LAGRANGE OSCILLANT SPACES

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Abstract

This paper studies the possibility to obtain in the convex interior of a given closed curve a Lagrange structure. The most important result is that this special Lagrange structure is given only by the elementary geometric properties of the given closed curve. An interesting class of metric spaces is highlighted so that the distance between two close points has the same Lagrangean form as those described by the two "special" tangent circles. As a particular result the Cayley hyperbolic distance between two closed points of the interior of a given circle from the Euclidean plane leads to a generalized Lagrange metric, which is not reductible to a Lagrange or a Finsler one. In this way the Poincaré model of the hyperbolic geometry of Lobacevski becomes an example of generalized Lagrange space.

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We shall consider in the two-dimensional Euclidean plane known the concepts: curve, closed curve, convex set, tangent in a point to a curve. Let P_0 be a fixed point belonging to a given curve $c : I \subset \mathbf{R} \mapsto \mathbf{R}$ and let P be a variable point on c in the neighborhood $U(P_0) \cap c$ of P_0.

Definition 1 The curve c is called *dual derivable* if the limit of the intersection of the tangents in P_0 and P, when P is moving on curve to P_0, is the fixed point P_0.

We can observe that the dual derivability excludes the existence of rectilinear components of the curve. Obviously, a simple closed curve having its interior as convex set is not dual derivable.

Definition 2 We shall call *parallel derivable* curve any simple closed curve belonging to the two- dimensional Euclidean plane which satisfies the conditions:

i) for any direction it allows just two tangents parallel with a given direction,

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ii) the tangents described above do not intersect again the interior of the curve.

The parallel derivability does not imply necessary a dual derivability for a curve, such that the following definition makes sense:

Definition 3 We shall call that the curve K is *i-derivable* if K is a simple closed curve of the two- dimensional Euclidean plane which proceeds from a parallel and dual derivable curve K^* by geometrical inversion of an arbitrary power with respect to an arbitrary point as pole, pole which is contained in the interior of K^*.

Lemma 1 *A parallel and dual derivable curve K^* has as interior a convex set.*

Proof. If not, there exist $M, N \in K^*$ such that the segment line $M\,N$ intersects K^*. That means K^* has points in the both sides of $M\,N$. In each side there exists, using Lagrange's Theorem, tangent lines parallel with $M\,N$. It is obvious that one of this tangent will intersect the interior of K^*, in collision with the parallel derivability of K^*. □

Lemma 2 *An i-derivable curve is dual and parallel derivable and its interior is a convex set.*

Proof. Consider a point A contained in the interior of the given i-derivable curve denoted by K. Taking into account that the inverse K^* is dual and parallel derivable, the geometric inversion $I\,(A, \mu)$ of arbitrary power will conserve both the angles between curves and the tangency, that means that K will be a dual and parallel derivable curve. The convex interior of K^* in Lemma 1 will be transformed into a convex set bounded by the initial curve K, with respect to $I\,(A, \mu)$. □

Lemma 3 *In any point A situated in its interior, an i-derivable curve K permits a pair of circles both mutually tangent in A and being each one also tangent in a unique point at K. The common tangent line in A of the two circles may have any direction.*

Proof. The i-derivable curve K proceeds from the inversion of K^* with respect to A, an interior point of K^*. The parallel lines having a given direction Δ are transformed in tangent circles passing by A with the tangent line in A parallel with Δ. Taking into account Lemma 2, the circles tangent in A will intersect each one K in only one point. □

Denote by s, S the tangent points at K of the circles described by Lemma 3 and by r, R the length of the radii of the same circles. We can observe that r, R depend on the point A and by the direction Δ.

Lemma 4 *An i-derivable curve allows a Lagrangean structure in its interior.*

Proof. We will consider the arclength element described using $x_1,\ x_2$ as usual coordinates in the plane of the curve:

$$ds = M\,(x_1, x_2, \dot{x}_1, \dot{x}_2)\,\sqrt{dx_1^2 + dx_2^2},$$

where we denote by $M(x_1, x_2, \dot{x}_1, \dot{x}_2)$ the expression $\dfrac{1}{2}\left(\dfrac{1}{r}+\dfrac{1}{R}\right)$ according to the special circle determined by A and $\Delta = \dfrac{\dot{x}_2}{\dot{x}_1}$. $\square$

We shall show that a particular distance that we shall introduce in the interior of an i- derivable curve leads to the same Lagrangean metric as the one introduced by the simple circles of i-derivable curves.

Consider A and B as fixed points in the interior of the i-derivable curve denoted by K and P an arbitrary point on K. The Euclidean distances $|PA|$, $|PB|$ determine a function $f(P) := \dfrac{|PA|}{|PB|}$, $f : K \mapsto \mathbf{R}^*$, which has a maximum M_{AB} and a minimum m_{AB} when P is moving on K. Then we can prove the following:

Theorem 1 *$d(A,B) := \ln M_{AB} \cdot m_{AB}^{-1}$ is a distance between A and B.*

Proof. If $A = B$ then $f(P) = \dfrac{|PA|}{|PB|} = 1$ for any $P \in K$ and that means $\ln \dfrac{M_{AB}}{m_{AB}} = \ln 1 = 0$. If $\ln \dfrac{M_{AB}}{m_{AB}} = 0$ for a pair A, B, then $M_{AB} = m_{AB}$ and that means that the function is constant. Or, if $A \neq B$, it results that P which belongs to K also belongs to the Apolloniu's circle of the pair A, B. But A and B are separated by the Apolloniu's circle which coincides with K in collision with A, $B \in int\, K$. For $d(A,B) = d(B,A)$, it is enough to observe that

$$\min_{P\in K} \frac{|PA|}{|PB|} = \frac{1}{\max\limits_{P\in K} \frac{|PB|}{|PA|}}.$$

We wish to prove that for any three points A, B, C in $int\, K$ we have:

$$d(A,B) + d(B,C) \geq d(A,C). \tag{1}$$

Let S_1, S_2, S_3 ; s_1, s_2, s_3, be the points for which the maximum and the minimum of the three ratios is reached:

$$\frac{\frac{|S_1A|}{|S_1B|}}{\frac{|s_1A|}{|s_1B|}} = \frac{M_{AB}}{m_{AB}}\,;\ \frac{\frac{|S_2B|}{|S_2C|}}{\frac{|s_2B|}{|s_2C|}} = \frac{M_{BC}}{m_{BC}}\,;\ \frac{\frac{|S_3A|}{|S_3C|}}{\frac{|s_3A|}{|s_3C|}} = \frac{M_{AC}}{m_{AC}}\,.$$

Therefore, for the substitutions with minoring role S_1, $S_2 \to S_3$, ; s_1, $s_2 \to s_3$, we obtain $\dfrac{M_{AB}}{m_{AB}} \cdot \dfrac{M_{BC}}{m_{BC}} \geq \dfrac{|S_3A|}{|S_3C|} : \dfrac{|s_3A|}{|s_3C|} = \dfrac{M_{AC}}{m_{AC}}$, equivalently with (1). See also [2], [3]. $\square$

These method for obtaining distances is known as the *logarithmic oscillation method of metrization* and was studied firstly by the Romanian geometer Dan Barbilian (see [1]). The previous distance is called *a Barbilian distance.*

Let A be a point belonging to the interior of the i-derivable curve K, Δ be a given direction and $A + dA$ be another point in a small neighborhood of A such that dA

is orthogonal to Δ. Let us denote by R, r the radii of the circles which appear in Lemma 3, and by ds the infinitesimal distance established by Theorem 1 between the points A and $A+dA$, i.e. $ds = d(A, A+dA) = \ln \dfrac{\max\limits_{P\in K} \dfrac{|PA|}{|P(A+dA)|}}{\min\limits_{P\in K} |PA||P(A+dA)|}$. Let us denote by $d\sigma$ the Euclidean distance between the points A, $A+dA$. Then we can prove:

Theorem 2 *The previous distance between two close points A, $A+dA$ from the interior of an i- derivable curve has the same form as the Lagrangean arclength determined by Lemma 4.*

Proof. We have to prove that $ds = \dfrac{1}{2}\left(\dfrac{1}{R}+\dfrac{1}{r}\right) d\sigma$.

In the given conditions $ds = \dfrac{M_{A(A+dA)} - m_{A(A+dA)}}{m_{A(A+dA)}}$. For A, $A+dA$, P with the coordinates (x_1, x_2), (x_1^1, x_2^1), (x^1, x^2) the Apolloniu's circle determined by $A, A+dA$ and the constant $\sqrt{\lambda}$ has the equation:

$$\sum_{i=1}^{2}\left((x^i - x_i)^2 - \lambda(x^i - x_i^1)^2\right) = 0.$$

Its radius will be:

$$\rho^2 = \frac{\lambda}{(1-\lambda)^2}\sum_1^2 (x_i - x_i^1)^2 .$$

For the maximum $M_{A(A+dA)}$ and the minimum $m_{A(A+dA)}$ of the expression $\frac{PA}{P(A+dA)}$, it appears:

$$R^2 = \frac{M_{A(A+dA)}}{(1-M_{A(A+dA)})^2} d\sigma^2\,,\ r^2 = \frac{m_{A(A+dA)}}{(1-m_{A(A+dA)})^2} d\sigma^2$$

and it results:

$$\frac{M_{A(A+dA)} - m_{A(A+dA)}}{m_{A(A+dA)}} = \frac{2\left(\sqrt{d\sigma^2+4r^2}+\sqrt{d\sigma^2+4R^2}\right)d\sigma}{\left(-d\sigma+\sqrt{d\sigma^2+4R^2}\right)\left(d\sigma+\sqrt{d\sigma^2+4r^2}\right)}.$$

Taking into account that we can neglect small infinities of second order, we obtain $\dfrac{2d\sigma}{d\sigma+\sqrt{d\sigma^2+4a^2}} = \dfrac{d\sigma}{a}$ and also $ds = \dfrac{1}{2}\left(\dfrac{1}{R}+\dfrac{1}{r}\right) d\sigma$. $\square$

Taking into account the method of metrization of the interior of the i-derivable curves these Lagrange spaces can be called *Lagrange oscillant spaces*. Among these Lagrange oscillant spaces we shall highlight one which is very important for geometers.

We shall analyze the particular case of the circle because the previous Barbilian distance is in fact a modified Cayley distance of the Cayley-Poincaré model of Lobacevski hyperbolic geometry ([2]).

Theorem 3 *The Barbilian distance between two closed points $A_1(x,y)$, $B_1(x+\dot{x}, y+\dot{y})$ from the interior of the circle Γ induces the generalized Lagrange metric which has as coefficients*

$$g_{11} = g_{22} = \frac{4\dot{y}^4(y\dot{x} - x\dot{y})^2}{(\dot{x}^2 + \dot{y}^2)\left(R^2\dot{x}^2 - (x^2+y^2)\dot{y}^2\right)^2},\ g_{12} = g_{21} = 0. \tag{2}$$

Proof. Let $A(x_0, y_0)$, $B(x_0 + dx, y_0 + dy)$ be two points from the interior of the circle Γ and consider the straight line $AB:\ y - y_0 = m(x - x_0)$, where $m := \frac{dy}{dx}$.

We put

$$ds := d(A,B) = \ln \frac{M_{AB}}{m_{AB}} = \ln\left(1 + \frac{M_{AB} - m_{AB}}{m_{AB}}\right) = \frac{M_{AB} - m_{AB}}{m_{AB}}. \tag{3}$$

Theorem 2 leads to

$$ds = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\sqrt{dx^2 + dy^2}. \tag{4}$$

We have to compute R_1 and R_2, which the radii of two circles tangent both to the Γ circle and to the straight line AB in A.

Let $O_1(x_1, y_1)$, $O_2(x_2, y_2)$ be the centres of the above two circles. Consider also T_1, T_2 the two tangent points of the two circles at Γ. We have:

$$O_1O_2\ :\ y - y_0 = -\frac{1}{m}(x - x_0).$$

Therefore O_1 will have the coordinates $O_1(x_1, y_0 - \frac{1}{m}(x_1 - x_0)$ and the square of the distance O_1A will be:

$$O_1A^2 = \frac{m^2+1}{m^2}(x_1 - x_0)^2.$$

Solving the system

$$\begin{cases} y = \dfrac{y_0 - \frac{1}{m}(x_1 - x_0)}{x_1}x \\ \\ x^2 + y^2 = R^2 \end{cases},$$

we will obtain the coordinates of the point T_1, that is:

$$T_1\left(\frac{Rx_1}{\sqrt{x_1^2 + \left(y_0 - \frac{1}{m}(x_1 - x_0)\right)^2}},\ \frac{R\left(y_0 - \frac{1}{m}(x_1 - x_0)\right)}{\sqrt{x_1^2 + \left(y_0 - \frac{1}{m}(x_1 - x_0)\right)^2}}\right).$$

If we consider the Euclidean distance O_1T_1, we will obtain:

$$O_1T_1^2 = \left(\sqrt{x_1^2 + \left(y_0 - \frac{1}{m}(x - x_0)\right)^2} - R\right)^2.$$

The condition $O_1A^2 = O_1T_1^2$ will lead to the equation:

$$4(x_1 - x_0)^2 \left(m^2n^2 - R^2(m^2+1)\right) + 4mn(x_1 - x_0)\left(R^2 - m^2(x_0^2 + y_0^2)\right) + \\ + \left(R^2 - m^2(x_0^2 + y_0^2)\right)^2 = 0,$$

(where $n = y_0 - mx_0$) which has as solutions

$$x_1 - x_0 = -\frac{1}{2} \cdot \frac{R^2 - m^2(x_0^2 + y_0^2)}{m\left(y_0 - mx_0 \pm R\sqrt{m^2+1}\right)}\ , \text{ that is:}$$

$$R_1 = \frac{\sqrt{m^2+1}}{2m} \cdot \frac{R^2 - m^2(x_0^2 + y_0^2)}{m\left(y_0 - mx_0 + R\sqrt{m^2+1}\right)},$$

$$R_2 = \frac{\sqrt{m^2+1}}{2m} \cdot \frac{R^2 - m^2(x_0^2 + y_0^2)}{m\left(y_0 - mx_0 - R\sqrt{m^2+1}\right)}.$$

Using (4) it results:

$$ds = \frac{m}{\sqrt{m^2+1}} \cdot \frac{2m(y_0 - mx_0)}{R^2 - m^2(x_0^2 + y_0^2)}\sqrt{dx^2 + dy^2}$$

and finally, replacing the particular point A by A_1, and m by $\frac{\dot{y}}{\dot{x}}$ it follows:

$$ds^2 = \frac{4\dot{y}^4(y\dot{x} - x\dot{y})^2}{(\dot{x}^2 + \dot{y}^2)\left(R^2\dot{x}^2 - (x^2+y^2)\dot{y}^2\right)^2}(dx^2 + dy^2).$$

The matrix of the coefficients $g_{ij}(x, y, \dot{x}, \dot{y})$ of the metric is

$$\begin{pmatrix} \dfrac{4\dot{y}^4(y\dot{x} - x\dot{y})^2(\dot{x}^2 + \dot{y}^2)}{\left(R^2\dot{x}^2 - (x^2+y^2)\dot{y}^2\right)^2} & 0 \\ 0 & \dfrac{4\dot{y}^4(y\dot{x} - x\dot{y})^2}{(\dot{x}^2 + \dot{y}^2)\left(R^2\dot{x}^2 - (x^2+y^2)\dot{y}^2\right)^2} \end{pmatrix}.$$

The rank $[g_{ij}] = 2$. So, $g_{ij}(x, y, \dot{x}, \dot{y})$ is a distinguished tensor field. Therefore it determines a generalized Lagrange space GL^2. $\square$ (see [9])

Definition 4 The space GL^2 with the fundamental tensor field g_{ij} from (2) is called a *Barbilian Space.*

These space, suggested by a problem of hyperbolic geometry can be important in applications. We prove:

Theorem 4 *A Barbillan Space is a generalized Lagrange space which is not reducible to a Lagrange or a Finsler space.*

Proof. By absurdum, we admit the existence of a Lagrangian $L(x, y, \dot{x}, \dot{y})$ for which

$$g_{ij}(x, y, \dot{x}, \dot{y}) = \frac{1}{2} \cdot \frac{\partial^2 L}{\partial y^i \partial y^j}, \tag{5}$$

where g_{ij} is the fundamental tensor of a Barbilian Space.

From (5) it follows that the Cartan tensor $C_{ijk} = \dfrac{1}{2}\dfrac{\partial g_{ij}}{\partial y^k}$ is totally symmetric. But from (2) we deduce $\dfrac{\partial g_{11}}{\partial \dot{y}} \neq \dfrac{\partial g_{12}}{\partial \dot{x}}$. Consequently, the equality, (5) is an impossibility. □

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