SHARPLY 2–TRANSITIVE LIE GROUPS
and their translation structures

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Abstract

In this note, we give a characterization of the sharply 2–transitive Lie groups: Let Γ be a (connected) nonabelian Lie group and let $\mathcal{P}$ be an invariant partition of Γ into closed subgroups of the dimension $\frac{1}{2}\dim\Gamma$. We require $\mathcal{P}$ to be compact in the Grassmannian topology. If $\mathcal{P}$ contains an abelian normal subgroup, or if Γ is not solvable, then Γ is a sharply 2–transitive Lie group or, for $\dim\Gamma = 2$, the connected component of $\mathrm{AGL}_1\mathbb{R}$ and $\mathcal{P}$ is the Frobenius partition.

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1 Introduction

Let Γ be a sharply 2–transitive transformation group of the manifold M. Tits shows in [16] that Γ is isomorphic to the affine group $\mathrm{AGL}_1\mathbb{F}$ for some topological nearfield $\mathbb{F}$.

Thus, we have the Frobenius partition $\mathcal{P}$ of Γ, which consists precisely of the stabilizers Γ_x (with $x \in M$) and the normal subgroup of all fixed point free elements of Γ.

From this group with partition, we construct a geometry with point set Γ and line set $\mathcal{L} = \Gamma \cdot \mathcal{P} = \{\gamma \cdot L\,;\, \gamma \in \Gamma \text{ and } L \in \mathcal{P}\}$. This *translation structure* of the pair $(\Gamma, \mathcal{P})$ is a symmetric plane (see section 4 for further information) and thus closely related to the classical symmetric planes: the affine, projective and hyperbolic planes over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$.

Using this geometric structure (and general results of Plaumann and Strambach concerning Lie groups with partitions), we characterize the sharply 2–transitive Lie groups by their Frobenius partition, see Theorem 5.1.

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2 Topological nearfields

A nearfield is an algebraic structure which satisfies nearly all axioms of a (skew) field; only one of the distributive laws may not be fulfilled. To be more precise:

Definition 2.1 A *(planar) nearfield* is a triple $= (F, +, \circ)$ such that

(1) $(F, +)$ is an abelian group. We denote the zero element of this group by 0.
(2) $(F^\times, \circ)$ is a group (with unit element 1), where $F^\times = F \setminus \{0\}$. Moreover, $0 \circ x = x \circ 0 = 0$ holds for every $x \in F$.
(3) $x \circ (y + z) = x \circ y + x \circ z$ for every x, y, $z \in F$.
(4) The equation $a \circ x - b \circ x = c$ has a unique solution x for every a, b, $c \in F$ with $a \neq b$ (planarity axiom).

A nearfield $\mathbb{F}$ with locally compact connected Hausdorff space F is called a *topological nearfield* if

(1) The maps $(a; b) \mapsto a - b$ and $(a; b) \mapsto a \circ b$ (from $F \times F$ to F) are continuous and the map $a \mapsto a^{-1}$ (from $F \setminus \{0\}$ to $F \setminus \{0\}$) is continuous, as well.
(2) The solution x of the equation $a \circ x - b \circ x = c$ depends continuously on $a \neq b; c \in F$.

Examples of topological nearfields are constructed from the quaternions: Let $\mathbb{H}$ be the skew field of the quaternions. By '+' and '·' we denote the usual addition and multiplication, respectively, of $\mathbb{H}$. Consider a continuous homomorphism $\varphi : \mathbb{R}_{\mathrm{pos}} \to \mathbb{H}^\times$ with $\|\varphi(r)\| = r$ for all $r \in \mathbb{R}_{\mathrm{pos}}$. Define a new multiplication on $\mathbb{H}$ by

$$x \circ y = x \cdot \varphi(\|x\|)^{-1} \cdot y \cdot \varphi(\|x\|) \text{ if } x \neq 0. \tag{1}$$

Then $\mathbb{F}_\varphi = (\mathbb{H}, +, \circ)$ is a topological nearfield. We refer to $\mathbb{F}_\varphi$ as the *Kalscheuer nearfield* defined by φ.

If φ equals the homomorphism $\mathbb{R}_{\mathrm{pos}} \to \mathbb{H}^\times; \smallsetminus \mapsto \smallsetminus \exp(\mathbb{R}\sqsupset \ln \smallsetminus)$ for some $R \in \mathbb{R}$, then we will write $\mathbb{F}_\mathbb{R}$ instead of $\mathbb{F}_\varphi$. Note that $\mathbb{F}_{\nvDash}$ is isomorphic to the field of quaternions.

Kalscheuer classified all topological (locally compact connected) nearfields in [3]. We state his result here:

Theorem 2.1 (Kalscheuer 1940) *Every topological nearfield is isomorphic to a Kalscheuer nearfield $_R$ for some R, or is isomorphic to the field of real or complex numbers.*

Remark: The Kalscheuer nearfields $\mathbb{F}_\mathbb{R}$ and $\mathbb{F}_\mathbb{S}$ are isomorphic if, and only if, $|R| = |S|$.

3 Groups with partitions and their geometries

Definition 3.1 A *partition* of a group Γ is a set $\mathcal{P}$ of nontrivial[1] subgroups of Γ with

[1] In this context, a nontrivial subgroup means a subgroup which is neither the entire group Γ nor the subgroup $\{e\}$.

(1) the group Γ is covered by $\mathcal{P}$, and
(2) $X \cap Y = \{e\}$ for any two distinct elements X, Y of $\mathcal{P}$.

A very interesting class of groups with partitions can be constructed from nearfields: Let $\mathbb{F}$ be a nearfield. The set $\mathrm{AGL}_1\mathbb{F}$ of all maps $\mathbb{F} \to \mathbb{F}; \curvearrowleft \mapsto \eth \circ \curvearrowleft +$ forms a group which is isomorphic to a semidirect product $\mathbb{F}^\times \ltimes \mathbb{F}$. Clearly, $\mathrm{AGL}_1\mathbb{F}$ is a sharply 2–transitive transformation group of $\mathbb{F}$.

Let N denote the normal subgroup $\{x \mapsto x+b\,;\, b \in \mathbb{F}\}$ and let $K = \{x \mapsto a\circ x\,;\, a \in \mathbb{F}^\times\}$. Then $\mathcal{P} = \{N\} \cup \{\gamma K \gamma^{-1}\,;\, \gamma \in \mathrm{AGL}_1\mathbb{F}\}$ is a partition of $\mathrm{AGL}_1\mathbb{F}$. Note, that $CP \setminus \{N\}$ consists precisely of the stabilizers of the action of $\mathrm{AGL}_1\mathbb{F}$ on $\mathbb{F}$. We refer to this special partition as the *Frobenius partition* of the group $\mathrm{AGL}_1\mathbb{F}$.

We return now to the case of an arbitrary group Γ with a partition $\mathcal{P}$ and assign a geometry to the pair $(\Gamma, \mathcal{P})$. These geometries are special examples of linear spaces:

Definition 3.2 Let P be a set and let $\mathcal{L}$ be a family of nonempty subsets of P. Then $(P, \mathcal{L})$ is called a *linear space* (with *point space* P and *line space* $\mathcal{L}$), if

(1) For every two distinct points $p, q \in P$ there exists exactly one line $L = p \vee q \in \mathcal{L}$ which contains both p and q.
(2) Every line contains at least two points.
(3) There exists a quadrangle, i. e. four points, no three of which are on the same line.

A *parallelism* of a linear space is an equivalence relation on the line set such that (distinct) parallel lines have no common point.

An *automorphism* of a linear space is a bijective map of the point set which maps lines onto lines.

Two distinct lines K and L of a linear space may not meet. If they do, we denote their unique common point by $K \wedge L$.

Proposition 3.1 *Let Γ be a group and let $\mathcal{P}$ be a partition of Γ. Define*

$$\mathcal{L} = \Gamma\mathcal{P} = \{\gamma L\,;\, \gamma \in \Gamma \text{ and } L \in \mathcal{P}\}$$

to be the family of all cosets γL of elements of $\mathcal{P}$. Then $(\Gamma, \mathcal{L})$ is a linear space.

Moreover,

$$\delta K \parallel \gamma L \text{ if, and only if, } K = L \text{ (with } K, L \in \mathcal{P} \text{ and } \gamma, \delta \in \Gamma)$$

defines a parallelism on $(\Gamma, \mathcal{L})$.

Then Γ operates via left translation on itself as a sharply transitive group of automorphisms on the linear space $(\Gamma, \mathcal{L})$. Furthermore, every element of Γ is —regarded as an automorphism of $(\Gamma, \mathcal{L})$— fixed point free and preserves the parallelism.

The proof of this proposition is very easy and thus omitted here.

Remarks: (a) The linear space $(\Gamma, \mathcal{L} = \Gamma\mathcal{P}, \|)$ with parallelism is called the *translation structure* of $(\Gamma, \mathcal{P})$ and denoted by $T(\Gamma, \mathcal{P})$.

(b) Among the linear spaces with parallelism, the translation structures of groups with partitions can be characterized as follows:

Let $(P, \mathcal{L}, \|)$ be a linear space with parallelism and let $o \in P$ be an arbitrary point. Assume that there exists a sharply transitive group Γ of automorphisms, such that $\gamma(L) \parallel L$ holds for every $\gamma \in \Gamma$, $L \in \mathcal{L}$. Then the set $\mathcal{P} = \{\Gamma_L\,;\, L \in \mathcal{L} \text{ with } o \in L\}$ is a partition of Γ and $(P, \mathcal{L}, \|)$ is isomorphic to the translation structure $T(\Gamma, \mathcal{P})$.

Examples: (a) Let $\mathbb{F}$ be a field. We consider a partition $\mathcal{P}$ of $\mathbb{F}^{\nvDash\ltimes}$ into n–dimensional linear subspaces (a so–called *spread*). Then the translation structure $T(\mathbb{F}^{\nvDash\ltimes}, \mathcal{P})$ is an affine translation plane. Conversely, every affine translation plane is construcded in this way, cf. André [1].

(b) Let $\mathbb{F}$ be a nearfield and let $\mathcal{P}$ the Frobenius partition of $\mathrm{AGL}_1\mathbb{F}$. By definition, $\mathcal{P}$ is invariant under conjugation. This is equivalent to the fact that the map ι : $\mathrm{AGL}_1\mathbb{F} \to \mathrm{AGL}_{\nVdash}\mathbb{F}; \gamma \mapsto \gamma^{-\nVdash}$ is an automorphism of the linear space $T(\mathrm{AGL}_1\mathbb{F}, \mathcal{P})$. Such translation structures are called *kinematic spaces*.

The translation structure $T(\mathrm{AGL}_1\mathbb{F}, \mathcal{P})$ is closely related to the affine plane over the nearfield $\mathbb{F}$:

Let $\mathbb{F}$ be a nearfield. The *kernel* $\mathrm{kern}(\mathbb{F})$ is the set of elements $c \in \mathbb{F}$ such that $(x + y) \circ c = x \circ c + y \circ c$ holds for every $x, y \in \mathbb{F}$. Of course $K = \mathrm{kern}(\mathbb{F})$ is a subfield of $\mathbb{F}$ and $\mathbb{F}$ is a right vector space over K. Moreover, the multiplication is K–linear. Thus, $L_a = \{a \circ x\,;\, x \in \mathbb{F}\}$ is a K–linear subspace of $\mathbb{F}^{\nvDash}$ and $\mathcal{S} = \{L_a\,;\, a \in \mathbb{F}\} \cup \{\nvdash \times \mathbb{F}\}$ is a spread, i. e. a partition of $\mathbb{F}^{\nvDash}$ into vector subspaces.

In fact, we obtain the usual affine plane $A_2\mathbb{F} = \mathbb{T}(\mathbb{F}^{\nvDash}, \mathcal{S})$ over the nearfield $\mathbb{F}$. By adding the parallel classes as new points and the set of all these new points as a new line we get the projective plane $P_2\mathbb{F}$ over $\mathbb{F}$.

Now, let $\gamma : \mathbb{F} \to \mathbb{F} : \curvearrowleft \mapsto \eth\circ\curvearrowleft+$ be an element of $\mathrm{AGL}_1\mathbb{F}$. Setting $\Phi(\gamma) = b + L_a$, we define a map from the point space $\mathrm{AGL}_1\mathbb{F}$ of the translation structure (where $\mathcal{P}$ denotes the Frobenius partition of $\mathrm{AGL}_1\mathbb{F}$) to the line space of $P_2\mathbb{F}$. A boring computation shows that the image of a line is contained in a line pencil of $P_2\mathbb{F}$. This means that Φ is an embedding of the linear space $T(\mathrm{AGL}_1\mathbb{F}, \mathcal{P})$ into the dual projective plane $P_2^d\mathbb{F}$ over $\mathbb{F}$. A closer look at the image of Φ shows that we can obtain $T(\mathrm{AGL}_1\mathbb{F}, \mathcal{P})$ by removing two lines from $P_2^d\mathbb{F}$.

4 Stable translation structures

For a topological nearfield $\mathbb{F}$, the projective translation plane $P_2\mathbb{F}$ and its dual are topological (locally compact connected) projective planes. This means that there exist topologies on the point set and the line set, such that the operations of joining points and of intersecting lines, respectively, are continuous. For an introduction to topological projective planes see [15].

At the end of the last section we have learned that the translation structure $T(\mathrm{AGL}_1\mathbb{F}, \mathcal{P})$ (where $\mathcal{P}$ denotes the Frobenius partition) can be obtained from $P_2^d\mathbb{F}$ by removing two lines. Since lines of a topological projective plane are allways closed subsets of the point set, we infer that $T(\mathrm{AGL}_1\mathbb{F}, \mathcal{P})$ is an open substructure of $P_2^d\mathbb{F}$. Such geometries are special examples for so–called stable planes:

Definition 4.1 A *stable plane* is a linear space $(P, \mathcal{L})$ whose point and line space are endowed with locally compact Hausdorff topologies, such that the following axioms are satisfied:

(1) The operation $\vee$ of joining points and $\wedge$ of intersecting lines are continuous, where defined.

(2) The domain of definition of the operation $\wedge$ is open in $\mathcal{L} \times \mathcal{L}$. (axiom of stability)

In addition, we require the covering dimension $\dim P$ of the point set to be positive and finite.

Famous examples of stable planes are —beside the topological projective or affine planes— the hyperbolic planes over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Other examples can be derived from these: Let $(P, \mathcal{L})$ be a stable plane and let U be an open subset of P. Then $(U, \mathcal{L}|_U)$ is a stable plane, where $\mathcal{L}|_U = \{L \cap U\,;\, L \in \mathcal{L} \text{ and } L \cap U \neq \emptyset\}$ is endowed with the topology induced by $\mathcal{L}$.

For an introduction into the theory of stable planes, the reader is refered to [2] or [6]. Here, we only want to give one of the main results:

Theorem 4.1 (Löwen 1983) *Let $(P, \mathcal{L})$ be a stable plane. Then the covering dimension $\dim P$ of P is equal to 2, 4, 8 or 16. Moreover, every line $L \in \mathcal{L}$ is closed in P and $\dim P = 2 \dim L$. For any point $p \in P$, the line pencil $\mathcal{L}_p = \{L \in \mathcal{L}\,;\, p \in L\}$ is compact in the topology induced by $\mathcal{L}$.*

We close this section with a short discussion of stable translation structures. Firstly, we treat the case of the affine planes:

Theorem 4.2 (Löwen 1989) *Let $\mathcal{S}$ be a partition of $\mathbb{R}^{\nvDash\ltimes}$ into vector subspaces of dimension n. If $\mathcal{S}$ is compact in the Grassmannian topology, then $T(\mathbb{R}^{\nvDash\ltimes}, \mathcal{S})$ is a topological affine translation plane. Moreover, every topological affine translation plane is isomorphic to one of these examples.*

This theorem is the main tool for the general case: Consider a locally compact connected group Γ and a partition $\mathcal{P}$ of Γ. Assume that the line space $\mathcal{L} = \Gamma \cdot \mathcal{P}$ of the translation structure $T(\Gamma, \mathcal{P})$ carries a topology, such that $T(\Gamma, \mathcal{P})$ becomes a stable plane. The subgroups Ł $\in \mathcal{P}$ of Γ are lines of $\mathbb{E}$ and therefore closed in Γ. Moreover, $\dim \Gamma = 2 \dim L$ holds for every line $L \in \mathcal{L}$.

From [14, 7.3 Satz] we infer that Γ is a Lie group. We define $\mathcal{S}$ to be the set of all subalgebras $T_e L$ of the Lie algebra $T_e \Gamma$ with $L \in \mathcal{P}$. Then $\mathcal{S}$ is a partition of $T_e \Gamma$ —regarded as a vector space— into subspaces of half the dimension. This means that $T(T_e \Gamma, \mathcal{S})$ is an affine translation plane. We refer to this plane (together with the structure of a Lie algebra on $T_e \Gamma$) as the *infinitesimal modell* of $T(\Gamma, \mathcal{P})$.

Theorem 4.3 (Maier 1995) *Let Γ be a connected Lie group with $\dim \Gamma = 2n$ and let $\mathcal{P}$ be a partition of Γ into n–dimensional closed Lie subgroups. Let $\mathcal{L} = \Gamma \cdot \mathcal{P}$ be the line space of the translation structure $T(\Gamma, \mathcal{P})$. Define $\mathcal{S} = \{T_e L\,;\, L \in \mathcal{P}\}$. Then there exists a topology on $\mathcal{L}$ such that $T(\Gamma, \mathcal{P})$ becomes a stable plane if, and only if, $\mathcal{S}$ is compact in the Grassmannian topology.*

Remarks: (a) In the situation of the theorem, we will call the partition $\mathcal{P}$ *stable* and the stable plane $T(\Gamma, \mathcal{P})$ a *stable translation structure.*

(b) Every abelian Lie group admitting a (nontrivial) partition into closed subgroups is a vector group, cf. [14]. Consequently, the class of stable translation structures whose underlying Lie group is abelian coincides with the class of topological affine translation planes.

(c) The theorem enables us to give a very large class of examples of stable (nonabelian) translation structures: Let $\Gamma = \mathbb{R}_{\mathrm{pos}} \ltimes \mathbb{R}^{\gtrdot}$, where the semidirect product is given by the homomorphism $\varphi : \mathbb{R}_{\mathrm{pos}} \to \mathrm{GL}_{\gtrdot}\mathbb{R}$ with $\varphi(r)(x) = rx$. Every vector subspace of the Lie algebra $T_e\Gamma$ is a Lie subalgebra. Moreover, the exponential function of Γ is a diffeomorphism. Consider a partition $\mathcal{S}$ of $T_e\Gamma$ in linear subspaces. Then $\exp \mathcal{S} = \{\exp L\,;\, L \in \mathcal{S}\}$ is a partition of γ.

Let now the dimension of Γ be 2, 4, 8 or 16. Choose a compact spread $\mathcal{S}$ in $T_e\Gamma$, i.e. a partition into subspaces of dimension $\frac{1}{2}\dim\Gamma$ which is compact in the Grassmannian topology. Then $\exp \mathcal{S}$ is a stable partition of the nonabelian group Γ.

In the case of an *invariant* stable partition $\mathcal{P}$ of a connected Lie group Γ we know that the map $s_e : \Gamma \to \Gamma; y \mapsto y^{-1}$ is an automorphism of the stable translation structure $T(\Gamma, \mathcal{P})$. In fact, s_e is a reflection with center $e \in \Gamma$, i.e. s_e is an involutive automorphism which fixes every line through e. Therefore, the map $s_x = x s_e x^{-1} : \Gamma \mapsto \Gamma; y \mapsto xy^{-1}x$ is a reflection at x for every $x \in \Gamma$.

On the other hand, s_x is a symmetry of the symmetric space[2] $(\Gamma\,;\,\bullet)$ (with $x \bullet y = s_x(y) = xy^{-1}x$). This means that the symmetric and the geometric structure are compatible. We make this precise this in the next definition:

Definition 4.2 A *symmetric plane* is a triple $(P, \mathcal{L}, \{s_x\,;\, x \in P\})$ with

(1) $(P, \mathcal{L})$ is a stable plane.

(2) $(P, \{s_x\})$ is a symmetric space in the sense of Loos (cf. [11]), where s_x denotes the symmetry at $x \in P$.

(3) Every symmetry s_x of the symmetric space is an automorphism of the stable plane $(P, \mathcal{L})$.

The group generated by all maps $s_x s_y$ is called the *motion group* of the symmetric plane.

Examples for symmetric planes are the projective or the hyperbolic planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. As we noticed above, the translation structure of a Lie group with an invariant stable partition is a symmetric plane, too. Thus, every topological affine translation plane is an example. In this case the motion group coincides with the translation group and the symmetric structure does not give more information about the plane. The affine translation planes (regarded as symmetric planes) are characterized by its abelian motion group. For this reason, they are often called ‘abelian symmetric planes’.

For details and further information we refer to [7]. For a classification of the symmetric planes with 2– and 4–dimensional point spaces, see [8].

[2]Here, we use the notion of a symmetric space in the sense of Loos (cf. [11]). Therefore, we do not require any Riemannian structure on the point space of a symmetric space.

5 The translation structure of a sharply 2–transitive Lie group

Let Γ be a sharply 2–transitive Lie group. By [16], Γ is isomorphic to $\mathrm{AGL}_1\mathbb{F}$ for some topological nearfield $\mathbb{F}$. Therefore, we may assume $\Gamma = \mathrm{AGL}_1\mathbb{F}$. We denote the element $\mathbb{F} \to \mathbb{F}; \curvearrowleft \mapsto \eth \circ \curvearrowleft +$ of $\mathrm{AGL}_1\mathbb{F}$ by $\gamma_{a,b}$.

The Frobenius partition of $\mathrm{AGL}_1\mathbb{F}$ consists of

(1) the normal subgroup $N = \{\gamma_{1,y}\,;\, y \in\}$ of fixed point free elements of $\mathrm{AGL}_1\mathbb{F}$, and

(2) the stabilizers $\Gamma_c = \{\gamma_{x,c-x\circ c}\,;\, x \in^\times\}$, where $c \in$.

Therefore, the lines of the translation structure $T(\mathrm{AGL}_1\mathbb{F}, \mathcal{P})$ are the subsets

(1) $\gamma_{a,b}N = \{\gamma_{a,y}\,;\, y \in\}$, and

(2) $\gamma_{a,b}\Gamma_c = \{\gamma_{x,a\circ c-x\circ c+b}\,;\, x \in^\times\}$

of $\mathrm{AGL}_1\mathbb{F}$. In fact, this translation structure is a well known stable plane, namely the symmetric plane over the nearfield $\mathbb{F}$, cf. [5]. We may see this fact directly: Consider the dual affine nearfield plane over $\mathbb{F}$. This plane has a point space homeomorphic to $\mathbb{F} \times \mathbb{F}$. The lines are defined by the equations $y = x \circ a + b$ and $x = c$, respectively. Therefore, we infer that $T(\mathrm{AGL}_1\mathbb{F}, \mathcal{P})$ is an open subplane of this topological affine plane and hence is a stable plane.

As we have seen in the discussion in the last section, $T(\mathrm{AGL}_1\mathbb{F}, \mathcal{P})$ is a symmetric plane with the symmetries $s_x(y) = xy^{-1}x$. For a detailed discussion of this symmetric plane, we refer to [5].

Finally, we give a characterization of the translation structures of the topological nearfields in terms of their infinitesimal models. First, we need a definition:

Definition 5.1 A partition $\mathcal{P}$ of a group Γ is called *split*, if

(1) Γ is not abelian and $\mathcal{P}$ is invariant.

(2) $\mathcal{P}$ contains an abelian normal subgroup of Γ

Since $N \in \mathcal{P}$ is an abelian normal subgroup of $\mathrm{AGL}_1\mathbb{F}$, the Frobenius partition of $\mathrm{AGL}_1\mathbb{F}$ splits. Of course the partition of the connected component of $\mathrm{AGL}_1\mathbb{R}$ into the one parameter subgroups splits, too. By an examination of all 2–dimensional Lie groups we infer that there are no more examples in this dimension. We therefore restrict ourselves to the higher dimensional cases:

Theorem 5.1 (a) *Let Γ be a nonabelian connected Liegroup with* $\dim\Gamma > 2$*, and let $\mathcal{P}$ be an invariant stable partition of Γ. Then $\mathcal{P}$ splits if, and only if, Γ is a sharply 2–transitive Lie group and $\mathcal{P}$ is the Frobenius partition of Γ.*

(b) *Let $\mathcal{P}$ be an invariant stable partition of the connected Lie group Γ. If Γ is not solvable, then Γ is isomorphic to the group* $\mathrm{AGL}_1\mathbb{F}_\mathbb{R}$ *over one of the Kalscheuer nearfields $_R$, and $\mathcal{P}$ is the Frobenius partition.*

Proof: (a) Let Γ be a connected Lie group with $2n = \dim\Gamma > 2$. Let $\mathcal{P}$ be an invariant stable partition of Γ. By $\mathcal{S}$ we denote the compact spread $\{T_eL\,;\, L \in \mathcal{P}\}$ of $T_e\Gamma$. Since $\mathcal{P}$ is invariant, we infer that $\mathrm{Ad}\Gamma = \exp \mathrm{ad}T_e\Gamma$ leaves $\mathcal{S}$ invariant.

Therefore, every $\gamma \in \exp \mathrm{ad} T_e\Gamma$ is an automorphism of the affine translation plane $T(T_e\Gamma, \mathcal{S})$.

If $\mathcal{P}$ splits, then there exists a normal abelian subgroup $S \in \mathcal{P}$. We fix another element $W \in \mathcal{P} \setminus \{S\}$. It follows that the Lie algebra $T_e\Gamma$ is a semidirect sum $T_e\Gamma = T_eW \ltimes T_eS$. We identify the vector spaces $T_e\Gamma$ and $\mathbb{R}^{\ltimes} \times \mathbb{R}^{\ltimes}$, such that $T_eW = \mathbb{R}^{\ltimes} \times \nvdash$ and $T_eS = 0 \times \mathbb{R}^{\ltimes}$ holds. Since $\mathcal{S}$ is a spread, it follows that for every line $L \in \mathcal{S} \setminus \{W, S\}$ there exists a matrix $M_L \in \mathrm{GL}_n\mathbb{R}$ with $L = \{(x, M_L x)^{\mathrm{tr}}\,;\, x \in \mathbb{R}^{\ltimes}\}$. This is also true for T_eW if we set $M_{T_eW} = 0$.

Consider an element $(0, y)^{\mathrm{tr}}$. Since T_eS is an abelian ideal, we infer that

$$\exp \mathrm{ad} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} E & 0 \\ A(y) & E \end{pmatrix},$$

where E denotes the $n \times n$–unit matrix and $A(y)$ is an $n \times n$–matrix. This automorphism of the affine plane $T(T_e\Gamma, \mathcal{S})$ is in fact a shear and therefore $A(y) = M_L$ for some $L \in \mathcal{S}$. In particular, $A(y)$ is regular or $A(y) = 0$.

With the same arguments as in [4, Satz 4.2.3] one can show $\dim \exp \mathrm{ad} T_eS = n$. This implies that $\exp \mathrm{ad} T_eS$ operates sharply transitive on $\mathcal{S} \setminus \{T_eS\}$. By [14, 6.4 Satz], S is connected and hence $\exp \mathrm{ad} T_eS = \mathrm{Ad} S$. From these facts we infer that S operates (via conjugation) sharply transitively on $\mathcal{P} \setminus \{S\}$. Thus Γ operates sharply 2–transitively on Γ/W and obviously $\mathcal{P}$ is the Frobenius partition. This finishes the proof of (a).

(b) Let Γ be a connected Lie group and let $\mathcal{P}$ be an invariant stable partition of Γ. Then Γ is the point set of the stable plane $T(\Gamma, \mathcal{P})$ and hence $\dim \Gamma \in \{2, 4, 8, 16\}$.

If Γ is not solvable, then Γ is a Frobenius group with Frobenius complement $\mathrm{SU}_2(\mathbb{C})$ or $\mathbb{H}^\times$, see [14, 6.3 Satz] (the groups listed in part (i) of this theorem are excluded because of their dimensions). Here, a *Frobenius group* (with kernel N and complement F) means a semidirect product $\Gamma = F \ltimes N$ with

(1) $\{N\} \cup \{\gamma F \gamma^{-1}\,;\, \gamma \in N\}$ is a partition of Γ (the *Frobenius partition*, and

(2) $\gamma F \gamma^{-1} \cap \delta F \delta^{-1} = \{e\}$ for $\gamma \neq \delta \in N$.

Now [13, Thm B] shows that N is a vector group with $\dim N \equiv 0(4)$. If $F \cong \mathrm{SU}_2(\mathbb{C})$, then $\dim \Gamma \equiv 3(4)$ and hence Γ can not be the point space of any stable plane. Thus, $F = \mathbb{H}^\times$ and the kernel N has to be isomorphic to $\mathbb{R}^{\ntrianglerighteq}$ or $\mathbb{R}^{\nVdash\nvDash}$. In the latter case, $\mathrm{Ad}N$ operates trivially on the 12–dimensional subspace T_eN of $T_e\Gamma$. Since every element of $\mathrm{Ad}N$ is an automorphism of the infinitesimal model of $T(\Gamma, \mathcal{P})$ (regarded as a topological affine translation plane), it follows that $\mathrm{Ad}N$ contains only the identitiy and hence that N is contained in the center of Γ. This is impossible, because Γ is a Frobenius group.

In the remaining case ($F = \mathbb{H}^\times$ and $N = \mathbb{R}^{\ntrianglerighteq}$) we see easily that $\Gamma = \mathrm{AGL}_1\mathbb{F}_{\mathbb{R}}$ for some $R \geq 0$ by checking all faithfull representations of F on N. Moreover, a direct computation shows that every invariant partition of $\mathrm{AGL}_1\mathbb{F}_{\mathbb{R}}$ into 4–dimensional subgroups is the Frobenius partition $\mathcal{F}$ and hence $\mathcal{P} = \mathcal{F}$. This completes the proof.

In a forthcoming article we will see that in fact *every* invariant stable partition of a nonabelian Lie group splits. This classifies the symmetric planes among the stable translation structures.

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