

# THE SPECTRUM OF THE LAPLACE OPERATOR FOR BOUNDED DOMAINS IN $\mathbb{R}^n$

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## Abstract

Let  $M$  be a bounded domain in  $\mathbb{R}^n$ . The aim of the present paper is to study the influence of the spectra of the Laplace operator obtained by using Dirichlet's and Neumann's conditions on the geometry of the  $\partial M$  and  $M$ .

**AMS Subject Classification:** 58G25.

**Key words:** Bounded domain, Dirichlet's condition, Neumann's condition, Laplace operator and boundary of a manifold.

## 1 Introduction.

Let  $(M, g)$  be a bounded domain in  $\mathbb{R}^n$  which is a Riemannian manifold with boundary. To this manifold we can associate the vector spaces  $\Lambda^q(M, \mathbb{R})$ ,  $q = 0, 1, \dots, n = \dim M$ . Let  $\Delta_q$  be the Laplace operator acting on  $\Lambda^q(M, \mathbb{R})$ . It is known that  $\Lambda^q(M, \mathbb{R})$  can be decomposed in two parts. One part contain the  $q$ -forms tangential to  $\partial M$ , the other part includes the  $q$ -forms, which vanish on  $\partial M$ . The Laplace operator acting on these two spaces gives two spectra. We study the influence of these two spectra on the geometry of the manifold  $(M, g)$  and its boundary  $\partial M$ .

The whole paper contain three sections. Each of them is analyzed as follows.

The first section is the introduction. The theory of bounded domains in  $\mathbb{R}^2$  with the relation of the spectrum of the Laplace operator is study in the second section. This is a special Riemannian manifold with boundary. The third section contains a similar problem as in the second section but for  $\mathbb{R}^n$ ,  $n \geq 3$ .

## 2 .

Let  $M$  be a bounded domain in  $\mathbb{R}^2$  referred to an orthogonal coordinate system  $Oxy$ .

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The Laplace operator  $\Delta$  on  $\mathbb{R}^2$  takes the form:

$$\Delta = - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

We consider the set of functions  $f$  on  $M$  with the property

$$S_1(M) = \{f : M \rightarrow \mathbb{R} \mid f|_{\partial M} = 0\} \text{ (Dirichlet's condition).}$$

We also consider the set of functions  $f$  on  $M$  with the property

$$S_2(M) = \{f : M \rightarrow \mathbb{R} \mid [f]_N = 0\} \text{ (Neumann's condition),}$$

where  $[f]_N$  is the derivative of  $f$  in the direction of the normal  $N$  to the curve  $c = \partial M$ . Neumann's conditions can be generalized as follows:

$$[f]_N + \varphi = 0 \text{ instead of } [f]_N = 0 \text{ (Neumann's condition),}$$

where  $\varphi$  is a given function on  $\mathbb{R}^2$ . In many cases we study with the condition  $\varphi = 0$ .

The Laplace operator is a linear operator on each of the vector spaces  $S_1(M)$  and  $S_2(M)$ .  $\Delta$  is a self adjoint elliptic operator having discrete spectrum

$$Sp(M)_d = \{0 = \lambda_{0,d} < \lambda_{1,d} = \dots < \dots < \lambda_{k,d} = \dots < \infty\} \quad (1)$$

acting on  $S_1(M)$  and

$$Sp(M)_n = \{0 = \lambda_{1,n} < \lambda_{1,n} = \dots < \dots < \lambda_{j,n} = \dots < \infty\} \quad (2)$$

acting on  $S_2(M)$ .

In (1) and (2) each eigenvalue has finite multiplicity.

In order to study the influence of (1), (2) or both we construct the following functions:

$$f_d(t) = \sum_{i=0}^{\infty} \mu_{i,d} e^{-\lambda_{i,d} t} \text{ and } f_n(t) = \sum_{i=0}^{\infty} \mu_{i,n} e^{-\lambda_{i,n} t}, \quad (3)$$

where  $\mu_{i,d}$  and  $\mu_{i,n}$  are the multiplicities of  $\lambda_{i,d}$  and  $\lambda_{i,n}$  respectively.

The asymptotic expansions of these functions are:

$$f_d(t) = \sum_{i=0}^{\infty} \mu_{i,d} e^{-\lambda_{i,d} t} \equiv (4\pi t)^{-1/2} \left\{ \alpha_{0,d} + \alpha_{1/2,d} t^{1/2} + \alpha_{1,d} t + \alpha_{3/2,d} t^{3/2} + \dots \right\}, \quad (4)$$

$$f_n(t) = \sum_{i=0}^{\infty} \mu_{i,n} e^{-\lambda_{i,n} t} \equiv (4\pi t)^{-1/2} \left\{ \alpha_{0,n} + \alpha_{1/2,n} t^{1/2} + \alpha_{1,n} t + \alpha_{3/2,n} t^{3/2} + \dots \right\}, \quad (5)$$

where

$$\alpha_{0,d} = \frac{1}{4\pi} \int_M dM = \frac{1}{4\pi} \text{Area}(M), \quad (6)$$

$$\alpha_{1/2,d} = \frac{1}{8\sqrt{\pi}} \int_{\partial M} \partial M ds, \quad (7)$$

$$\alpha_{1,d} = \frac{1}{24\pi} \int_{\partial M} R ds, \quad (8)$$

$$\alpha_{3/2,d} = \frac{1}{256\sqrt{\pi}} \int_{\partial M} R^2 ds, \quad (9)$$

$$\alpha_{2,d} = \frac{1}{315\pi} \int_{\partial M} R^{3/2} ds, \quad (10)$$

$$\alpha_{0,n} = \frac{1}{4\pi} \int_M dM = \frac{1}{4\pi} \text{Area}(M), \quad (11)$$

$$\alpha_{1/2,n} = \frac{1}{8\sqrt{\pi}} \int_{\partial M} ds, \quad (12)$$

$$\alpha_{1,n} = \frac{1}{24\pi} \int_{\partial M} R ds, \quad (13)$$

$$\alpha_{3/2,n} = \frac{5}{4\sqrt{\pi}} \int_{\partial M} R^2 ds, \quad (14)$$

$$\alpha_{2,n} = \frac{4}{45\pi} \int_{\partial M} R^{3/2} ds, \quad (15)$$

where  $R$  is the curvature of  $c$ .

Now we can put some problems.

1st **PROBLEM.** Compute the spectra  $Sp(M)_d$  and  $Sp(M)_n$  when  $M$  is a given bounded domain in  $\mathbb{R}^2$ . It is obvious that  $Sp(M)_n$  depends on  $\varphi$ . In the above case we have used  $\varphi=0$ .

2nd **PROBLEM.** What is the influence of  $Sp(M)_d$  or  $Sp(M)_n$  or both on the geometry of  $M$ ?

3rd **PROBLEM.** Let  $M_1$  and  $M_2$  be two bounded domains of  $\mathbb{R}^2$ . If we assume

$$Sp(M_1)_d = Sp(M_2)_d \quad \text{or} \quad Sp(M_1)_n = Sp(M_2)_n$$

or both equalities are valid, is there an isometry  $r$  on  $\mathbb{R}^2$  such that  $r(M_1) = M_2$ ?

4th **PROBLEM.** Which is the influence of  $Sp(M)_d$  or  $Sp(M)_n$  or both of these on the topological structure of the bounded domain  $M$ ?

5th **PROBLEM.** Compute the other coefficients  $\alpha_{\nu/2,d}, \nu = 5, 6, \dots$  and  $\alpha_{\nu/2,n}, \nu = 5, 6, \dots$  in the asymptotic expansions (4) and (5) respectively.

We have some partial results to the above problems, but they still remain many open questions.

We must notice that these coefficients

$$\alpha_{\nu/2,d} \text{ and } \alpha_{\nu/2,n}, \quad v = 0, 1, \dots$$

are isospectral invariants, that means, if we have:

$$Sp(M_1)_d = Sp(M_2)_d \text{ or } Sp(M_1)_n = Sp(M_2)_n$$

then

$$\alpha_{\nu/2,d}(M_1) = \alpha_{\nu/2,d}(M_2), \quad v = 0, 1, 2, \dots$$

or

$$\alpha_{\nu/2,n}(M_1) = \alpha_{\nu/2,n}(M_2), \quad v = 0, 1, 2, \dots$$

Now, we state some of the known results.

**Theorem 2.1** *Let  $M_1$  and  $M_2$  be two bounded domains in  $\mathbb{R}^2$ . If  $Sp(M_1)_d = Sp(M_2)_d$  or  $Sp(M_1)_n = Sp(M_2)_n$ , then  $Area(M_1) = Area(M_2)$ .*

*Proof.* This is a consequence of the formula (6) or (13).□

**Theorem 2.2** *Let  $M_1$  and  $M_2$  be two bounded domains in  $\mathbb{R}^2$ . If  $Sp(M_1)_d = Sp(M_2)_d$  or  $Sp(M_1)_n = Sp(M_2)_n$ , then  $length(\partial M_1) = length(\partial M_2)$ .*

*Proof.* We can prove this using the formulas (7) and (12).□

**Theorem 2.3** *We consider two bounded domains  $M_1$  and  $M_2$  in  $\mathbb{R}^2$ . If  $Sp(M_1)_d = Sp(M_2)_d$  or  $Sp(M_1)_{n_1} = Sp(M_2)_n$ , then the topological structure of  $M_1$  is the same with  $M_2$ .*

*Proof.* We have:

$$\iint_M K dM + \int_{\partial M} k_g ds = \sum_{\tau=1}^k i_\tau - \pi$$

and since  $K = 0$ , the above formula becomes

$$\int_{\partial M} (k_g = R) ds = \sum_{\tau=1}^{\sigma} i_\tau - \pi. \quad \square$$

**Theorem 2.4** *Let  $M_1$  and  $M_2$  be two bounded domains in  $\mathbb{R}^2$ . If  $Sp(M_1)_d = Sp(M_2)_d$  or  $Sp(M_1)_n = Sp(M_2)_n$  and  $M_1$  is a disc, then so is  $M_2$ .*

*Proof.* Since  $M_1$  is a disc, then  $\partial M_1$  is a circle. From (8), (9) and (15) we obtain that

$$R_1 = c \text{ and } R_1 = R_2 = c$$

and therefore  $M_2$  is a disc and there is an isometry  $r$  on  $\mathbb{R}^2$  such that  $r(M_1) = M_2$ . □

However for a long time it was open the following conjecture.

**Conjecture 2.5** Let  $M_1$  and  $M_2$  be two bounded domains in  $\mathbb{R}^2$  with the property  $Sp(M_1)_d = Sp(M_2)_d$  or  $Sp(M_1)_n = Sp(M_2)_n$ . This means that  $M_1$  and  $M_2$  are isospectral. Can we construct  $M_1$  and  $M_2$  in a such way so that  $M_1$  is not congruent onto  $M_2$ ?

**Theorem 2.6** There are two bounded domains  $M_1$  and  $M_2$  in  $\mathbb{R}^2$ , which are described in [12] with the property  $Sp(M_1)_d = Sp(M_2)_d$ , but they are not congruent.

6th **PROBLEM.** Construct more examples with the property of the theorem 2.6.

### 3 .

We consider  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . Let  $\{e_1, \dots, e_n\}$  be an orthonormal base of  $\mathbb{R}^n$  and  $(x_1, \dots, x_n)$  its natural coordinate system.

Let  $M$  be a bounded domain in  $\mathbb{R}^n$ . The Laplace operator  $\Delta$  on  $\mathbb{R}^n$  has the form:

$$\Delta = - \left( \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right). \quad (16)$$

From the set  $D^0(\mathbb{R}^n) = \{f : \mathbb{R}^n \rightarrow \mathbb{R}\}$  of all differentiable functions on  $\mathbb{R}^n$  we construct the following subsets  $S(M)_d$  and  $S(M)_n$  defined by:

$$S(M)_d = \{f \in D^0(\mathbb{R}^n) \mid f|_{\partial M} = 0\} \text{ (Dirichlet's condition),} \quad (17)$$

$$S(M)_n = \{f \in D^0(\mathbb{R}^n) \mid [f]_N + \varphi = 0\} \text{ (Neumann's condition),} \quad (18)$$

where  $\varphi$  is an arbitrary function on  $\mathbb{R}^n$  and  $[f]_N$  is the derivative of the function  $f$  to the direction of the normal  $N$ .

From these vector spaces  $S(M)_d$  and  $S(M)_n$ , by means of the Laplace operator  $\Delta$ , we obtain the spectra:

$$Sp(M)_d = \{0 < \lambda_{1,d} = \lambda_{1,d} = \dots < \lambda_{2,d} = \dots < \lambda_{k,d} = \dots < \infty\}, \quad (19)$$

$$Sp(M)_n = \{0 < \lambda_{1,n} = \lambda_{1,n} = \dots < \lambda_{2,n} = \dots < \lambda_{j,n} = \dots < \infty\} \quad (20)$$

respectively.

We construct the two functions obtaining their asymptotic expansions

$$f_d(t) = \sum_{i=0}^{\infty} \mu_{i,d} e^{-\lambda_{i,d} t} \cong (4\pi t)^{-n/2} \{\alpha_{0,d} + \alpha_{1/2,d} t^{1/2} + \alpha_{1,d} t + \alpha_{3/2,d} t^{3/2} + \alpha_{2,d} t^2 + \dots\}, \quad (21)$$

$$f_n(t) = \sum_{i=0}^{\infty} \mu_{i,n} e^{-\lambda_{i,n} t} \cong (4\pi t)^{-n/2} \{\alpha_{0,n} + \alpha_{1/2,n} t^{1/2} + \alpha_{1,n} t + \alpha_{3/2,n} t^{3/2} + \alpha_{2,n} t^2 + \dots\}, \quad (22)$$

where

$$\alpha_{0,d} = (4\pi)^{-n/2} \int_M dM = (4\pi)^{-n/2} Vol(M), \quad (23)$$

$$\alpha_{1/2,d} = -4^{-1}(4\pi)^{-(n-1)/2} \int_{\partial M} d(\partial M) 4^{-1}(4\pi)^{-(n-1)/2} Vol(M), \text{ if } n \geq 4, \quad (24)$$

or

$$-4^{-1}(4\pi)^{-(n-1)/2} Area(\partial M), \text{ if } n = 3, \quad (25)$$

$$\alpha_{1,d} = 6^{-1}(4\pi)^{-n/2} \int_{\partial M} 2L_{aa}d(\partial M), \quad (26)$$

$$\alpha_{3/2,d} = (384)^{-1}(4\pi)^{-(n-1)/2} \int_{\partial M} (-7L_{aa}L_{bb} + 10L_{ab}L_{ab})d(\partial M), \quad (27)$$

$$\begin{aligned} \alpha_{2,d} = \\ (360)^{-1}(4\pi)^{-4/2} \int_{\partial M} \frac{1}{21}(40L_{aa}L_{bb}L_{cc} - 264L_{ab}L_{ab}L_{cc} + 320L_{ab}L_{ab}L_{bc}L_{ac})d(\partial M), \end{aligned} \quad (28)$$

$$\alpha_{0,n} = (4\pi)^{-n/2} \int_M dM = (4\pi)^{-n/2} Vol(M), \quad (29)$$

$$\alpha_{1/2,n} = 4^{-1}(4\pi)^{(n-1)/2} \int_{\partial M} d(\partial M) = -4^{-1}(4\pi)^{(n-1)/2} Vol(\partial M), \quad (30)$$

if  $n \geq 4$  or

$$4^{-1}(4\pi)^{(n-1)/2} Area(M), \text{ if } n = 3, \quad (31)$$

$$\alpha_{1,n} = 6^{-1}(4\pi)^{-n/2} \int_{\partial M} L_{aa}d(\partial M), \quad (32)$$

$$\alpha_{3/2,n} = (384)^{-1}(4\pi)^{-(n-1)/2} \int_{\partial M} (13L_{aa}L_{bb} + 2L_{ab}L_{ab})d(\partial M), \quad (33)$$

$$\alpha_{2,n} = \frac{(360)^{-1}(4\pi)^{-n/2}}{21} \int [180L_{aa}L_{bb}L_{cc} + 168L_{ab}L_{ab}L_{cc} + L_{ab}L_{bc}L_{ca}]d(\partial M), \quad (34)$$

where  $L = (L_{ab})$  is the second fundament form on the boundary.

It must be noticed that the local coordinate system  $(x_1, \dots, x_n)$  is taken such that  $\frac{\partial}{\partial x_n}$  is the normal to  $\partial M$ .

Now, we can put the same problems as in the case of  $\mathbb{R}^2$ , that means we substitute  $\mathbb{R}^2$  by  $\mathbb{R}^n$ ,  $n \geq 3$ .

Now, we have the following theorems.

**Theorem 3.1** *Let  $M_1$  and  $M_2$  be two bounded domains in  $\mathbb{R}^n$ . If  $Sp(M_1)_d = Sp(M_2)_d$  or  $Sp(M_1)_n = Sp(M_2)_n$ , then  $Vol(M_1) = Vol(M_2)$ .*

*Proof.* This is obtained from (16) since the coefficients  $\alpha_{\nu/2,d}$ ,  $\nu = 0, 1, \dots$  and  $\alpha_{\nu/2,n}$  are isospectral invariants.  $\square$

**Theorem 3.2** We consider two bounded domains in  $\mathbb{R}^n$ . We assume that

$$Sp(M_1)_d = Sp(M_2)_d \text{ or } Sp(M_1)_n = Sp(M_2)_n.$$

Then  $Vol(\partial M_1) = Vol(\partial M_2)$  if  $n \geq 4$  or  $Area(\partial M_1) = Area(\partial M_2)$  if  $n = 3$ .

*Proof.* Since we have:

$$\alpha_{1/2,d}(M_1) = \alpha_{1/2,d}(M_2) \text{ or } \alpha_{1/2,n}(M_1) = \alpha_{1/2,n}(M_2),$$

we obtain that  $Vol(\partial M_1) = Vol(\partial M_2)$  if  $n \geq 4$  or  $Area(M_1) = Area(M_2)$ , if  $n = 3$ .  $\square$

**Theorem 3.3** Let  $M_1$  and  $M_2$  be two bounded domains in  $\mathbb{R}^n$ . If we assume that

$$Sp(M_1)_d = Sp(M_2)_d \text{ and } Sp(M_1)_n = Sp(M_2)_n \quad (35)$$

and  $\partial M_1$  is totally geodesic, then so is  $\partial M_2$ .

*Proof.* From (35) we conclude that

$$\alpha_{\nu/2,d}(M_1) = \alpha_{\nu/2,d}(M_2), \quad \nu = 0, 1, 2, \dots, \quad (36)$$

$$\alpha_{\nu/2,n}(M_1) = \alpha_{\nu/2,n}(M_2), \quad \nu = 0, 1, 2, \dots, \quad (37)$$

which by means of (26), (27), (28) and (32), (33) and (34) we obtain:

$$\int_{\partial M_1} L_{aa} d(\partial M_1) = \int_{\partial M_2} L'_{aa} d(M_2), \quad (38)$$

$$\int_{\partial M_1} (-7L_{aa}L_{bb} + 10L_{ab}L_{ab})d(\partial M_1) = \int_{\partial M_2} (-7L'_{aa}L'_{bb} + 10L'_{ab}L'_{ab})d(\partial M_2), \quad (39)$$

$$\begin{aligned} & \int_{\partial M_1} (40L_{aa}L_{bb}L_{cc} - 264L_{ab}L_{ab}L_{cc} + 320L_{ab}L_{bc}L_{ca})d(\partial M_1) = \\ & = \int_{\partial M_2} (40L'_{aa}L'_{bb}L'_{cc} - 264L'_{ab}L'_{ab}L'_{cc} + 320L'_{ab}L'_{bc}L'_{ca})d(\partial M_2), \end{aligned} \quad (40)$$

$$\int_{\partial M_1} L_{aa} d(\partial M_1) = \int_{\partial M_2} L'_{aa} d(\partial M_2), \quad (41)$$

$$\int_{\partial M_1} (13L_{aa}L_{bb} + 2L_{ab}L_{ab})d(\partial M_1) = \int_{\partial M_2} (13L'_{aa}L'_{bb} + 2L'_{ab}L'_{ab})d(\partial M_2), \quad (42)$$

$$\int_{\partial M_1} (180L_{aa}L_{bb}L_{cc} + 168L_{ab}L_{ab}L_{cc} + 224L_{ab}L_{bc}L_{ca})d(\partial M_1) =$$

$$= \int_{\partial M_2} (180L'_{aa}L'_{bb}L'_{cc} + 168L'_{ab}L'_{bc}L'_{ca})d(\partial M_2), \quad (43)$$

where  $L_{ab}$  and  $L'_{ab}$  are the coefficients of the second fundamental forms on  $\partial M_1$  and  $\partial M_2$  respectively.

Since the manifold  $M_1$  is totally geodesic, it implies

$$L \equiv 0 \iff L_{ab} = 0. \quad (44)$$

The relations (38)-(42) by means of (43) yield:

$$\int_{\partial M_2} L'_{aa}d(\partial M_2) = 0, \quad (45)$$

$$\int_{\partial M_2} (-7L'_{aa}L'_{bb} + 10L'_{ab}L'_{ab})d(\partial M_2) = 0, \quad (46)$$

$$\int_{\partial M_2} (40L'_{aa}L'_{bb}L'_{cc} - 264L'_{ab}L'_{ab}L'_{cc} - 320L'_{ab}L'_{bc}L'_{ca})d(\partial M_2) = 0, \quad (47)$$

$$\int_{\partial M_2} (13L'_{aa}L'_{bb} + 2L'_{ab}L'_{ab})d(\partial M_2) = 0, \quad (48)$$

$$\int_{\partial M_2} (180L'_{aa}L'_{bb}L'_{cc} + 168L'_{ab}L'_{ab}L'_{cc} + 224L'_{ab}L'_{bc}L'_{ca})d(\partial M_2) = 0. \quad (49)$$

From (45) and (47) we conclude that

$$\int_{\partial M_2} L'_{aa}L'_{bb}d(\partial M_2) = 0 \text{ and } \int_{\partial M_2} L'_{ab}L_{ab}d(\partial M_2) = 0. \quad (50)$$

The relations (49) yield:

$$L'_{ab} = 0 \iff L' \equiv 0, \quad (51)$$

which means that  $\partial M_2$  is totally geodesic.  $\square$

**Lemma 3.1** *Let  $L = (L_{ab})$  be the second fundamental form on  $\partial M$ , where  $M$  is a bounded domain in  $\mathbb{R}^n$ . If we consider the function*

$$\delta(L) = (n-1)L_{ab}L_{ab} - L_{aa}L_{bb}, \quad (52)$$

*then we have:*

$$\delta(L) \geq 0 \quad (53)$$

*and*

$$\delta(L) = 0 \quad (54)$$

*if, and only if,  $\partial M$  is totally umbilical.*

**Theorem 3.4** *Let  $M_1$  and  $M_2$  be two bounded domains in  $\mathbb{R}^n$  with the properties  $Sp(M_1)_d = Sp(M_2)_d$  and  $Sp(M_1)_n = Sp(M_2)_n$ . If  $\partial M_1$  is totally umbilical, then so is  $\partial M_2$ .*

*Proof.* From the property

$$Sp(M_1)_d = Sp(M_2)_d \text{ and } Sp(M_1)_n = Sp(M_2)_n \quad (55)$$

we conclude that

$$\alpha_{3/2,d}(M_1) = \alpha_{3/2,d}(M_2), \quad (56)$$

$$\alpha_{3/2,n}(M_1) = \alpha_{3/2,n}(M_2), \quad (57)$$

which by means of (27) and (33) imply:

$$\int_{\partial M_1} (-7L_{aa}L_{bb} + 10L_{ab}L_{ab})d(M_1) = \int_{\partial M_2} (-7L'_{aa}L'_{bb} + 10L'_{ab}L'_{ab})d(\partial M_2), \quad (58)$$

$$\int_{\partial M_1} (13L_{aa}L_{bb} + 2L_{ab}L_{ab})d(\partial M_1) = \int_{\partial M_2} (13L'_{aa}L'_{bb} + 2L'_{ab}L'_{ab})d(\partial M_2) \quad (59)$$

respectively.

The relations (58) and (59) can be written:

$$-7X_1 + 10X_2 = 0, \quad (60)$$

$$13X_1 + 2X_2 = 0 \quad (61)$$

respectively, where

$$X_1 = \int_{\partial M_1} (L_{aa}L_{bb}d(\partial M_1) - \int_{\partial M_2} L'_{aa}L'_{bb}d(\partial M_2)), \quad (62)$$

$$X_2 = \int_{\partial M_2} L_{ab}L_{ab}d(\partial M_1) - \int_{\partial M_2} L'_{ab}L'_{ab}d(\partial M_2). \quad (63)$$

From the system (60) and (61) and since

$$\begin{vmatrix} -7 & 10 \\ 13 & 2 \end{vmatrix} = -14 - 130 \neq 0,$$

we conclude that

$$X_1 = X_2 = 0,$$

which by means of (19) and (63) yield:

$$\int_{\partial M_1} L_{aa}L_{bb}d(\partial M_1) = \int_{\partial M_2} L'_{aa}L'_{bb}d(\partial M_2), \quad (64)$$

$$\int_{\partial M_1} L_{ab}L_{ab}d(\partial M_1) = \int_{\partial M_2} L'_{ab}L'_{ab}d(\partial M_2). \quad (65)$$

From (64) and (65) we obtain:

$$\int_{\partial M_1} ((n-1)L_{aa}L_{bb} - L_{ab}L_{ab})d(\partial M_1) = \int_{\partial M_2} \{(n-1)L'_{aa}L'_{bb} - L'_{ab}L'_{ab}\}d(\partial M_2). \quad (66)$$

Since  $\partial M_1$  is totally umbilical we conclude that

$$\delta(L) = (n-1)L_{aa}L_{bb} - L_{ab}L_{ab} = 0. \quad (67)$$

The relation (66) by means of (67) implies:

$$\int_{\partial M_2} \delta(L')d(\partial M_2) = \int_{\partial M_2} \{(n-1)L'_{aa}L'_{bb} - L'_{ab}L'_{ab}\}d(\partial M_2) = 0. \quad (68)$$

Since we have:

$$\delta(L') \geq 0 \quad (69)$$

then (68) gives:

$$\delta(L') = (n-1)L'_{aa}L'_{bb} - L'_{ab}L'_{ab} = 0, \quad (70)$$

which means that  $\partial M_2$  is totally umbilical.  $\square$

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