

ON THE HIGHER ORDER RANDERS SPACES

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Abstract

The paper follows Miron's approach in the general Randers space and his theory is extended to the higher order setting.

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1 Introduction

The Randers spaces were introduced by G.Randers (1941) as "Riemannian spaces with affine connection". In 1953, Stephenson and Kilmister considered the spaces with Randers metric.

The Finsler spaces having the fundamental function

$$F(x, y) = \sqrt{\gamma_{ij}(x)y^i y^j} + b_i(x)y^i,$$

where $\gamma_{ij}(x)$ is a Riemannian metric tensor were introduced by R.G.Ingarden. H.Eliopoulos finds the movement equations on the Finsler spaces with Randers metric and after that, he studied the general Randers metrics. In the volume "Lagrange and Finsler geometry" edited by R.Miron and P.L.Antonelli, [4], R.Miron considered the general Randers space GR^n with the metric

$$(1) \quad F(x, y) = \alpha(x, y) + \beta(x, y)$$

where α is a fundamental function of a Finsler space and $\beta = b_i(x)y^i$ is an 1-form. He develops a theory of Randers spaces completely different from the classical one.

This paper follows Miron's approach and his theory is extended to the higher order setting. First, we define the Finsler space $F^{(k)n} = (M, F)$, where M is a smooth manifold and F is from (1), starting with the prolongation of Riemann space $Prol^k R^n$.

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Secondly, we determine the coefficients of a k-spray and the nonlinear connection. Since in $k = 2$ case the expression of the coefficients of the nonlinear connection N is very simple, we give it and the adapted basis to it.

Let M be a real, smooth manifold of dimension n . We denote by $R^n = (M, \gamma_{ij}(x))$ the Riemann space and by $Prol^k R^n = (Osc^k M, G)$ its prolongation of order k .

We consider the Liouville d-vector fields, [2]:

$$(1.1) \quad \begin{aligned} z^{(1)m} &= y^{(1)m}, \\ z^{(2)m} &= \frac{1}{2}[\Gamma z^{(1)m} + \gamma_{ij}^m z^{(1)i} z^{(1)j}], \dots, \\ z^{(k)m} &= \frac{1}{k}[\Gamma z^{(k-1)m} + \gamma_{ij}^m z^{(1)j} z^{(k-1)i}], \end{aligned}$$

where

$$\Gamma = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}}.$$

We have [5] the following Theorem:

Theorem 1.1 *The function $\alpha^2 = \gamma_{ij} z^{(k)i} z^{(k)j}$ is a differential Lagrangian that has the properties:*

- 1) α^2 is global defined on $Osc^k M$;
- 2) α^2 is a regular Lagrangian;
- 3) α^2 depends on the metric γ_{ij} and by $Osc^k M$, only;
- 4) The fundamental tensor field of α^2 is given by

$$(1.2) \quad \frac{1}{2} \frac{\partial \alpha^2}{\partial y^{(k)i} \partial y^{(k)j}} = \gamma_{ij}(x).$$

Let us consider the 1-forms field

$$(1.3) \quad \beta(x, y^{(1)}, \dots, y^{(k)}) = b_i(x) z^{(k)i},$$

where $b_i(x)$ is the electromagnetic potential and has physics means.

Thus the function $F : Osc^k M \rightarrow R$ which is given by $F = \alpha + \beta$ or

$$(1.4) \quad F(x, y^{(1)}, \dots, y^{(k)}) = \sqrt{\gamma_{ij}(x) z^{(k)i} z^{(k)j}} + b_i(x) z^{(k)i}.$$

We can give the Theorem:

Theorem 1.2 *The fundamental tensor field of the Lagrangian $F^2 = (\alpha + \beta)^2$ is given by*

$$(1.5) \quad g_{ij} = (p\gamma_{ij} + l_i l_j) - p \overset{\circ}{l}_i \overset{\circ}{l}_j,$$

where $\overset{\circ}{l}_i = \frac{\partial \alpha}{\partial y^{(k)i}}$, $l_i = \overset{\circ}{l}_i + b_i$, $p = \frac{\alpha + \beta}{\alpha}$.

Proof. Since $\overset{\circ}{l}_i = \frac{1}{\alpha} \gamma_{ij} z^{(k)j}$ we have $\gamma^{ij} \overset{\circ}{l}_i \overset{\circ}{l}_j = 1$.

We put: $\overset{\circ}{l}^i = \frac{1}{\alpha} z^{(k)i}$ and $\tilde{z}_i^{(k)} = \gamma_{ij} z^{(k)j}$. Using this notation we obtain:

$$\begin{aligned} g_{ij} &= \frac{1}{2} \frac{\partial^2 F^2}{\partial y^{(k)i} \partial y^{(k)j}} = (\alpha + \beta) \frac{\partial^2(\alpha + \beta)}{\partial y^{(k)i} \partial y^{(k)j}} + \frac{\partial(\alpha + \beta)}{\partial y^{(k)i}} \frac{\partial(\alpha + \beta)}{\partial y^{(k)j}} = \\ &= \frac{\alpha + \beta}{\alpha} \left[\alpha \frac{\partial^2 \alpha}{\partial y^{(k)i} \partial y^{(k)j}} \right] + l_i l_j = p \left[\alpha \frac{\partial^2 \alpha^2}{\partial y^{(k)i} \partial y^{(k)j}} - \frac{\partial \alpha}{\partial y^{(k)i}} \frac{\partial \alpha}{\partial y^{(k)j}} \right] + l_i l_j = \\ &= p(\gamma_{ij} - \overset{\circ}{l}_i \overset{\circ}{l}_j) + l_i l_j \end{aligned}$$

□

We prove now, the following Theorem:

Theorem 1.3 *The pair $F^{(k)n} = (M, F)$ is a Finsler space of order k .*

Proof. We must to show the following properties:

1. F is of C^∞ -class on $\tilde{E} = Osc^k M \setminus \{0\}$ and continuous on the null section;
2. F is positive on an open set, where $\beta \geq 0$;
3. F is k -homogeneous on the fibres of E ;
4. The Hessian with the elements:

$$(1.6) \quad g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^{(k)i} \partial y^{(k)j}}$$

is positively defined.

F is of C^∞ -class on \tilde{E} and continuous on the null section of the canonical surjection $\pi : E \rightarrow M$ because α and β has these properties. It is known that $z^{(k)i}$ is k -homogeneous on the fibres of $Osc^k M$, i.e. $h_t z^{(k)i} = t^k z^{(k)i}$. To prove 4. we need the following Lemmas, whose proofs are not difficult:

Lemma 1.1 *Let $\|A_{ij}\|$, $(i, j = 1, \dots, n)$ a real nonsingular matrix, having $\|A_{ij}\|^{-1} = \|A^{ij}\|$. Then the matrix $\|B_{ij}\|$ with the elements $B_{ij} = A_{ij} + c_i c_j$, such that $1 + c^2 \neq 0$, $c^2 = A^{ij} c_i c_j$, is nonsingular. Its determinant is $\det\|B^{ij}\| = (1 + c^2) \det\|A_{ij}\|$ and $\|B_{ij}\|^{-1}$ has the elements $B^{ij} = A^{ij} - \frac{1}{1+c^2} c^i c^j$, ($c^i = A^{ij} c_j$).*

Lemma 1.2 *If $\|A_{ij}\|$, $(i, j = 1, \dots, n)$ is a real nonsingular matrix, having A^{ij} as elements of its inverse and d_i ($i = 1, \dots, n$) are real numbers for which $1 - d^2 \neq 0$, $d^2 = A^{ij} d_i d_j$, then the matrix with the elements $B_{ij} = A_{ij} - d_i d_j$ is nonsingular. It has the determinant $\det\|B^{ij}\| = (1 - d^2) \det\|A_{ij}\|$ and its inverse has the elements $B^{ij} = A^{ij} + \frac{1}{1-d^2} d^i d^j$, ($d^i = A^{ij} d_j$).*

Applying Lemma 1.1 to the tensor

$$B_{ij} = p\gamma_{ij} + l_i l_j$$

and taking $\tilde{l}^2 = \frac{1}{p} \gamma^{ij} l_i l_j \geq 0$ we get $1 + \tilde{l}^2 > 0$.

Therefore $\|B_{ij}\|$ is nonsingular,

$$\det\|B_{ij}\| = (1 + \tilde{l}^2)p^n \det\|\gamma_{ij}\|$$

and

$$B^{ij} = \frac{1}{p}\gamma^{ij} - \frac{1}{1 + \tilde{l}^2}\tilde{l}^i\tilde{l}^j,$$

where $\tilde{l}^i = \frac{1}{p}\gamma^{mi}l_m$.

Applying Lemma 1.2 to g_{ij} we get

$$g^{ij} = \frac{1}{p}\gamma^{ij} - \frac{1}{1 + \tilde{l}^2}\tilde{l}^i\tilde{l}^j + \frac{1}{1 - \tilde{d}^2}\tilde{d}^i\tilde{d}^j,$$

where $\tilde{d}^i = B^{ij}\sqrt{p} \overset{\circ}{l}_j$. Since $1 - \tilde{d}^2 = \frac{p}{1 + \tilde{l}^2}$ we obtain

$$\det\|g_{ij}\| = (1 - \tilde{d}^2)\det\|B_{ij}\| = (1 - \tilde{d}^2)(1 + \tilde{l}^2)p^n \det\|\gamma_{ij}\| = p^{n+1}\det\|\gamma_{ij}\|,$$

where $p > 0$ and γ_{ij} is positively defined. \square .

The Finsler space $F^{(k)n} = (M, F)$ is called the higher order Randers space.

Proposition 1.1 *The contravariant g^{ij} of g_{ij} is given by:*

$$(1.7) \quad g^{ij} = \frac{1}{p}\gamma^{ij} - \frac{1}{p^2}[\overset{\circ}{l}^i\overset{\circ}{l}^j (1 - \tilde{l}^2) + \overset{\circ}{l}^i b^j + \overset{\circ}{l}^j b^i],$$

where $\tilde{l}^2 = \frac{1}{p}\gamma^{ij}l_i l_j$.

2 k-Spray. Nonlinear Connection.

The nonlinear connection $\overset{\circ}{N}$ of the space $ProkR^n$, determined by R.Miron [5], has the dual coefficients:

$$(2.1) \quad \begin{cases} \overset{\circ}{M}_j^i = \frac{\partial \overset{\circ}{G}^i}{\partial y^{(k)j}} = \gamma_{jh}^i y^{(1)h}, \\ \overset{\circ}{M}_j^i = \frac{1}{2}(\Gamma \overset{\circ}{M}_j^i + \overset{\circ}{M}_m^i \overset{\circ}{M}_j^m), \dots \\ \overset{\circ}{M}_j^i = \frac{1}{k}(\Gamma \overset{\circ}{M}_j^i + \overset{\circ}{M}_m^i \overset{\circ}{M}_j^m), \end{cases}$$

where

$$(2.2) \quad \Gamma = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}}.$$

Starting with the canonical k -spray determined by I.Bucataru [2], given in the form:

$$(2.3) \quad (k+1) \overset{\circ}{G}^i = y^{(1)j} \overset{\circ}{M}_j^i + 2y^{(2)j} \overset{\circ}{M}_j^i + \dots + ky^{(k)j} \overset{\circ}{M}_j^i,$$

we consider the coefficients:

$$(2.4) \quad G^i = \overset{\circ}{G}^i - F_j^i(x) z^{(k)j},$$

where $F_j^i(x) = \gamma^{im}(x) F_{mj}(x)$ and

$$(2.5) \quad F_{mj}(x) = \frac{\partial b_j}{\partial x^m} - \frac{\partial b_m}{\partial x^j}$$

is the electromagnetic tensor.

Theorem 2.1 G^i from (2.4) are the local coefficients of a k -spray.

Proof. With respect to a transformation of a local coordinates, the coefficients G^i change as follows:

$$(2.6) \quad (k+1)\widetilde{G}^i = (k+1)G^j \frac{\partial \widetilde{x}^i}{\partial x^j} - (y^{(1)j} \frac{\partial \widetilde{y}^{(k)i}}{\partial x^j} + 2y^{(2)j} \frac{\partial \widetilde{y}^{(k)i}}{\partial y^{(1)j}} + \dots + ky^{(k)j} \frac{\partial \widetilde{y}^{(k)i}}{\partial y^{(k-1)j}}).$$

□

Having the k -spray, we can determine the coefficients of the nonlinear connection.

Theorem 2.2 The set of functions

$$(2.7) \quad \begin{cases} M_j^i = M_j^i - F_j^i, \\ M_j^i = M_j^i - F_j^i, \\ \dots \\ M_j^i = M_j^i + F_j^i, \end{cases}$$

give the dual coefficients of a nonlinear connection N , determined by the k -spray only with the coefficients G^i .

In the previous formula, F_j^i , ($\alpha = 2, \dots, k$) is given by the following formulae:

$$(2.8) \quad \alpha F_j^i = \Gamma(F_j^i) + M_m^i F_j^m + F_m^i M_j^m - F_m^i F_j^m.$$

Proof. We construct the dual coefficients, starting from the coefficients of k -spray (2.4):

$$M_{(1)}^i = \frac{\partial G^i}{\partial y^{(k)j}} = \frac{\partial \overset{\circ}{G}^i}{\partial y^{(k)j}} - F_j^i = M_{(1)}^i - F_j^i$$

$$M_{(2)}^i = \frac{1}{2} (\Gamma M_{(1)}^i + M_{(1)}^i M_{(1)}^m) = \frac{1}{2} [\Gamma(M_{(1)}^i - F_j^i) + (M_{(1)}^i - F_m^i)(M_{(1)}^m - F_j^m)] = M_{(2)}^i - F_j^i,$$

where

$$(2.9) \quad F_j^i = \frac{1}{2} [\Gamma(F_j^i) + \underset{(1)}{M_m^i} \overset{\alpha-1}{F_j^m} + F_m^i \underset{(1)}{M_j^m} - F_m^i F_j^m].$$

By induction over α we prove the relation (2.8). \square

Proposition 2.1 *In $k = 2$ case ($Osc^2 M$), we obtain a simple express to M_j^i if the nonlinear connection is integrable:*

$$(2.10) \quad \begin{cases} M_j^i = \underset{(1)}{M_j^i} - F_j^i, \\ M_j^i = \underset{(2)}{M_j^i} + \frac{1}{2} F_m^i F_j^m. \end{cases}$$

Proof. Starting with formula (2.9) we get:

$$\underset{(2)}{M_j^i} = \underset{(2)}{M_j^i} - \frac{1}{2} [\Gamma(F_j^i) + \underset{(1)}{M_m^i} \overset{\alpha-1}{F_j^m} + F_m^i \underset{(1)}{M_j^m} - F_m^i F_j^m].$$

and

$$\Gamma(F_j^i) + \underset{(1)}{M_m^i} \overset{\alpha-1}{F_j^m} + F_m^i \underset{(1)}{M_j^m} = y^{(1)r} \frac{\partial F_j^i}{\partial x^r} + F_m^i \gamma_{rj}^m y^{(1)r} + F_j^m \gamma_{rm}^i y^{(1)r} = F_{j|r}^i y^{(1)r}.$$

The nonlinear connection being integrable we have [4], $F_{ij|r} = 0$ ($|r$ is the covariant derivative with respect to Levi-Civita connection of R^n). Since $F_j^i = \gamma^{im} F_{mj}$ we get $F_{j|r}^i = g^{im} |_{r} F_{mj} + g^{im} F_{mj|r} = 0$. It follows:

$$\underset{(2)}{M_j^i} = \underset{(2)}{M_j^i} + \frac{1}{2} F_m^i F_j^m$$

\square

Proposition 2.2 *In $k = 2$ case, the coefficients of nonlinear connection are looking as follows:*

$$(2.11) \quad \begin{cases} N_j^i = \underset{(1)}{M_j^i} = \underset{(1)}{N_j^i} - F_j^i, \\ N_j^i = \underset{(2)}{N_j^i} - \Gamma(F_j^i) - \frac{1}{2} F_m^i F_j^m, \end{cases}$$

One can prove, without difficulties, the following Proposition:

Proposition 2.3 *The adapted basis to the nonlinear connection N is given by:*

$$(2.12) \quad \begin{cases} \frac{\delta}{\delta x^i} = \frac{\overset{\circ}{\delta}}{\delta x^i} + F_i^j \frac{\overset{\circ}{\delta}}{\delta y^{(1)j}} + (\frac{1}{2} F_m^i F_j^m - \Gamma(F_j^i)) \frac{\partial}{\partial y^{(2)j}}, \\ \frac{\delta}{\delta y^{(1)i}} = \frac{\overset{\circ}{\delta}}{\delta y^{(1)i}} + F_i^j \frac{\partial}{\partial y^{(2)j}} \frac{\delta}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}, \end{cases}$$

where $\{ \frac{\overset{\circ}{\delta}}{\delta x^i}, \frac{\overset{\circ}{\delta}}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}} \}$ is the adapted basis to the nonlinear connection $\overset{\circ}{N}$.

The next problem is to give the N-linear connection. These will be resolved in a forthcoming paper.

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