SYMMETRIC INVARIANT NON-DEGENERATE BILINEAR FORMS ON NILPOTENT LIE ALGEBRAS

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Abstract

Let g be a Nilpotent Lie algebra of finite dimension. The aim of the present work is to study symmetric invariant non-degenerate bilinear forms on g.

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1 Introduction

Let g be a nilpotent Lie algebra of finite dimension n over the field $K = \mathbf{R}$ or $K = \mathbf{C}$. It is known that there is a complete classification of g until dimension 7 ([2]) and, in some cases, of dimension 8 and 9 ([3, 4, 5, 6]). The existence of a symmetric invariant non-degenerate bilinear form on a nilpotent Lie algebra is an open problem.

The aim of the present paper is to find out which of these known nilpotent Lie algebras admit a symmetric invariant non-degenerate bilinear form.

The whole paper contains three sections. The first section contains the introduction. Some known results are given in the second section. From the complete classification of nilpotent Lie algebras up to dimension 7 ([2]) we determine which of them admit a symmetric, invariant, non-degenerate bilinear form. This is the content of the last section.

2 Some Known results

Let g be a nilpotent Lie algebra of finite dimension n over a field K of characteristic zero. A symmetric bilinear form

$$\varphi : g \times g \to K$$

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is called *non-degenerate*, when $N\varphi = \{ 0 \}$, where

$$N\varphi = \{x \in g \mid \varphi(x, v) = 0, \ \forall v \in g\}.$$

This φ is called *invariant*, if we have:

$$\varphi\left([x, y], z\right) = \varphi\left(x, [y, z]\right) \quad \forall x, y, z \in g.$$

The pair (g, φ) is called *regular quadratic Lie algebra*.

The classification of regular quadratic Lie algebras of dimension less or equal than 7 is given in ([2]) and there are the following four Lie algebras:

1. g_1 : $\dim g_1 = 1$, Abelian and φ_1 the standard inner product.

2. w_3 : $\dim w_3 = 5$, whose structure constants are:
$c_{12}^3 = -c_{21}^3 = 1$, $c_{13}^4 = -c_{31}^4 = 2$, $c_{23}^5 = -c_{32}^5 = -2$, all the others are zero.

3. w_4 : $\dim w_4 = 6$, whose structure constants are:
$c_{15}^4 = -c_{51}^4 = 1$, $c_{13}^2 = -c_{31}^3 = -1$, $c_{53}^6 = -c_{35}^6 = 1$, all the others are zero.

4. w_5 : $\dim w_5 = 7$, whose structure constants are:
$c_{12}^3 = -c_{21}^3 = 1$, $c_{13}^4 = -c_{31}^4 = 1$, $c_{14}^5 = -c_{41}^5 = 2$, $c_{15}^6 = -c_{51}^6 = -1$, $c_{25}^7 = -c_{52}^7 = 1$, $c_{43}^7 = -c_{34}^7 = 2$, all the others are zero.

3 Main results

All nilpotent Lie algebras up to dimension seven which are regular quadratic are given by the below theorem:

Theorem 1 *The nilpotent Lie algebra up to dimension seven, which are regular quadratic, are the following:*

(I) *All the Abelian with φ the common inner product.*

(II) *The indecomposable ones, which are:*

(II)$_a$ n_5^3 *isometric to* (w_3, φ_3), $\dim n_5^3 = 5$;

(II)$_b$ n_6^{21} *isometric to* (w_4, φ_4), $\dim n_6^{21} = 6$;

(II)$_c$ n_7^{12} *isometric to* (w_5, φ_5), $\dim n_7^{12} = 7$.

(III) *The decomposable ones, which are:*

(III)$_a$ $n_6^{32} = n_5^3 \oplus g_1$, *with* $\varphi = \varphi_1 + \varphi_3$, $\dim n_6^{32} = 6$;

(III)$_b$ $n_7^{153} = n_6^{21} \oplus g_1$, *with* $\varphi = \varphi_1 + \varphi_4$, $\dim n_7^{153} = 7$;

(III)$_c$ $n_7^{143} = n_5^3 \oplus g_2$, *with* $\varphi = \varphi_2 + \varphi_3$, $\dim n_7^{143} = 7$, *where g_k , $k = \overline{1,2}$ are the Abelian Lie algebras with the standard symmetric, invariant, non-degenerate bilinear forms on g_k , $k = \overline{1,2}$, respectively, the Lie algebras n_5^3 , n_6^{21} , n_7^{12} , n_6^{32} , n_7^{153} , and n_7^{143} are given in the classification ([2]), φ_1 , φ_2 are the standard Euclidean inner product and φ_3 , φ_4 , φ_5 are the symmetric, invariant, non-degenerate bilinear forms on n_5^3 , n_6^{21} and n_7^{12} , respectively.*

Proof. We distinguish three cases: Abelian, indecomposable and decomposable.

(I) ABELIAN CASES

Let g_k be an Abelian Lie algebra of dimension k over the field $\mathbf{R}$. Let $\{e_1, e_2, \dots, e_k\}$ be a base of g_k. If a and b are two elements of g_k, which can be written:

$$a = \sum_{i=1}^{k} a_i e_i \text{ and } b = \sum_{j=1}^{k} b_j e_j,$$

the symmetric, invariant, non-degenerate bilinear form φ_k on g_k is defined by

$$\varphi_k : g_k \times g_k \mapsto \mathbf{R}\,,\ \varphi_k\ : (a,b)\ \mapsto \varphi_k\,(a,b) = \sum_{i=1}^{k}\sum_{j=1}^{k} a_i b_j .$$

In order to study the non-Abelian nilpotent Lie algebras g, which admit a symmetric, invariant, non-degenerate, bilinear form φ, we will use the classification which is given in [2, 102-124].Therefore we try to determine the regular quadratic Lie algebras (g, φ) . We study separately each dimension.

(II) INDECOMPOSABLE CASES

(II)' $\dim g = 3$. There is only the n_3^1 . This is not regular quadratic.

Indeed: Let $\{e_1, e_2, e_3\}$ be a base of n_3^1 such that $[e_1, e_2] = e_3$ and the other Lie brackets are zero. Let φ be the required quadratic form on n_3^1. Therefore we have:

$\varphi(e_3, e_1) = \varphi([e_1, e_2], e_1) =$ (φ is invariant) $= \varphi(e_1, [e_2, e_1]) = \varphi(e_1, -e_3) \Rightarrow 2\varphi(e_3, e_1) = 0$.

It can be also proved that

$$\varphi(e_3, e_2) = 0 \text{ and } \varphi(e_3, e_3) = 0$$

and finally

$$\varphi(e_3, a_1e_1 + a_2e_2 + a_3e_3) = 0\,.$$

The above relation implies that $e_3 \in N\varphi = \{\,0\,\}$, which means that n_3^1 is not regular quadratic.

(II)" $\dim g = 4$. The only nilpotent Lie algebras not decomposable is n_4^1. This is not regular quadratic.

Indeed: We prove this in the same way as for n_3^1.

$(II)_a$ $\dim g = 5$. From the indecomposable Lie algebras of dimension five, which are: n_5^1, n_5^2, n_5^3, n_5^4, n_5^5, n_5^6 , the n_5^3 is the unique regular quadratic.

Indeed: Using the same technique as in the cases (II)' we conclude that the only indecomposable nilpotent Lie algebra which is regular quadratic is n_5^3. All the other are not regular quadratic. It is obvious that n_5^3 is isometric onto w_3 provided with the symmetric, invariant, non-degenerate bilinear form φ_3, which is given in [1].

(II)$_b$ $\dim g = 6$. It is known that the indecomposable Lie algebras of dimension six are n_6^i, $i = 1, 2, \ldots, 24$. The only regular quadratic is n_6^{21}.

Indeed: We use the same technique as the previous ones and we conclude that n_6^{21} is the only regular quadratic. All the others are not regular quadratic. This nilpotent Lie algebra n_6^{21} is isometric onto w_4, provided with symmetric, invariant, non-degenerate bilinear form φ_4, which is given in [1].

(II)$_c$ $\dim g = 7$. The indecomposable nilpotent Lie algebras of dimension seven are the following:

- $n_7^{1,a}$;
- n_7^i, $i = 2, \ldots, 11$;
- $n_7^{13,a}$;
- n_7^i, $i = 14, \ldots, 41, 47, \ldots, 52, 60, 61$;
- $n_7^{62,a}$;
- n_7^i, $i = 63, \ldots, 69$;
- $n_7^{70,a}$;
- n_7^i, $i = 71, \ldots, 77$;
- $n_7^{78,a}$;
- n_7^i, $i = 79, \ldots, 131$;
- $n_7^{132,a}$;
- n_7^i, $i = 133, 142, 147, \ldots, 152, 157$,

where $a \neq 0$ is a parameter defined in ([2]). The n_7^{12} is the only regular quadratic.

Indeed: If we use the same methods as in the case $\dim g = 3$, then we obtain that n_7^{12} is the only regular quadratic. All the others are not regular quadratic. This nilpotent Lie algebra n_7^{12} is isometric onto w_5 provided with the symmetric, invariant, non-degenerate bilinear form φ_5 which is given in [1].

(III) DECOMPOSABLE CASES

Let $(g_1, \neq f_1)$ and (g_2, f_2) be two regular quadratic Lie algebras with Lie brackets $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$, respectively. We consider the direct sum

$$g_1 \oplus g_2 \,,$$

which is a Lie algebra with Lie bracket

$$[X,Y]=[X_1,Y_1]_1+[X_2,Y_2]_2\,,$$

where $X=X_1+X_2$ and $Y=Y_1+Y_2$, $X_1,Y_1\in g_1$ and $X_2,Y_2\in g_2$.

On the Lie algebra $g=g_1\oplus g_2$ we define a symmetric bilinear form f as follows:

$$f:g\times g\mapsto K,$$

$$f:(X=X_1+X_2,Y=Y_1+Y_2)\mapsto f(X,Y)=f_1(X_1,Y_1)+f_2(X_2,Y_2)\,,$$

where $X_1,Y_1\in g_1$ and $X_2,Y_2\in g_2$.

It can be easily seen that f is invariant and non-degenerate. Therefore $(g=g_1\oplus g_2,\ f=f_1\oplus f_2)$ is regular quadratic.

Below we study the decomposable nilpotent Lie algebras of dimension less or equal than seven which are regular quadratic.

(III)' $\dim g=4$. There is only one such Lie algebra which is $n_4^2=n_3^1\oplus g_1$. This is not regular quadratic.

Indeed: If $\{e_1,e_2,e_3\}$ is a base of n_3^1 and $\{f_1\}$ is a base of g_1, then $\{(e_1,0),\,(e_2,0),\,(e_3,0),\,(0,f_1)\}$ is a base of $n_4^2=n_3^1\oplus g_1$.

The Lie brackets of n_4^2 are the following :

$$[(e_i,0),(0,f_1)]=(0,0),\ i=1,2,3 \text{ and } [(e_1,0),(e_2,0)]=(e_3,0)$$

and all the others are zero.

If φ is a symmetric, invariant, bilinear form on n_4^2, then we have:

$$\varphi\left([(x_1,y_1)\,,\,(x_2,y_2)]\,,\,(z,w)\right)=\varphi\left((x_1,y_1)\,,\,[(x_2,y_2)\,,\,(z,w)]\right),$$

for all $x_1,x_2,z\in n_3^1$ and $y_1,y_2,w\in g_1$. It can be easily proved that

$$\varphi\left((e_3,0)\,,\,(0,f_1)\right)=\varphi\left([(e_1,0)\,,\,(e_2,0)]\,,(0,f_1)\right)=$$

$$=\varphi\left((e_1,0)\,,\,(0,0)\right)=0,$$

$$\varphi\left((e_3,0)\,,\,(e_1,0)\right)=\varphi\left((e_3,0)\,,\,(e_2,0)\right)=$$

$$=\varphi\left((e_3,0)\,,\,(e_3,0)\right)=0.$$

Therefore we obtain that

$$(e_3,0)\,\in\,N\varphi=\{\,(0,0)\,\},$$

which implies that n_4^2 is not regular quadratic.

(III)" $\dim g=5$. All the decomposable nilpotent Lie algebras of dimension five are n_5^7, n_5^8 and n_5^9, which are not regular quadratic.

Indeed: This can be proved with the same method as in (III)'.

(III)$_a$ $\dim g=6$. The decomposable nilpotent Lie algebras of dimension 6 are n_6^i, $i=25,\ldots,35$. The n_6^{32} is the only regular quadratic.

Indeed: Using the same method as in (III)' we can prove that only $n_6^{32} = n_5^3 \oplus g_1$ with the bilinear form $\varphi = \varphi_1 + \varphi_3$ is regular quadratic and the others n_6^i, $i = 25, \ldots, 31, 33$ are not regular quadratic.

$\mathbf{(III)}_b + \mathbf{(III)}_c$, $\dim g = 7$. The following Lie algebras: n_7^i, $i = 42, \ldots, 46, 53, \ldots, 59, 134, \ldots, 141$; $i = 143, \ldots, 146$; $i = 153, \ldots, 156$; $i = 158$; $i = 159$ of dimension seven are decomposable. The n_7^{153} and n_7^{143} are regular quadratic.

Indeed: Using the techniques of (III)' we obtain that only $n_7^{153} = n_6^{21} \oplus g_1$, with bilinear form $\varphi_1 + \varphi_4$ and n_7^{143} with the bilinear form $\varphi_2 + \varphi_3$ are regular quadratic. All the others are not regular quadratic. □

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References

[1] Favre G., Santharoubane L. J., *Symmetric, invariant, non-degenerate bilinear form on a Lie algebra*, Journal of Algebra, Vol.105, No 2, 1987, 451- 464.

[2] Piu P., Goze M., *Gruppi e Algebre di Lie*, appunti per un seminario, Universita degli studi di Cagliari, Dipartimento di Mathematica, 1991.

[3] Tsagas Gr., Kobotis A., *Classification of special nilpotent Lie algebras of dimension eight and their invariants*, Tensor N.S., Vol. 47, No 2, 1988, 116-118.

[4] Tsagas Gr., Kobotis A., *Classification of special nilpotent Lie algebras of dimension nine and their invariants*, Tensor N.S., Vol. 50, No 1, 1991, 56-59.

[5] Tsagas Gr., Kobotis A., *Characteristic elements of a category of nilpotent Lie algebras of dimension eight*, Algebras, Groups and Geometries 9, 1992, 137- 256.

[6] Tsagas Gr., Kobotis A., *Dynkin diagrams and other elements of class of nilpotent Lie algebras of dimension nine* , Algebras, Groups and Geometries.

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