ON THE LAGRANGE SPACES WITH (α, β)-METRICS

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Abstract

One considers a Lagrange space with the fundamental function (1.1) and one determines the fundamental tensor field, canonical nonlinear connection and canonical metrical linear N-connection.

1. Introduction

Let M be a real, n-dimensional C^∞ manifold and $\tau : TM \to M$ its tangent bundle. We shall consider, together with a Finsler space $F^n = (M, F(x,y))$, a covector field $A_i(x)$ defined on M, or on open set of M, and "the electromagnetic" 1-form $\beta(x,y) = A_i(x)y^i$. So, we can define the real function on TM:

$$(1.1) \quad L(x,y) = \alpha^2(x,y) + a\beta(x,y) + b\beta^2(x,y), \quad \forall (x,y) \in TM,$$

where $\alpha(x,y) = F(x,y)$.

This function is C^∞-differentiable on $\widetilde{TM} = TM - \{O\}$ and continuous on the null section $O : M \to TM$. Obviously $L(x,y)$ is not homogeneous with respect to (y^i). We prove that $L(x,y)$ is a regular Lagrangian and that $L^n = (M, L(x,y))$ is a Lagrange space [2]. We study the Lagrange space L^n with the fundamental function (1.1) and determine the fundamental tensor field, canonical nonlinear connection and canonical metrical linear N-connection.

2. The Lagrange space $L^n = (M, L)$

According to (1.1) the fundamental tensor field of the Lagrange space L^n, $g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$, is given by:

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Proposition 2.1 *The fundamental tensor field of the Lagrange space* $L^n = (M, L)$ *is:*

(2.1) $g_{ij}(x,y) = \gamma_{ij}(x,y) + bA_i(x)A_j(x),$

where $\gamma_{ij} = \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial y^i \partial y^j}$ *is the fundamental tensor field of the Finsler space* F^n.

In order to deduce that g_{ij}, from (2.1), is nonsingular we distinguish two cases: $b > 0$ and $b < 0$. In this respect we reproduce the following Lemmas from the paper [2].

Lemma 2.2 *Let* $\|A_{ij}\|$, $(i,j = 1,...,n)$ *a real nonsingular matrix, having* $\|A_{ij}\|^{-1} = \|A^{ij}\|$. *Then the matrix* $\|B_{ij}\|$ *with the elements* $B_{ij} = A_{ij} + c_i c_j$, *such that* $1 + c^2 \neq 0$, $c^2 = A^{ij}c_i c_j$, *is nonsingular. Its determinant is* $det\|B_{ij}\| = (1+c^2)det\|A_{ij}\|$ *and* $\|B_{ij}\|^{-1}$ *has the elements* $B^{ij} = A^{ij} - \dfrac{1}{1+c^2}c^i c^j$, $(c^i = A^{ij}c_j)$.

Lemma 2.3 *If* $\|A_{ij}\|, (i,j = 1,...,n)$ *is a real nonsingular matrix, having* A^{ij} *as elements of its inverse and* $d_i, (i = 1,...,n)$ *are real numbers for which* $1 - d^2 \neq 0$, $d^2 = A^{ij}d_i d_j$, *then the matrix with the elements* $B_{ij} = A_{ij} - d_i d_j$ *is nonsingular. It has the determinant* $det\|B_{ij}\| = (1-d^2)det\|A_{ij}\|$ *and its inverse has the elements* $B^{ij} = A^{ij} + \dfrac{1}{1-d^2}d^i d^j, d^i = A^{ij}d_j$.

Applying these Lemmas we obtain:

Theorem 2.1 1^0 *The d-tensor field* g_{ij} *has the following properties:*

$$(2.2)\quad \begin{array}{l} If\ b > 0, \quad then \quad det\|g_{ij}\| = (1+c^2)det\|\gamma_{ij}\|, \\ \text{where } c^2 = b\gamma^{ij}(x,y)A_i(x)A_j(x). \\ If\ b < 0, \quad then \quad det\|g_{ij}\| = (1-d^2)det\|\gamma_{ij}\|, \\ \text{where } d^2 = -b\gamma^{ij}(x,y)A_i(x)A_j(x). \end{array}$$

2^0 *The contravariant tensor of* (g_{ij}) *is as follows:*

$$(2.3)\quad \begin{array}{l} If\ b > 0, \quad then\ g^{ij}(x,y) = \gamma^{ij}(x,y) - \frac{1}{1+c^2}A^i(x,y)A^j(x,y), \\ \text{where } A^i(x,y) = \sqrt{b}\gamma^{ij}(x,y)A_j(x). \\ If\ b < 0, \quad then\ g^{ij}(x,y) = \gamma^{ij}(x,y) + \frac{1}{1-d^2}A^i(x,y)A^j(x,y), \\ \text{where } A^i(x,y) = \sqrt{-b}\gamma^{ij}(x,y)A_j(x). \end{array}$$

Now, we can state:

Theorem 2.2 *The pair* $L^n = (M, L(x,y))$, *where* $L(x,y)$ *is given by (1.1), is a Lagrange space.*

Remark. The classical case is obtained when $\gamma_{ij}(x,y) = \gamma_{ij}(x)$, (Lorentz metric).

The space L^n is called *the Lagrange space of generalized electrodynamics* and F^n *the associated Finsler space* to L^n.

3. Variational problem

Let $c : [0,1] \to M$ be a smooth curve in M expressed in a local chart (U, φ) on the base manifold M by $x^i = x^i(t), t \in [0,1], Im\ c \subset U$. The length of c in the Lagrange space L^n is

(3.1) $I(c) = \int_0^1 L(x, \dot{x})dt.$

The variational problem concerning $I(c)$, leads to the Euler-Lagrange equation:

$$(3.2)\ \begin{cases} \frac{d}{dt}\{\frac{\partial L}{\partial y^i}\} - \frac{\partial L}{\partial x^i} = 0 \\ y^i = \frac{dx^i}{dt}. \end{cases}$$

We denote the electromagnetic tensor field, determined by the covector field $A_i(x)$,by

(3.3) $F_{ij}(x) = \dfrac{\partial A_j}{\partial x^i} - \dfrac{\partial A_i}{\partial x^j}$

and consider its mixed form

(3.4) $F^i_j(x,y) = \gamma^{ik}(x,y)F_{kj}(x).$

After usual calculation we get:

Theorem 3.1 *The Euler-Lagrange equations in variational problem concerning the functional (3.1) are given by*

$$(3.5)\ \begin{cases} \frac{d^2x^i}{dt^2} + 2(G^i(x,y) + H^i(x,y)) = 0, \\ y^i = \frac{dx^i}{dt}, \end{cases}$$

where $G^i(x,y) = \frac{1}{2}\gamma^i_{rs}y^r y^s$, $H^i(x,y) = \dfrac{1}{2}(\dfrac{a+2b\beta}{2}F^i_h y^h + A^i\overline{B})$, *if* $b > 0$, $\overline{B} = \dfrac{\sqrt{b}}{2}\tilde{F} - \dfrac{1}{\sqrt{b}(1+c^2)}A_k\gamma^k_{rs}\dfrac{dx^r}{dt}\dfrac{dx^s}{dt} - \dfrac{a+2b\beta}{2(1+c^2)}A^iF_{ih}\dfrac{dx^h}{dt} - \dfrac{b}{2(1+c^2)}\tilde{F}A$, $\tilde{F} = (\dfrac{\partial A_r}{\partial x^s} + \dfrac{\partial A_s}{\partial x^r})\dfrac{dx^s}{dt}\dfrac{dx^r}{dt}$, $A = A^iA_i$ *and* $H^i(x,y) = \dfrac{1}{2}(\dfrac{a+2b\beta}{2}F^i_h y^h + A^i\underline{B})$, *if* $b < 0$, $d^2 \neq 1$, $\underline{B} = \dfrac{\sqrt{-b}}{2}\tilde{F} + \dfrac{1}{\sqrt{-b}(1-d^2)}A_k\gamma^k_{rs}\dfrac{dx^r}{dt}\dfrac{dx^s}{dt} + \dfrac{a+2b\beta}{2(1-d^2)}A^iF_{ih}\dfrac{dx^h}{dt} + \dfrac{b}{2(1-d^2)}\tilde{F}A.$

The equations (3.5) determine a spray defined only by the Lagrangian $\mathcal{L}$ from (1.1), so we can develop the geometry of the Lagrange space
$L^n = (M, L)$ using this canonical spray only. We have then:

Theorem 3.2 *The canonical nonlinear connection of the Lagrange space L^n is given by:*

$$(3.6)\ \begin{cases} N^i_j = \overset{o}{N}{}^i_j - \underset{(1)}{\overline{A}{}^i_j},\ if\ b > 0, \\ N^i_j = \overset{o}{N}{}^i_j - \underset{(1)}{\underline{A}{}^i_j},\ if\ b < 0, \end{cases}$$

where $\overline{\underset{(1)}{A}}{}^i_j = \frac{1}{2}\frac{\partial}{\partial y^j}\{\frac{a+2b\beta}{2}F^i_h y^h + A^i\overline{B}\}$, $\underline{\underset{(1)}{A}}{}^i_j = \frac{1}{2}\frac{\partial}{\partial y^j}\{\frac{a+2b\beta}{2}F^i_h y^h + A^i\underline{B}\}$.

We can prove now:

Theorem 3.3 The linear N-connection on the Lagrange space L^n is:

$$(3.7)\quad \begin{cases} N^i_j = \overset{o}{N}{}^i_j - \overline{\underset{(1)}{A}}{}^i_j,\ L^i_{jk} = \overset{o}{F}{}^i_{jk} + \overset{o}{C}{}^i_{jm}\overline{\underset{(1)}{A}}{}^m_k + \overline{A}^i_{jk},\ C^i_{jk} = \overset{o}{C}{}^i_{jk} + \overline{\underset{(1)}{C}}{}^i_{jk} \end{cases}$$

in the case $b > 0$,

$$(3.8)\quad \begin{cases} N^i_j = \overset{o}{N}{}^i_j - \underline{\underset{(1)}{A}}{}^i_j,\ L^i_{jk} = \overset{o}{F}{}^i_{jk} + \overset{o}{C}{}^i_{jm}\underline{\underset{(1)}{A}}{}^m_k + \underline{A}^i_{jk},\ C^i_{jk} = \overset{o}{C}{}^i_{jk} + \underline{\underset{(1)}{C}}{}^i_{jk} \end{cases}$$

in the case $b < 0$, where $\overline{A}^i_{jk} = \frac{b}{2}g^{ih}(\frac{\partial(A_jA_h)}{\partial x^k} + \frac{\partial(A_kA_h)}{\partial x^j} - \frac{\partial(A_jA_k)}{\partial x^h}) - \frac{A^iA^h}{2(1+c^2)}(\frac{\delta\gamma_{jh}}{\delta x^k} + \frac{\delta\gamma_{kh}}{\delta x^j} - \frac{\delta\gamma_{jk}}{\delta x^h})$, $\underline{A}^i_{jk} = \frac{b}{2}g^{ih}(\frac{\partial(A_jA_h)}{\partial x^k} + \frac{\partial(A_kA_h)}{\partial x^j} - \frac{\partial(A_jA_k)}{\partial x^h}) + \frac{A^iA^h}{2(1-d^2)}(\frac{\delta\gamma_{jh}}{\delta x^k} + \frac{\delta\gamma_{kh}}{\delta x^j} - \frac{\delta\gamma_{jk}}{\delta x^h})$, $\overline{\underset{(1)}{C}}{}^i_{jk} = -\frac{1}{2(1+c^2)}A^iA^h(\frac{\partial\gamma_{hj}}{\partial y^k} + \frac{\partial\gamma_{kh}}{\partial y^j} - \frac{\partial\gamma_{jk}}{\partial y^h})$, $\underline{\underset{(1)}{C}}{}^i_{jk} = \frac{1}{2(1-d^2)}A^iA^h(\frac{\partial\gamma_{hj}}{\partial y^k} + \frac{\partial\gamma_{kh}}{\partial y^j} - \frac{\partial\gamma_{jk}}{\partial y^h})$.

Then the whole geometry of the Lagrange space L^n can be developed only on the base of canonical linear connection given by Theorem 3.3. This connection is a canonical one because it is determined only by the fundamental function (1.1) of the Lagrange space $L^n = (M, L)$.

References

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