoreals $\hat{R} = \hat{R}(\hat{n}, +, \hat{\times})$ with $N \times N$-dimensional isounit $\hat{I} = (\hat{I}^i_j)$, $i, j = 1, 2, \ldots, N$, of Kadeisvili [13] Class I (nowhere degenerate, symmetric, real valued and positive-definite with Class C^∞ elements) and local chart $\hat{x} = \{\hat{x}^k\}$. A *tangent isovector* $\hat{X}(\hat{m})$ at a point $\hat{m} \in M(\hat{E})$ is an isofunction defined in the neighborhood $N(\hat{m})$ of $\hat{m}$ with values in $\hat{R}$ satisfying the *isolinearity conditions*

$$\hat{X}_m \left(\hat{\alpha} \hat{\times} \hat{f} + \hat{\beta} \hat{\times} \hat{g}\right) = \hat{\alpha} \hat{\times} \hat{X}_{\hat{m}} \left(\hat{f}\right) + \hat{\beta} \hat{\times} \hat{X}_{\hat{m}} (\hat{g}) , \; \hat{X}_{\hat{m}} \left(\hat{f} \hat{\times} \hat{g}\right) = \hat{f}(\hat{m}) \hat{\times} \hat{X}_{\hat{m}} (\hat{g}) + \hat{g}(\hat{m}) \hat{\times} \hat{X}_{\hat{m}} \left(\hat{f}\right) , \tag{113}$$

for all $\hat{f}$, $\hat{g} \in M(\hat{E})$ and $\hat{\alpha}$, $\hat{\beta} \in \hat{R}$, where $\hat{\times}$ is the isomultiplication in $\hat{R}$ and the use of the symbol $\hat{}$ means that the quantities are defined on isospaces.

The collection of all tangent isovectors at $\hat{m}$ is called the *tangent isospace* and denoted $TM(\hat{E})$. The *tangent isobundle* is the $2N$-dimensional union of all possible tangent isospaces when equipped with an isotopic structure (see below). The *cotangent isobundle* $T^*M(\hat{E})$ is the dual of the tangent isobundle and it is defined with respect to the isounit $\hat{I}_2 = diag.\,(\hat{I}, \hat{T}) = diag.\,(\hat{T}^{-1}, \hat{I}^{-1})$, with the understanding pointed our in the preceding section that more general isounits of the type $\hat{I}_2 = diag.\,\left(\hat{I}, \hat{W}^{-1}\right)$, $\hat{W} \neq \hat{I}$, are possible because of the independence of $\hat{x}$ and $\hat{p}$.

Let $\hat{b} = \{\hat{b}^\mu\} = \{\hat{x}^k, \hat{p}_k\}$, $\mu = 1, 2, \ldots, 2N$, be a local chart of $\hat{T}^*M(\hat{E})$. An *isobasis* of $T^*M(\hat{E})$ is, up to equivalence, the (ordered) set of isoderivatives $\hat{\partial} = \{\hat{\partial}/\hat{\partial}\hat{b}^\mu\} = \{\hat{T}^\nu_{2\mu} \partial/\partial b^\nu\}$. A generic element $\hat{X} \in T^*M(\hat{E})$ can then be written $\hat{X} = \hat{X}^\mu(\hat{m}) \hat{\partial}/\hat{\partial}\hat{b}^\mu$.

The *fundamental one-isoform* on $T^*M(\hat{E})$ is given in the local chart $\hat{b}$ by

$$\hat{\theta} = \hat{R}^\circ_\mu \left(\hat{b}\right) \hat{d}\hat{b}^\mu = \hat{R}^\circ_\mu (b)\, \hat{I}^\mu_{2\nu}\, d\hat{b}^\nu = \hat{p}_k \hat{d}\hat{x}^k = \hat{p}_k \hat{I}^k_i d\hat{x}^i, \; \hat{R}^\circ = \left\{\hat{p}, \hat{0}\right\}. \tag{114}$$

The space $T^*M(\hat{E})$, when equipped with the above one-form, is an isobundle denoted $T^*_1 M(1)$. The isoexact, nowhere degenerate, *isosymplectic two-isoform* in isocanonical realization is given by

$$\hat{\omega} = \hat{d}\hat{\theta} = \hat{d}\left(\hat{R}^\circ_\mu \hat{d}\hat{b}^\mu\right) = \omega_{\mu\nu} \hat{d}\hat{b}^\mu \wedge \hat{d}\hat{b}^\nu =$$

$$2\; \hat{d}\hat{x}^k \wedge \hat{d}\hat{p}_k = \hat{I}^k_i d\hat{x}^i \wedge \hat{T}^j_k d\hat{p}_j \equiv d\hat{x}^k \wedge d\hat{p}_k \equiv \omega. \tag{115}$$

The isospace $T^*M(\hat{E})$, when equipped with the above two-isoform, is an *isosymplectic isomanifold* in isocanonical realization denoted $T^*_2 M(\hat{E})$. The *isosymplectic geometry* is the geometry of the isosymplectic isomanifolds. The last identity in (115) show that the *isosymplectic isocanonical two-isoform* $\hat{\omega}$ *formally coincides with the conventional symplectic canonical two-form* ω.

The abstract identity of the symplectic and isosymplectic geometries is then evident. This illustrates on geometric grounds Bruck's [5] statement to the effect that "the isotopies are so natural to keep in un-noticed". However, one should remember that the underlying metric is isotopic, that $\hat{p}_k = \hat{T}^i_k p_i$, where p_i is the variable of the conventional canonical realization of the symplectic geometry, and that identity (115) no longer holds for the more general isounits $\hat{I}_2 = diag.(\hat{I}, \hat{W}^{-1})$. Also, the symplectic geometry is local-differential, while the isosymplectic geometry admits nonlocal-integral terms when embedded in the isounit.

A vector isofield $\hat{X}(\hat{m})$ defined on the neighborhood $N(\hat{m})$ of a point $\hat{m} \in T_2^* M(\hat{E})$ with local coordinates $\hat{b}$ is called *isohamiltonian* when there exists an isofunction $\hat{H}$ on $N(\hat{m})$ over $\hat{R}$ such that $\hat{X} \rfloor \hat{\omega} = -\hat{d}\hat{H}$, i.e.,

$$\omega_\mu \hat{X}^\nu (\hat{m})\, \hat{d}\hat{b}^\mu = \hat{d}\hat{H}(\hat{m}) = \left(\hat{d}\hat{H} / \hat{\partial}\hat{b}^\mu\right) \hat{d}\hat{b}^\mu, \tag{116}$$

which are equivalent to isohamilton equations (83). The isosymplectic geometry is therefore the geometry underling the isohamiltonian mechanics.

It is straighforward to construct isoforms Φ_p of arbitrary order p. The proof of the following property then follows from the properties of the isodifferential calculus.

Lemma 1 (**Isopoincaré Lemma**)*: Under the assumed smoothness and regularity conditions, isoexact p-isoforms are closed, i.e.,*

$$\hat{d}\hat{\Phi}_p = \hat{d}\left(\hat{d}\hat{\Phi}_{p-1}\right) \equiv 0. \tag{117}$$

For the two-dimensional case (see, e.g., [18] or [27]), the conventional Poincaré lemma is known to provide the necessary and sufficient conditions in geometric form for the contravariant version $\omega^{\mu\nu} = [(\omega_{\alpha\beta})^{-1}]^{\mu\nu}$ of the canonical symplectic tensor $\omega_{\mu\nu}$ to be Lie, i.e., for brackets (95) to satisfy the Lie algebra axioms. In this way, the symplectic geometry is the geometry underlying Lie's theory.

The isopoincaré lemma for the two-dimensional case provides the necessary and sufficient conditions for the same contravariant tensor $\omega^{\mu\nu}$ to be , this time, Lie-isotopic, i.e., for the isobrackets (97) to verify the Lie axioms in isospaces over isofields [12, 18, 32, 35]. The isosymplectic geometry is therefore the geometry underlying the Lie-Santilli isotheory.

The *general one-isoform* in the local chart $\hat{b}$ is given by

$$\hat{\Theta} = \hat{R}_\mu \left(\hat{b}\right) \hat{d}\hat{b}^\mu = \hat{R}_\mu \left(\hat{b}\right) \hat{I}^\mu_{2\nu} \left(t, \hat{b}, \hat{d}\hat{b}/\hat{d}\hat{t}, \ldots\right) d\hat{b}^\nu, \quad \hat{R} = \left\{\hat{P}\left(\hat{x}, \hat{p}\right), \hat{Q}\left(\hat{x}, \hat{p}\right)\right\}. \tag{118}$$

The general *isosymplectic isoexact two-isoform* in the same chart is then given by

$$\hat{\Omega} = \hat{d}\left(\hat{R}_\mu \left(\hat{b}\right) \hat{d}\hat{b}^\mu\right) = \hat{\Omega}_{\mu\nu} \left(\hat{t}, \hat{b}, \hat{d}\hat{b}/\hat{d}\hat{t}, \ldots\right) \hat{d}\hat{b}^\mu \wedge \hat{d}\hat{b}^\nu, \tag{119}$$

$$\hat{\Omega}_{\mu\nu} = \frac{\hat{\partial}\hat{R}_\nu}{\hat{\partial}\hat{b}^\mu} - \frac{\hat{\partial}\hat{R}_\mu}{\hat{\partial}\hat{b}^\nu} = \hat{T}^\alpha_{2\mu} \frac{\hat{\partial}\hat{R}_\nu}{\hat{\partial}\hat{b}^\alpha} - \hat{T}^\alpha_{2\nu} \frac{\hat{\partial}\hat{R}_\mu}{\hat{\partial}\hat{b}^\alpha}. \tag{120}$$

One can see that, while at the canonical level exact the two-form and its isotopic extension $\hat{\omega}$ formally coincide, *this is no longer the case for exact, but arbitrary two forms* Ω and $\hat{\Omega}$ in the same local chart.

Note that the isoform $\hat{\Omega}$ is isoexact, $\hat{\Omega} = \hat{d}\hat{\Theta}$, and therefore isoclosed, $\hat{d}\hat{\Omega} \equiv 0$ in isospace. However, if the same isoform $\hat{\Omega}$ is projected in ordinary space and called

Ω, it is no longer necessarily exact and, therefore, it is not generally closed, $\mathrm{d}\Omega = 0$. These properties prove the following lemma.

Lemma 2 (General Lie-Santilli Isobrackets). *Let $\hat{\Omega} = \hat{\Omega}_{\mu\nu}\hat{d}\hat{b}^{\mu} \wedge \hat{d}\hat{b}^{\nu}$ be a general exact two-isoform, $\hat{\Omega} = \hat{d}\hat{\Theta} = \hat{d}(\hat{R}_{\mu}\hat{d}\hat{b}^{\mu})$. Then the brackets among sufficiently smooth and regular isofunctions $\hat{A}(\hat{b})$ and $\hat{B}(\hat{b})$ on $\hat{T}_2^* M(\hat{E})$*

$$\left[\hat{A}, \hat{B}\right]_{isot.} = \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{b}^{\mu}}\hat{\Omega}^{\mu\nu}\frac{\partial\hat{B}}{\hat{\partial}\hat{b}^{\nu}} , \tag{121}$$

$$\Omega^{\mu\nu} = \left[\left(\frac{\hat{\partial}\hat{R}_{\alpha}}{\hat{\partial}\hat{b}^{\beta}} - \frac{\hat{\partial}\hat{R}_{\beta}}{\hat{\partial}\hat{b}^{\alpha}}\right)^{-1}\right]^{\mu\nu} , \tag{122}$$

satisfy the Lie-Santilli axioms in isospace (but not necessarily the same axioms when projected in ordinary spaces).

An important property of the symplectic geometry is Darboux's Theorem [7] which expresses the capability of reducing arbitrary symplectic two-forms to the canonical form or, equivalently, the reduction of Birkhoff's to Hamilton's equations. The following additional property completes the axiom-preserving character of the isotopies of the symplectic geometry.

Theorem 8 (Isodarboux Theorem): *A $2N$-dimensional isocotangent bundle $\hat{T}_2^* M(\hat{E})$ equipped with a nowhere degenerate, exact, $\hat{C}^{\infty}$ two-isoform $\hat{\Omega}$ in the local chart $\hat{b}$ is an isosymplectic manifold if and only if there exists coordinate transformations $\hat{b} \to \hat{b}'(\hat{b})$ under which $\hat{\Omega}$ reduces to the isocanonical two- isoform $\hat{\omega}$, i.e.,*

$$\frac{\hat{\partial}\hat{b}^{\mu}}{\hat{\partial}\hat{b}'^{\,\alpha}}\hat{\Omega}_{\mu\nu}\left(\hat{b}\left(\hat{b}'\right)\right)\frac{\hat{\partial}\hat{b}^{\nu}}{\hat{\partial}\hat{b}'^{\,\beta}} = \omega_{\alpha\beta} . \tag{123}$$

Proof. Suppose that the transformation $\hat{b} \to \hat{b}'(\hat{b})$ occurs via the following intermediate transform $\hat{b} \to \hat{b}''(\hat{b}) \to \hat{b}'(b''(b))$. Then there always exists a transform $\hat{b} \to \hat{b}''$ such that

$$\left(\hat{\partial}\hat{b}^{p}/\hat{\partial}\hat{b}''^{\sigma}\right)\left(\hat{b}''\right) = \hat{I}^{\rho}_{\sigma}\left(\hat{b}\left(\hat{b}''\right)\right) , \tag{124}$$

under which the general isosymplectic tensor $\hat{\Omega}_{\mu\nu}$ reduces to the Birkhoffian form when recompute in the $\hat{b}$ chart

$$\frac{\hat{\partial}\hat{b}^{\mu}}{\hat{\partial}\hat{b}''^{\,\alpha}}\hat{\Omega}_{\mu\nu}\left(\hat{b}\left(\hat{b}''\right)\right)\frac{\hat{\partial}\hat{b}^{\nu}}{\hat{\partial}\hat{b}''^{\,\beta}}_{|\hat{b}''} = \left(\frac{\partial\hat{R}_{\nu}}{\partial\hat{b}^{\alpha}} - \frac{\partial\hat{R}_{\mu}}{\partial\hat{b}^{\nu}}\right)_{|\hat{b}''} = \omega_{\alpha\beta\,|\hat{b}''} . \tag{125}$$

The existence of a second transform $\hat{b}'' \to \hat{b}'$ reducing $\Omega_{\alpha\beta}$ to $\omega_{\alpha\beta}$ is then known to exist (see, e.g., [27]). This proves the necessity of the isodarboux chart. The sufficiency is proved as in the conventional case. □

The isotopies of the remaining aspects of the symplectic geometry (Lie derivative, global treatment, etc.) can be constructed along the preceding lines and are omitted for brevity.

Remark 1. The symplectic geometry in canonical realization can geometrize in the given b-chart only a subclass of Newtonian systems, namely, conservative systems plus a restricted class of nonconservative systems called *nonessentially nonselfadjoint* [26]. The remaining systems can only be geometrized via their representation with respect to an arbitrary symplectic two-form and its reduction to the canonical form via the Darboux's transforms. However, the Darboux transforms are nonlinear and therefore, as recalled earlier, they cannot be realized in laboratory and imply the loss of conventional relativities because of the loss of the inertial character of the original frame.

Remark 2. The direct *universality*, of the *conventional* symplectic geometry for the characterization of all possible local, analytic and regular Newtonian systems (universality) in the frame of the experimenter (direct universality), was proved in ref. [27] via the use of the general one-forms on the ordinary cotangent bundle $T^*M(E) = T^*M[E(x, \delta, R)]$ in the local chart

$$\Theta = R_\mu\,(b)\,db^\mu, \tag{126}$$

with corresponding general, exact, symplectic two-form

$$\Omega = \Omega\,(b)_{\mu\nu}\,db^\mu \wedge db^\nu, \tag{127}$$

where $\Omega_{\mu\nu}$ is the Birkhoffian tensor (91). A vector field $X(m)$ in the neighborhood of a point $m \in T^*M(E)$ which is not Hamiltonian in the given chart b results to be always Birkhoffian in the same chart, i.e., when a function H on $N(m)$ such that $X\rfloor\omega = -dH$ does not exist in the b-chart, there always exists a Birkhoffian tensor $\Omega_{\mu\nu}\,(b)$ such that $X\rfloor\Omega = -dH$. The maps within a fixed b-chart $\theta \to \Theta$ and $\omega \to \Omega$ were identified in ref. [25] as a first form of isotopies of the symplectic geometry in canonical realization.

Remark 3. Despite the achievement of the above direct universality, the symplectic geometry continues to be insufficient for recent applications owing to its local-differential character. This is due to the recent emergence in Newtonian mechanics, particle physics and other disciplines of nonlocal-integral systems, such as a space-ship during re-entry in our atmosphere with necessary terms involving integrals over the surface of the space-ship. In fact, two space-ships with the same weight and the same speed at the initiation of the re-entry in atmosphere but different shapes have different re-entry trajectories.

A second isotopy of the symplectic geometry for the characterization of the additional nonlocal, integral terms was submitted by this author [29] via the lifting of the unit and of the associative product while preserving the conventional differential calculus. For instance, the isocanonical one-form on $T^*M(\hat{E})$ in the above formulation is given by

$$\hat{\theta} = \hat{R}^\circ_\mu \left(\hat{b}\right) \hat{I}^\mu_{2\nu} d\hat{b}^\nu \tag{128}$$

and, as such, it coincides with one-isoform (114). In fact, the isotopic degrees of freedom of the product of the former are merely transferred to the isotopic degrees of freedom of the differentials in the latter. However, the two-isoforms of these approaches are different, as one can verify (see ref. [32], Sect. 5.4 for brevity).

The above second isotopy of the symplectic geometry preserves all conventional axioms, including the Poincaré Lemma, the Darboux's Theorem, etc. Also, the latter theorems hold in both isospaces as well as in their projection into the conventional spaces. In particular, the generalized brackets were Lie-isotopic in both isospace and in their projection in the conventional space.

The drawback of the above isotopy is that it implies the loss of the basic unit $\hat{I}_2$ in the transition from one- to two-isoforms evidently due to the use of the conventional calculus (see also ref.s [32], Sect. 5.4 for brevity).

In this section we have introduced the third isotopy of the symplectic geometry, this time based on the isodifferential calculus. Its main advantage over the preceding isotopies is its remarkable simplicity, as well as the preservation of the basic unit $\hat{I}_2 = diag.(\hat{I}, \hat{T})$ for isoforms of arbitrary order. *Another advantage is that the conventional coordinate-free treatment of the symplectic geometry can be preserved in its entirety for the characterization of the isosymplectic geometry submitted in this section and merely subjected to a more general realization of the symbols such as dx, dH etc.*

Remark 4. The isosymplectic geometry of this section is particularly suited for the isotopies of symplectic quantization first studied by Lin [16] and then treated in [33]. These isotopies are significant for the study of nonlocal-integral and nonhamiltonian interactions in particle physics, superconductivity and other fields.

Remark 5. The nonlinear, nonlocal and noncanonical character of the isotopies is evident from the preceding analysis. It is important to point out that linearity is reconstructed in isospace and called *isolinearity*, as shown in Eq. (113). Locality is equally reconstructed in isospace and called *isolocality*, because one- and two-isoforms are based on the local isodifferentials $\hat{d}\hat{x}$ and $\hat{d}\hat{p}$. Similarly, canonicity is reconstructed in isospace, and called *isocanonicity*, because the canonical form $p_k dx^k$ is preserved by the isotopic form $\hat{p}_k \hat{d}\hat{x}^k$ in isospace. The nonlinear, nonlocal and noncanonical character of isotopic theories solely emerge when they are projected in the original spaces.

Numerous other reconstruction of original properties in isospaces occur under isotopies. As an example, it is easy to see that isogroups (107) are characterized by *nonunitary* transforms in an ordinary Hilbert space $\widehat{\mathcal{H}}$, i.e., for $U = \exp\{i\hat{H}\hat{T}t\}$, $UU^\dagger \neq I$ owing to the noncommutativity of $\hat{H}$ and $\hat{T}$. However, these transforms do verify the axiom of unitarity when written in the isohilbert space $\widehat{\mathcal{H}}$, Eq.s (109). At any rate, any nonunitary transform $UU^\dagger = \hat{I} \neq I$ can always be rewritten in the isounitary form $\hat{U}\hat{U}^\dagger = \hat{I}$ via the factorization $U = \hat{U}\hat{T}^{1/2}$.

The latter point illustrates again the lack of equivalence between conventional and isotopic theories which are connected at the classical level by noncanonical transforms and at the operator level by nonunitary transforms (see [33] for details).

7 Isoriemannian geometry

The Riemannian geometry [23] is *exactly valid* for the *exterior gravitational problem* in vacuum, because an extended body moving in the homogeneous and isotropic vacuum (such as Jupiter in its planetary trajectory around the Sun) can be effectively approximated as a massive point, thus providing the physical foundations of the local-differential character of the geometry.

The Riemannian geometry is only *approximately valid* for *interior gravitational problems* (such as a space-ship during re-entry in our inhomogeneous and anisotropic atmosphere) because the shape of the body considered affects its trajectory and the local- differential treatment is no longer exact. Also, interior problems imply contributions which are nonlinear in the velocities and/or wavefunctions (in addition to the conventional nonlinearity in the coordinates) as well as nonlocal-integral. These latter effects are beyond the descriptive capacities of the Riemannian geometry in its current formulation (for a study of these limitation see Santilli [33], Ch. 9; an independent appraisal was also provides by E. Cartan [6]).

In the final analysis, astrophysical bodies undergoing gravitational collapse are not composed of ideal points (as necessary for the exact validity of the Riemannian geometry but instead of extended and hyperdense hadrons in conditions of total mutual penetration and compression in large numbers into small regions of space. The need under the latter conditions of a generalization of the Riemannian geometry which is arbitrarily nonlinear and nonlocal- integral is beyond scientific doubts.

Numerous deformations-generalizations of the Riemannian geometry have been studied during the last decades to represent more general conditions, but they generally imply the abandonment of the Einsteinian axioms in favor of yet un-identified axioms (because deformed Riemannian spaces are no longer isomorphic to the original space). This author submitted in 1988 [30] (see [32], Ch. 5 and [33], Ch. 9, for a comprehensive presentation) the isotopies of the Riemannian geometry, called *isoriemannian geometry*, to achieve the desired representation of arbitrary nonlinear and nonlocal effects while preserving the original Riemannian, and therefore Einsteinian axioms. The isogeometry was constructed via the isotopic lifting of the unit and of the product of the original geometry while preserving the conventional differential calculus. The emerging generalized geometry did result to be an isotopy of the original one, that is, preserving the original Riemannian axioms, while permitting the representation of nonlinear and nonlocal effects via their embedding in the generalized unit.

In this section we shall present, apparently for the first time, the isoriemannian geometry formulated via the isotopy of the differential calculus (rather than that of the product), and show that the latter formulation is more conducive to a single, unified, abstract formulation of the geometry with different realizations, the conventional local-differential one for the exterior problem in vacuum and the more general nonlocal-integral isotopic one for interior problems within physical media. Our study will be again in local realizations representing the fixed inertial reference frame of the observer while all abstract treatments are left to the interested reader. For the conventional case we shall assume all topological assumptions of Lovelock and Rund [18] of which we shall preserve the symbols for clarity in the comparison of the results. For the isotopic case we shall assume the topological assumption by Tsagas and Sourlas [35] which are also tacitly implied hereon. Our presentation will be made, specifically, for the $(3+1)$-dimensional space-time, the extension to arbitrary dimensions and signatures being elementary.

Let $\mathcal{R} = \mathcal{R}(x, g, R)$ be a $(3+1)$-dimensional Riemannian space over the reals $R(n, +, \times)$ [18] with local coordinates $x = \{x^\mu\} = \{r, x^4\}$, $x^4 = c_0 t$, $\mu = l, 2, 3, 4$, where c_0 is the speed of light in vacuum, nowhere singular, symmetric and real-valued metric $g(x) = (g_{\mu\nu})$ with tangent Minkowski space $M(x, \eta, R)$ with metric $\eta = diag.(1, 1, 1, -1)$. The interval the familiar expression $x^2 = x^\mu g_{\mu\nu}(x) x^\nu \in R$ with infinitesimal line element $ds^2 = dx^\mu g_{\mu\nu}(x) dx^\nu$ and related formalism (covariant derivative, Christoffel's symbols, etc. [18]).

Let $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}(\hat{x}, \hat{g}, \hat{R})$ be an isotopic image of $\mathcal{R}$, called *isoriemannian space*, first introduced by this author in ref. [28] of 1983, with local coordinates $\hat{x} = \{\hat{x}^\mu\} (= \{x^\mu\})$ and *isometric* $\hat{g} = \hat{T} g$, where $\hat{T} = (\hat{T}_\mu^\nu)$ is a nowhere singular, symmetric, real valued and positive-definite 4×4 matrix with C^∞ elements. The isospace $\widehat{\mathcal{R}}$ is defined over the isoreals $\hat{R} = \hat{R}(\hat{n}, +, \hat{\times})$ with isounit $\hat{I} = (\hat{I}_\nu^\mu) = \hat{T}^{-1}$. The lifting $\mathcal{R} \to \widehat{\mathcal{R}}$ leaves unrestricted the functional dependence of the isounit/isotopic element, which can therefore depend in an arbitrarity nonlinear and nonlocal-integral way on the coordinates $\hat{x}$, velocities $\hat{v} = d\hat{x}/dt$, accelerations $\hat{a} = d\hat{v}/dt$ and any needed additional quantity of the interior medium, such as density μ, temperature τ, etc. By recalling that the original unit of $\mathcal{R}$ is $I = diag.(1, 1, 1, 1)$, the lifting $\mathcal{R} \to \widehat{\mathcal{R}}$ is characterized by

$$I = diag.(1, 1, 1, 1) \to \hat{I}\,(\hat{x}, \hat{v}, \hat{a}, \mu, \tau, \ldots) = T^{-1}, \; g(x) \to \hat{g}\,(\hat{x}, \hat{v}, \hat{a}, \mu, \tau, \ldots) = \hat{T} g. \tag{129}$$

We then have the *isoline element*

$$\hat{x}^2 = [\hat{x}^\mu \hat{g}_{\mu\nu}\,(\hat{x}, \hat{v}, \hat{a}, \mu, \tau, \ldots)\,\hat{x}^\nu]\,\hat{I} \in \hat{R}, \tag{130}$$

with infinitesimal version

$$\hat{d}\hat{s}^2 = \left(\hat{d}\hat{x}^\alpha \hat{g}_{\alpha\beta} \hat{d}\hat{x}^\beta\right) \hat{I} \in \hat{R}. \tag{131}$$

The capability of representing arbitrarily nonlinear and nonlocal effects of the interior problem is therefore embedded *ab initio* in the isoriemannian geometry.

The isonormal coordinates $\hat{y}$ occur when the isometric $\hat{g}$ is reduced, not to the Minkowski metric η, but rather to its isotopic image, i.e., $\hat{g}_{\hat{x}} \to \hat{\eta}_{\hat{y}} = \hat{T}_{\hat{y}} \eta_{\hat{y}}$ and, as such, they are the conventional normal coordinates. In different terms, the correct tangent space is not the conventional space $M(x, \eta, R)$, but the isominkowskian space $\hat{M}(\hat{x}, \hat{\eta}, \hat{R})$ first submitted in ref. [25]. In particular, *the isounit and related isotopic element are the same for both the isoriemannian spaces and its tangent isominkowskian spaces.* Under these conditions, the isonormal coordinates only reduce the g-component in $\hat{g} = \hat{T} g$ to the η-component of $\hat{\eta} = \hat{T}\eta$. As a result, *isonormal coordinates coincide with the conventional normal coordinates.*

It is easy to see that, despite the arbitrary functional dependence of the isometric $\hat{g}$, *all infinitely possible isotopic images* $\widehat{\mathcal{R}}(\hat{x}, \hat{g}, \hat{R})$ *of a Riemannian space* $\mathcal{R}(x, g, R)$ *are locally isomorphic to the latter*, i.e., for each given metric g, $\widehat{\mathcal{R}} \approx \mathcal{R}$ for all infinitely possible $\hat{I}$ of Kadeisvili's Class I [13]. This is first due to the preservation by $\hat{I}$ of the axioms of I, as a result of which the field R and its isotopic image $\hat{R}$ lose any distinction at the abstract level [31]. Second, the local isomorphism $\widehat{\mathcal{R}} \approx \mathcal{R}$ follows from the fact that, in conjunction with the deformation of the metric elements $g_{\mu\nu} \to \hat{g}_{\mu\nu} = \hat{T}_{\mu}^{\alpha} g_{\alpha\nu}$, the corresponding unit has been deformed by the *inverse* amount, $I_{\alpha}^{\mu} \to \hat{I}_{\alpha}^{\mu} = (\hat{T}_{\mu}^{\alpha})^{-1}$, thus preserving the original geometric characteristics. In particular, the isospace $\mathcal{R}$ is *isocurved*, that is (unlike the case for the isoeuclidean spaces), curvature exists in the original space and persists under isotopy.

To have an idea of the various applications under study with isoriemannian spaces, the diagonal isotopic element

$$\hat{T} = diag.\left(n_1^{-2}, n_2^{-2}, n_3^{-2}, n_4^{-2}\right), \ n_\mu > 0, \ m = 1, 2, 3, 4, \tag{132}$$

permits the representation of the locally varying speed $c = c_0/n_4$ of electromagnetic waves within physical media, which occurs via the forth component of the isoline element

$$\hat{x}^4 \hat{g}_{44} \hat{x}^4 = tc\,(\hat{x}, \mu, \tau, \ldots)\, g_{44}\,(x)\, t, \ c = c_0/n_4(\hat{x}, m, \tau, \ldots) \tag{133}$$

where g_{44} is the ordinary metric element and n_4 is the familiar index of refraction. This permits a gravitational treatment of the locally varying speed of light in interior conditions. As an example, light propagating in our atmosphere has a dependence on the density, and then assumes yet different values when propagating in water, glasses, etc. It is evident that the representation of the locally varying speed of light is not possible with the Riemannian geometry or with its tangent Minkowskian geometry. Also, the decrease of the speed of light within inhomogeneous and anisotropic media has novel effects, such as a shift of light frequency toward the red which cannot be predicted via the Riemannian or Minkowskian geometries, but which is quantitatively treatable in accordance with available experimental data via the isogeometries [33].

The representation technically occurs via the *isolight cone* $\hat{d}\hat{s}^{\hat{2}} = \hat{d}\hat{x}^{\mu} \hat{g}_{\mu\nu} \hat{d}\hat{x}^{\nu} = 0$ [33] which is the image in isospace of the *deformed* light cone in our space-time, as generated by a locally varying speed of light. In a way similar to the fact that the isosphere is a perfect sphere in isospace (Sect. 1), *the isolight cone is a perfect cone in isospace* (see ref. [33], Ch. 8, for details). This occurrence is not a mere mathematical curiosity because it is important for numerical examples, such as the correct calculations of the gravitational horizon in the hyperdense astrophysical chromospheres where it is well known that the speed of light is locally varying with the temperature, density, etc. Note that the conventional exterior motion in vacuum is a particular case of the isoriemannian geometry occurring for $\hat{I} = I$.

In the first formulation of the isoriemannian geometry [30], differentials of contravariant isofields $\hat{X}^{\beta}$ on $\widehat{\mathcal{R}}$ where defined by $d\hat{X} = (\partial\hat{X}) * d\hat{x} = (\partial_{\mu}\hat{X})\hat{T}_{\nu}^{\mu} d\hat{x}^{\nu} \neq dX = (\partial_{\mu}X)dx^{\mu}$, $\partial_{\mu} = \partial/\partial x^{\mu}$. The isodifferential calculus allows us to introduce the following alternative definition

$$\hat{d}\,\hat{X}^\beta = \left(\hat{\partial}_\mu \hat{X}^\beta\right) \hat{d}\hat{x}^\mu = \hat{T}^\rho_\mu \left(\partial_\rho \hat{X}^\beta\right)^\mu_\sigma d\hat{x}^\sigma \equiv \left(\partial_\mu \hat{X}^\beta d\hat{x}^\mu\right), \tag{134}$$

namely, *isodifferential of isovector fields coincide with ordinary differentials.*

The *isocovariant differential* can be defined by

$$\hat{D}\,\hat{X}^\beta = \hat{\partial}\,\hat{X} + \hat{\Gamma}^\beta_{\alpha\gamma} \hat{X}^\alpha\, \hat{d}\hat{x}^\gamma, \tag{135}$$

with corresponding *isocovariant derivative*

$$\hat{X}^\beta_{\hat{|}\mu} = \hat{\partial}_\mu \hat{X}^\beta + \hat{\Gamma}^\beta_{\alpha\mu} \hat{X}^\alpha, \tag{136}$$

where the *isochristoffel's symbols* are given by

$$\hat{\Gamma}_{\alpha\beta\gamma} = \frac{1}{2}\left(\hat{\partial}_\alpha \hat{g}_{\beta\gamma} + \hat{\partial}_\gamma \hat{g}_{\alpha\beta} - \hat{\partial}_\beta \hat{g}_{\alpha\gamma}\right) = \hat{\Gamma}_{\gamma\beta\alpha}\ , \tag{137}$$

$$\hat{\Gamma}^\beta_{\alpha\gamma} = \hat{g}^{\beta\rho} \hat{\Gamma}_{\alpha\rho\gamma} = \hat{\Gamma}^\beta_{\gamma\alpha}\ , \quad \hat{g}^{\beta\rho} = \left[(\hat{g}_{\mu\nu})^{-1}\right]^{\beta\rho}, \tag{138}$$

and one should note the abstract identity of the conventional and isotopic connections. The extension to covariant isofields and covariant or contravariant tensor isofields is consequential and it is hereon assumed (see also[36]).

The repetition of the proof of [18] pag. 80-81, yields to the following:

Lemma 3 (Isoricci Lemma) *Under the assumed conditions, the isocovariant derivatives of all isometrics on isoriemannian spaces are identically null,*

$$\hat{g}_{\alpha\beta\hat{|}\gamma} \equiv 0, \quad \alpha, \beta, \gamma = 1, 2, 3, 4. \tag{139}$$

Despite the similarities with the conventional case, the lack of equivalence of the Riemannian and isoriemannian geometries can be illustrated via the *isotorsion*

$$\hat{\tau}^\beta_{\alpha\gamma} = \hat{\Gamma}^\beta_{\alpha\gamma} - \hat{\Gamma}^\beta_{\gamma\alpha}\ , \tag{140}$$

which is identically null for the isoriemannian geometry here considered, but its projection in the original space $\mathcal{R}$ is not necessarily null. Interior gravitational models treated with the isoriemannian geometry are therefore theories with null isotorsion but generally non-null torsion as requested for a realistic treatment of interior problems.

The occurrence also illustrates the property, verified at subsequent levels later on, that departures from conventional geometric properties must be studied in the *projection* of isoriemannian spaces in the original spaces because, when treated in their respective spaces, the two geometries coincide. Stated in different terms, when using the conventional Riemannian geometry, exterior gravitation can only be studied in the spaces $\mathcal{R}$. On the contrary, when using the isogeometry, interior gravitation can be studies in *two* different spaces, the isoriemannian spaces $\widehat{\mathcal{R}}$ and their projection into $\mathcal{R}$.

Another way of identifying the differences between the Riemannian and isoriemannian geometries is by considering the following *isotopic Newton equations* in isoriemannian space

$$\frac{\hat{D}\hat{x}_\beta}{\hat{D}\hat{\tau}} = \frac{\hat{d}v_\beta}{\hat{d}\hat{\tau}} + \hat{\Gamma}_{\alpha\beta\gamma}(\hat{x}, \hat{v}, \hat{a}, \ldots) \frac{\hat{d}\hat{x}^\alpha}{\hat{d}\hat{\tau}} \frac{\hat{d}\hat{x}^\gamma}{\hat{d}\hat{\tau}} = 0, \tag{141}$$

where $\hat{v} = \hat{d}\hat{x}/\hat{d}\tau = \hat{I}^0_0 dx/d\tau$, $\hat{\tau}$ is the proper isotime and $\hat{I}^0_0$ the related isounit. The preceding equations must then be compared with the conventional equations

$$\frac{Dx_\beta}{Ds} = \frac{dv_\beta}{ds} + \hat{\Gamma}_{\alpha\beta\gamma}(x) \frac{dx^\alpha}{ds} \frac{dx^\gamma}{ds} = 0. \tag{142}$$

It is evident that the latter equations are at most quadratic in the velocities while the isotopic equations are arbitrarily nonlinear in the velocities, as it occurs already in a flat space (Sect. 3). Also, the latter equations are local-differential while the former admit nonlocal-integral term.

We now introduce: the *isocurvature tensor*

$$\hat{R}^\beta_{\alpha\gamma\delta} = \hat{\partial}_\delta \hat{\Gamma}^\beta_{\alpha\gamma} - \hat{\partial}_\gamma \hat{\Gamma}^\beta_{\alpha\delta} + \hat{\Gamma}^\beta_{\rho\delta} \hat{\Gamma}^\rho_{\alpha\gamma} - \hat{\Gamma}^\beta_{\rho\gamma} \hat{\Gamma}^\rho_{\alpha\delta} \ ; \tag{143}$$

the *isoricci tensor*

$$\hat{R}_{\mu\nu} = \hat{R}^\beta_{\mu\nu\beta} \ ; \tag{144}$$

the *isocurvature isoscalar*

$$\hat{R} = \hat{g}^{\alpha\beta} \hat{R}_{\alpha\beta} = \hat{R}^\mu_\mu \ ; \tag{145}$$

the *isoeinstein tensor*

$$\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2}\hat{R} \tag{146}$$

and the *isotopic isoscalar*

$$\hat{\Theta} = \hat{g}^{\alpha\beta} \hat{g}^{\gamma\delta} \left(\hat{\Gamma}_{\rho\alpha\delta} \hat{\Gamma}^\rho_{\gamma\beta} - \Gamma_{\rho\alpha\beta} \hat{\Gamma}^\rho_{\gamma\delta} \right) = \hat{\Gamma}_{\rho\alpha\beta} \hat{\Gamma}^\rho_{\gamma\delta} \left(\hat{g}^{\alpha\delta} \hat{g}^{\gamma\beta} - \hat{g}^{\alpha\beta} \hat{g}^{\gamma\delta} \right), \tag{147}$$

the latter one being new for the isoriemannian geometry (see below).

Tedious but simple calculations then yield the following basic properties of the isoriemannian geometry:

Property l: *Antisymmetry of the last two indices of the isocurvature tensor*

$$\hat{R}^\beta_{\alpha\gamma\delta} = -\hat{R}^\beta_{\alpha\delta\gamma} \ ; \tag{148}$$

Property 2: *Symmetry of the first two indices of the isocurvature tensor*

$$\hat{R}_{\alpha\beta\gamma\delta} = \hat{R}_{\beta\alpha\gamma\delta} \ ; \tag{149}$$

Property 3: *Vanishing of the totally antisymmetric part of the isocurvature tensor*

$$\hat{R}^\beta_{\alpha\gamma\delta} + \hat{R}^\beta_{\gamma\delta\alpha} + \hat{R}^\beta_{\delta\alpha\gamma} \equiv 0 \ ; \tag{150}$$

Property 4: *Isobianchi identity*

$$\hat{R}^{\beta}_{\alpha\gamma\delta\hat{|}\rho} + \hat{R}^{\beta}_{\alpha\rho\gamma\hat{|}\delta} + \hat{R}^{\beta}_{\alpha\delta\rho\hat{|}\gamma} \equiv 0 \; ; \tag{151}$$

Property 5: *Isofreud identity (see Freud [9] for the original form, Pauli [22] for a subsequent treatment, Rund [24] for a more recent presentation and Santilli [32], Ch. 5, for a general review)*

$$\hat{S}^{\alpha}_{\beta} = \hat{R}^{\alpha}_{\beta} = -\frac{1}{2}\delta^{\alpha}_{\beta}\hat{R} - \frac{1}{2}\delta^{\alpha}_{\beta}\hat{\Theta} = \hat{U}^{\alpha}_{\beta} + \hat{\partial}_{\rho}\hat{V}^{\alpha\rho}_{\beta} \; , \tag{152}$$

where $\hat{\Theta}$ *is the isotopic isoscalar (147) and*

$$\hat{U}^{\alpha}_{\beta} = -\frac{1}{2}\frac{\hat{\partial}\hat{\Theta}}{\hat{\partial}\hat{g}^{\alpha\beta}}_{\hat{|}\beta} \; , \tag{153}$$

$$\hat{V}^{\alpha\rho}_{\beta} = \frac{1}{2}[\hat{g}^{\gamma\delta}(\delta^{\alpha}_{\beta}\hat{\Gamma}^{\rho}_{\alpha\delta} - \delta^{\rho}_{\beta}\hat{\Gamma}^{\rho}_{\gamma\delta}) +$$
$$(\delta^{\rho}_{\beta}\hat{g}^{\alpha\gamma} - \delta^{\alpha}_{\beta}\hat{g}^{\rho\gamma})\hat{\Gamma}^{\delta}_{\gamma\delta} + \hat{g}^{\rho\gamma}\hat{\Gamma}^{\alpha}_{\beta\gamma} - \hat{g}^{\alpha\gamma}\hat{\Gamma}^{\alpha}_{\beta\gamma}] \; . \tag{154}$$

Note *the abstract identity of the conventional and isotopic properties.* This confirms that the conventional and isotopic geometries can be treated at the realization-free level via one single set of axioms, as desired.

The repetition of the proof of the Theorem of [18], p. 32l, leads to the following property first identified in 1988 [30] (see also [33]) and which is here recovered via the isodifferential calculus.

Theorem 9 (Fundamental Theorem for Interior Gravitation) *Under the assumed regularity and continuity conditions, the most general possible isolagrange equations* $\hat{E}^{\alpha\beta} = 0$ *along an actual path* $\hat{P}_0$ *on a* $(3+1)$*-dimensional isoriemannian space satisfying the properties:*

1. *Symmetry condition*

$$\hat{E}^{\alpha\beta} = \hat{E}^{\beta\alpha}, \tag{155}$$

2. *Contracted isobianchi identity*

$$\hat{E}^{\alpha\beta}{}_{\hat{|}\beta} \equiv 0 \tag{156}$$

3. *The isofreud identity*

$$\hat{S}^{\alpha}_{\beta} = \hat{R}^{\alpha}_{\beta} - \frac{1}{2}\delta^{\beta}_{\alpha}\hat{R} - \frac{1}{2}\delta^{\alpha}_{\beta}\hat{\Theta} = \hat{U}^{\alpha}_{\beta} + \hat{\partial}_{\rho}\hat{V}^{\alpha\rho}_{\beta} \; , \tag{157}$$

are given by

$$\hat{E}^{\alpha\beta} = \alpha\,\hat{g}^{\frac{1}{2}}\left(\hat{R}^{\alpha\beta} - \frac{1}{2}\hat{g}^{\alpha\beta}\hat{R} - \frac{1}{2}\hat{g}^{\alpha\beta}\hat{\Theta}\right) + \beta\hat{g}^{\alpha\beta} - \hat{g}^{\frac{1}{2}}\hat{D}^{\alpha\beta} = 0, \tag{158}$$

where $\hat{g}^{\frac{1}{2}} = (\det \hat{g})^{1/2}$, α *and* β *are constants and* $\hat{D}^{\alpha\beta}$ *is a source tensor. For* $\alpha = 1$ *and* $\beta = 0$ *the interior isogravitation field equations can be written*

$$\hat{S}^{\alpha\beta} = \hat{R}^{\alpha\beta} - \frac{1}{2}\hat{g}^{\alpha\beta} - \frac{1}{2}\hat{g}^{\alpha\beta}\hat{\Theta} = \hat{t}^{\alpha\beta} - \hat{\tau}^{\alpha\beta} = \hat{U}^{\alpha}_{\beta} + \hat{\partial}_{\rho}\hat{V}^{\alpha\rho}_{\beta} \ , \tag{159}$$

where $\hat{t}^{\alpha\beta}$ *is a source tensor and* $\hat{\tau}^{\alpha\beta}$ *is a stress-energy tensor.*

Note the appearance in Eq.s (159) of the isotopic isoscalar $\hat{\Theta}$ in the l.h.s and of source terms in the r.h.s., the latter ones originating from the isofreud identity. Additional studies not reported here for brevity (see [33], Ch. 9) have shown that the tensors $\hat{t}^{\alpha\beta}$ is nowhere null and of first order in magnitude. This implies that, at the isonormal coordinates the isometric $\hat{g}$ is indeed reduced to the tangent isominkowski metric $\hat{\eta} = T\eta$, but the source $\hat{t}^{\alpha\beta}$ cannot be rendered null (this occurrence is called *isoequivalence principle*).

A vector isofield $\hat{X}^{\beta}$ on $\widehat{\mathcal{R}}$ is said to be transported by *isoparallel displacement* from a point $\hat{m}(\hat{x})$ on a curve $\hat{C}$ on $\widehat{\mathcal{R}}$ to a neighboring point $\hat{m}'(\hat{x} + \hat{d}\hat{x})$ on $\hat{C}$ if

$$\hat{D}\,\hat{X}^{\beta} = \hat{d}\,\hat{X}^{\beta} + \hat{\Gamma}^{\beta}_{\alpha\gamma}\hat{X}^{\alpha}\hat{d}\hat{x}^{\gamma} \equiv 0, \tag{160}$$

or in integrated form

$$\hat{X}^{\beta}\,(\hat{m}') - \hat{X}^{\beta}\,(m) = \widehat{\int}_{\hat{m}}^{\hat{m}'} \frac{\hat{\partial}\hat{X}^{\beta}}{\hat{\partial}\hat{x}^{\alpha}}\frac{\hat{d}\hat{x}^{\alpha}}{\hat{d}s}\hat{d}s \ . \tag{161}$$

The isotopy of the conventional case [18] then yield the following:

Lemma 4 *Necessary and sufficient conditions for the existence of an isoparallel transport along a curve on a* $(3+1)$*-dimensional isoriemannian space are that all the following conditions are identically verified along* $\hat{C}$

$$\hat{R}^{\beta}_{\alpha\gamma\delta}\hat{X}^{\alpha} = 0, \quad \beta, \gamma, \delta = 1, 2, 3, 4. \tag{162}$$

Note, again, the abstract identity of the conventional and isotopic parallel transport. Along similar lines, we say that a smooth path $\hat{x}_{\alpha}$ on $\widehat{\mathcal{R}}$ with isotangent $\hat{v}_{\alpha} = \hat{d}\hat{x}_{\alpha}/\hat{d}\hat{s}$ is an *isogeodesic* when it is solution of the isodifferential equations

$$\frac{\hat{D}\hat{x}_{\beta}}{\hat{D}\hat{s}} = \frac{\hat{d}\hat{v}_{\beta}}{\hat{d}\hat{s}} + \hat{\Gamma}_{\alpha\beta\gamma}\frac{\hat{d}\hat{x}^{\alpha}}{\hat{d}\hat{s}}\frac{\hat{d}\hat{x}^{\gamma}}{\hat{d}\hat{s}} = 0. \tag{163}$$

It is easy to prove the following:

Lemma 5 *The isogeodesics of an isoriemannian space* $\mathcal{R}$ *are the curves verifying the isovariational principle*

$$\hat{\delta}\widehat{\int}\left[\hat{g}_{\alpha\beta}\,(\hat{x}, \hat{v}, \hat{a}, \mu, \tau, \ldots)\,\hat{d}\hat{x}^{\alpha}\hat{d}\hat{x}^{\beta}\right]^{1/2} = 0. \tag{164}$$

Finally, we point out the property which is inherent in the notion of isotopies as realized in this paper:

Lemma 6 *Geodesic trajectories in ordinary space remain isogeodesics in isospace.*

For instance, if a circle is originally a geodesic, its image under isotopy in isospace remains the perfect circle, the isocircle of Sect. 1, and the same happens for other curves. As it is the case for all other aspects, the differences between a geodesic and an isogeodesic emerge when projecting the latter in the space of the former. As recalled in Sect. 3, the projection of the isocircle in the conventional space becomes an ellipse under the assume topology (and can be a hyperbola when relaxing the positive-definite character of 1) [32].

We can say in figurative terms that interior physical media "disappear" under their isoriemannian geometrization, in the sense that actual trajectories under resistive forces due to physical media (which are not geodesics of a Riemannian space) are turned into isogeodesics in isospace with the shape of the geodesics in the absence of resistive forces. This property is inherent in the very conception of the isotopic Newton equations, e.g., in representation (42), and it is only re-expressed in this section in an isocurved space.

Remark l: A question raised in this section is: why use in interior problems the Riemannian geometry with metric $g(x)$ when the same axioms permit metrics $\hat{g}(\hat{x}, \hat{v}, \hat{a}, \ldots)$ with a more general functional dependence in the velocities and other variables as needed for interior conditions ? In fact, at the abstract level we have the identities $I \equiv \hat{I}$, $dx \equiv \hat{d}\hat{x}$, $R(n, +, \times) \equiv \hat{R}(\hat{n}, +, \hat{\times})$ and $\mathcal{R}(x, g, R) \equiv \widehat{\mathcal{R}}(\hat{x}, \hat{g}, \hat{R})$ with consequential *unique* abstract geometric axioms for both spaces $\mathcal{R}$ and $\widehat{\mathcal{R}}$. Within such a setting, $\mathcal{R}$ emerges as a simpler *realization* of the Riemannian axioms, and $\widehat{\mathcal{R}}$ as a more general realization.

Besides the representation of internal nonlinear, nonlocal and noncanonical effects, the isotopic treatment of gravitation permit novel advances, such as in gravitational collapse. In fact, under the decomposition $\hat{g} = \hat{T}\eta$, where η is the Minkowski metric, gravitational horizons are the zeros of the isotopic element $\hat{T}(\hat{x}, \hat{v}, \ldots = 0$, while gravitational singularities are the zeros of the isounit $\hat{I}(\hat{x}, \hat{v}, \ldots) = 0$. This illustrates the significance of the singular isotopies of Kadeisvili's Class IV [13].

Remark 2: Once the basic unit $I = diag.(1, 1, 1)$ of the Riemannian geometry has been lifted into an arbitrary 4×4 matrix $\hat{I}$, numerous possibilities emerge which are precluded by theories based on I. In this paper we have studied the simplest possible class of liftings $I \rightarrow \hat{I}$, the axiom-preserving isotopies of Kadeisvili's Class I. The isotopies of Kadeisvili's Class II are characterized by the *isodual map*

$$\hat{I} \rightarrow \hat{I}^d = -\hat{I} < 0, \tag{165}$$

which characterizes *isodual isofields* $\hat{R}^d(\hat{n}^d, +, \hat{\times}^d)$, with isodual isonumbers $\hat{n}^d = -\hat{n}$, *isodual isoproduct* $\hat{n}^d \hat{\times}^d \hat{m}^d = \hat{n}^{d_\alpha} T^{d_\alpha} \hat{m}^d = -\hat{n} \hat{\times} \hat{m}$, *isodual norm* $\hat{|}\hat{n}^d\hat{|} = | n | \hat{I}^d < 0$, *isodual isoriemannian spaces* $\widehat{\mathcal{R}}^d(\hat{x}, \hat{g}^d, \hat{R}^d)$ with *isodual metric* $\hat{g}^d = \hat{T}^d g = -\hat{g}$, *isodual isoseparation* $\hat{x}^{\hat{2}d} = (\hat{x}^\mu \hat{g}^d_{\mu\nu} \hat{x}^\nu)\hat{I}^d \equiv (\hat{x}^\mu \hat{g}_{\mu\nu} \hat{x}^\nu)\hat{I} = \hat{x}^{\hat{2}}$, *isodual isochristoffel symbols*, *isodual isocurvature*, etc.

The emerging *isodual isoriemannian geometry* has been studied in detail in [32, 33] because, when formulated at the operator level, *the isodual map is equivalent to charge conjugation.* As a result, the isoriemannian geometry can be used for the study of interior gravitational problems of matter, while the isodual isoriemannian geometry provides a novel representation of interior gravitational problems of antimatter.

The emerging predictions are intriguing. For instance, the isoriemannian geometry preserves the *attractive* character of gravitation for matter within the field of matter. Similarly, the isodual isoriemannian geometry preserve the *attractive* character of antimatter within the field of antimatter (this is due to the fact that curvature, which is now negative-definite, is referred to the underlying isofield with a negative-definite norm, thus resulting in attraction). However, the projection of the isodual isogeometry for antimatter in the space of matter results in *repulsion*, i.e., in the prediction that an antiparticle (such as a positron or an antineutron) experiences a repulsion when in the gravitational field of Earth [33].

The origin of this particular model of *antigravity* is significant per se. It is due to the fact that the isoriemannian treatment of interior gravitation implies the transition from the traditional description of the field in vacuum, to a theory on the *origin* of the gravitational field. Within such a setting, the old problem of the *unification* of the gravitational and electromagnetic fields is turned into their *identification.* As it is well known, the primary origin of the mass of elementary particles is of electromagnetic type (plus short range weak and strong corrections). This implies the presence of an electromagnetic field in the exterior of particles or of astrophysical masses which is of electromagnetic origin and of first-order in magnitude *even for bodies with null total charge* (otherwise the established electromagnetic origin of the mass of neutral particles such as the π° would be violated). In particular,. the $\hat{t}^{\mu\nu}$ source tensor in Eq.s (159) is precisely of electromagnetic origin, that is, it represents the established electromagnetic origin of mass. Antigravity is then a mere consequence. In fact, the identification of the gravitational and electromagnetic fields implies the equivalence in their behavior, i.e., the reversal of the sign of the force for particle-particle or antiparticle-antiparticle when passing to particle-antiparticle (see ref. [33] Ch. 8 for a relativistic treatment and Ch. 9 for the gravitational counterpart). Note the role plaid by the isodual isoriemannian geometry for the above prediction of antigravity.

Remark 3: The isotopic theories presented in this paper and their isodual images indicated above are only the beginning of a hierarchy of structural generalizations of contemporary mathematics which are permitted by the generalization of the unit. The next level of generalization is that in which the generalized unit is no longer symmetric. These latter theories were first submitted by the author [25] in ref. [25] under the name of *genotopies*, and they preserve the original axioms under a certain ordering and differentiation of the isomultiplication to the right and that to the left [31]. This implies the construction of *genofields, genospaces, genoalgebras, genogeometries, genomechanics,* etc., all admitting their isotopic counterpart as a particular case. As an illustration of the possibilities of the genotopies, note that the isotopies of the Riemannian geometry studied in this section preserve the symmetric character of the metric. By comparison, the genotopies of the Riemannian geometry, called *genoriemannian geornetry* [32] are capable of preserving the Riemannian axioms at the abstract level, yet they permit the relaxation of the symmetric character of the metric. The isoriemannian geometry results to per particularly suited for the treatment of interior gravitational problems with nonlinear, nonlocal and nonhamiltonian internal effects and *time-reversal invariant trajectories of their center of mass*, as it is the case, say, for the structure of Jupiter. The broader genoriemannian geometry results instead to be particularly suited for the study of open *irreversible* interior gravitational conditions, such as for the vortices in the Jovian atmosphere with continuously varying angular momenta while considering the rest of the system as external (see [33] for details). The genotopic extension of the isodifferential calculus and its applications to mechanics and geometries are contemplated for study in a next paper.

The genotopies themselves are particular cases of a more general level of mathematical formulations based on *multivalued hyperstructures* [37] with consequential multivalued *hyperfields, hyperspaces, hypermechanic, hypergeometries,* etc., all admitting the corresponding genotopic and isotopic structures as particular cases. In fact, the generalized unit of the isotopies and genotopies is *unique* per each considered system. Multivalued hypergeneralization occur when the generalized unit is constituted by a (finite or infinite and ordered or non-ordered) set of elements, and result to be particularly promising for the study of systems more complex than those appearing in physics, such as those in theoretical biology. The further multivalued extension of the genodifferential calculus and its expected applications are also contemplated for study at a future time.

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