ON A CLASS OF 3 - τ - MANIFOLDS

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Abstract

In the present communication we prove that every 3-dimensional contact metric manifold satisfying with $\nabla_\xi \tau = 0$ and η-parallel tensor field C (given by (2)) has constant scalar curvature in the direction of ξ. Also, every 3-dimensional Sasakian manifold with η-parallel tensor field C has constant curvature 1.

AMS Subject Classification: 53C15, 53C25.
Key words: Contact metric manifold, τ-manifold

1 INTRODUCTION

D. E. Blair and T. Koufogiorgos proved in [2] that a conformally flat contact metric manifold M^{2n+1} with $Q\varphi = \varphi Q$ is of constant curvature 1 if $n > 1$ and 0 or 1 if $n = 1$. Also, they proved that on a 3-dimensional contact metric manifold the condition $Q\varphi = \varphi Q$ implies $\ell\varphi = \varphi\ell$. D. Perrone proved in [5] that on every contact metric manifold the conditions $\ell\varphi = \varphi\ell$, $\nabla_\xi h = 0$, $\nabla_\xi \ell = 0$ and $\nabla_\xi \tau$ (where $\tau = L_\xi g$) are equivalent. A 3 - dimensional contact metric manifold satisfying $\nabla_\xi \tau = 0$ is called 3-τ - manifold.
In the present communication some new results on a class of 3-τ- manifolds are given.

2 PRELIMINARIES

A real contact manifold is a C^∞ manifold M^{2n+1} endowed with a 1- form η such that $\eta \wedge (d\eta)^n \neq 0$. In this case there always exists a unit vector field ξ, called

Editor Gr.Tsagas *Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, 1996,* 170-174

the characteristic vector field of the contact structure η, such that $d\eta(\xi, Y) = 0$ and $\eta(\xi) = 1$. The distribution defined by the subspace $\{X \in T_pM : \eta(X) = 0\}$, $p \in M$ is called *contact subbundle* or *contact distribution* and is denoted by B.

Given a contact manifold M_{2n+1} we can always globally define a Riemannian metric g and a $(1,1)$ tensor field φ which satisfies the conditions:

$$\varphi^2 = -id + \eta \otimes \xi\,,\ \eta(X) = g(X, \xi)\ ,\ d\eta(X, Y) = g(X, \varphi Y).$$

The metric g is called an *associated metric* and the structure defined on M^{2n+1} by the tensor fields (φ, ξ, η, g) a *contact metric structure*. For details we refer to [1]. Denoting by L the Lie derivation, we define the tensor field $h = \frac{1}{2}L_\xi\varphi$ which plays a fundamental role. h is symmetric and anticommutes with φ, also h vanishes if and only if ξ is Killing and $h\xi = 0$. When ξ is Killing, the contact metric structure is said to be *K-contact*. We also define a tensor field ℓ by $\ell X = R(X, \xi)\xi$, where R is the curvature tensor of the contact metric structure. On every contact metric manifold M^{2n+1} we have the following formulas, which involve the tensor fields h and ℓ (see e.g. [1]):

$$\nabla_Y\xi = -\varphi Y - \varphi hY\,,\ \varphi\ell\varphi - \ell = 2(\varphi^2 + h^2)\,,$$

$$\nabla_\xi h = \varphi - \varphi\ell - \varphi h^2\,,\ Tr\,\ell = g(Q\xi, \xi) = 2n - Tr\,h^2\,,$$

where ∇ is the Riemannian connection and Q is the Ricci operator. If f is a real function and the almost complex structure J on $M^{2n+1} \times \mathbf{R}$ defined by $J\left(Y, f\frac{d}{dt}\right) = \left(\varphi Y - f\xi, \eta(Y)\frac{d}{dt}\right)$, is integrable, then the structure is said to be *normal* and the manifold is called *Sasakian*. A Sasakian manifold may be characterized by the relation: $R(Y, Z)\xi = \eta(Z)Y - \eta(Y)Z$, for all vector fields Y, Z on the manifold. A Sasakian manifold is a K-contact manifold. The inverse is true only for dimension 3.

On every 3 - dimensional Riemannian manifold M the curvature tensor $R(Y, Z)W$ is given by

$$\begin{aligned} R(Y, Z)W &= g(Z, W)QY - g(Y, W)QZ + g(QZ, W)Y - g(QY, W)Z - \\ &\quad -\frac{S}{2}\left[g(Z, W)Y - g(Y, W)Z\right], \end{aligned} \tag{1}$$

where Q is the Ricci operator, $S = Tr\,Q$ is the scalar curvature and Y, Z and W are arbitrary vector fields. We also define on a contact metric manifold M^3 the tensor field C by

$$C(Y, Z) = (\nabla_Y Q)Z - (\nabla_Z Q)Y - \frac{1}{4}\left[(Y \cdot S)Z - (Z \cdot S)Y\right]. \tag{2}$$

On every 3 - τ - manifold the Ricci operator Q is given by [4]:

$$QY = aY + b\eta(Y)\xi + \eta(Y)Q\xi + \eta(QY)\xi', \tag{3}$$

where $a = \frac{1}{2}(S - Tr\ell)$ and $b = -\frac{1}{2}(S + Tr\ell)$. We will complete this section with some results of [4], which we will use in this paper.

Proposition 2.1 *Let M^3 be a non-Sasakian 3 -τ-manifold. If X is a unit-eigenvector of h orthogonal to ξ, with eigenvalue λ then:*

$$Tr\,\ell = 2(1-\lambda^2) \leq 2, \tag{4}$$

$$\begin{aligned}
\nabla_\xi X &= \nabla_\xi(\varphi X) = 0, \nabla_X \xi = -(\lambda+1)\varphi X, \nabla_{\varphi X}\xi = (1-\lambda)X, \\
\nabla_X X &= \frac{1}{2\lambda}\left[\varphi X \cdot \lambda + \eta(QX)\right]\varphi X, \\
\nabla_{\varphi X}(\varphi X) &= \frac{1}{2\lambda}\left[X \cdot \lambda + \eta(Q\varphi X)\right]X, \\
\nabla_X(\varphi X) &= -\frac{1}{2\lambda}\left[\varphi X \cdot \lambda + \eta(QX)\right]X + (\lambda+1)\xi, \\
\nabla_{\varphi X} X &= -\frac{1}{2\lambda}\left[X \cdot \lambda + \eta(Q\varphi X)\right]\varphi X + (\lambda-1)\xi.
\end{aligned} \tag{5}$$

Proposition 2.2 *On every 3-τ-manifold we have $\xi \cdot Tr\,\ell = 0$.*

3 MAIN RESULTS

Let M^3 be a 3 - dimensional Sasakian manifold. We consider a unit vector field $X \in B$. Then X, φX, ξ form an orthonormal frame of M^3. Differentiating the inner products two of X, φX and ξ with respect to X, φX, ξ respectively, we obtain:

$$\begin{array}{llll}
\nabla_X \xi = -\varphi X, & \nabla_{\varphi X}\xi = X, & \nabla_X X = \alpha\varphi X, & \nabla_{X\varphi} X = -\alpha X + \xi \\
\nabla_\xi X = \gamma\varphi X, & \nabla_\xi \varphi X = -\gamma X, & \nabla_{\varphi X} X = -\beta\varphi X - \xi, & \nabla_{\varphi X}\varphi X = \beta X,
\end{array} \tag{6}$$

where α, β and γ are smooth functions on M^3.
It is well known that

$$(\nabla_X Q)X + (\nabla_{\varphi X} Q)\varphi X + (\nabla_\xi Q)\xi = \frac{1}{2} grad\,S \tag{7}$$

for any unit X orthogonal to ξ. Using the relations (6) and (7) we prove that $\xi \cdot S = 0$. It is known that every contact metric 3-manifold has η-parallel tensor field C (given by (2)) if

$$g\left(\left(\nabla_W C\right)\left(\varphi Y, \varphi Z\right), \varphi V\right) = 0\,, \tag{8}$$

where Y, Z and V are arbitrary vector fields and $W \in B$.

After the above remarks we can prove the following:

Theorem 3.1 *A 3 - dimensional Sasakian manifold, with η- parallel tensor field C, has constant curvature 1.*

We now shall prove the following:

Proposition 3.1 *On every 3-τ-manifold with η-parallel tensor field C, we have $\xi \cdot S = 0$.*

Proof. The Proposition is true for Sasakian manifolds. We consider a non-Sasakian 3-τ-manifold with η-parallel tensor field. Let X be a unit eigenvector of h with eigenvalue λ. Then, φX is an eigenvector of h with eigenvalue $-\lambda$ and X, φX, ξ form an orthonormal frame. We denote:

$$\begin{aligned}
\Sigma &= \frac{1}{2\lambda}\left[\varphi X\cdot\lambda+\eta(QX)\right],\\
T &= \frac{1}{2\lambda}\left[X\cdot\lambda+\eta(Q\varphi X)\right],\\
A_1 &= \eta(\nabla_X QX)-\Sigma\eta(Q\varphi X)-\frac{1}{4}\xi\cdot S,\\
A_2 &= \eta(\nabla_{\varphi X} Q\varphi X)-T\eta(QX)-\frac{1}{4}\xi\cdot S,\\
B_1 &= \eta(\nabla_X Q\varphi X)+\Sigma\eta(QX)+2(\lambda+1)(\lambda^2-1),\\
B_2 &= \eta(\nabla_{\varphi X} QX)+T\eta(Q\varphi X)+2(\lambda-1)(\lambda^2-1),\\
C_1 &= (\lambda+3)\eta(Q\varphi X)-2\lambda(X\cdot\lambda)-\frac{1}{4}X\cdot S,\\
C_2 &= (\lambda-3)\eta(QX)-2\lambda(\varphi X\cdot\lambda)-\frac{1}{4}\varphi X\cdot S.
\end{aligned}\tag{9}$$

The relations (2) and (9) give:

$$\begin{aligned}
C(X,\xi) &= A_1X+B_1\varphi X+C_1\xi,\\
C(\varphi X,\xi) &= B_2X+A_2\varphi X+C_2\xi,\\
C(X,\varphi X) &= C_2X-C_1\varphi X+(B_1-B_2)\xi.
\end{aligned}\tag{10}$$

The relation (7) is equivalent to

$$\varphi\left(\left(\nabla_W C\right)(\varphi Y,\varphi Z)\right)=0\,,\tag{11}$$

for all vector fields Y and Z on M^3 and W of the contact distribution B. Putting in the last equation: a) $W=Y=X$, $Z=\varphi X$, b) $W=Y=\varphi X$, $Z=X$, we take

$$\begin{aligned}
X\cdot C_1 &= (\lambda+1)(B_2-2B_1)+\Sigma C_2,\\
X\cdot C_2 &= (\lambda+1)A_1-\Sigma C_1,\\
\varphi X\cdot C_1 &= (\lambda-1)A_2-TC_2,\\
\varphi X\cdot C_2 &= (\lambda-1)(B_1-2B_2)+TC_1.
\end{aligned}\tag{12}$$

Using (8), the second and third of equations (12) can be written

$$\begin{aligned}
&4\eta(\nabla_X QX)+3(X\cdot\lambda)(\varphi X\cdot\lambda)+2\lambda(X\cdot\varphi X\cdot\lambda)+\frac{1}{4}X\cdot\varphi X\cdot S=\\
&=\frac{1}{\lambda}\left[\varphi X\cdot\lambda+\eta(QX)\right]\left[(\lambda+2)\eta(Q\varphi X)-\frac{1}{8}X\cdot S\right]+\frac{\lambda+1}{4}\xi\cdot S,
\end{aligned}\tag{13}$$

$$\begin{aligned}
&4\eta(\nabla_{\varphi X} Q\varphi X)-3(X\cdot\lambda)(\varphi X\cdot\lambda)-2\lambda(\varphi X\cdot X\cdot\lambda)-\frac{1}{4}\varphi X\cdot X\cdot S=\\
&=\frac{1}{\lambda}\left[X\cdot\lambda+\eta(Q\varphi X)\right]\left[(2-\lambda)\eta(QX)-\frac{1}{8}\varphi X\cdot S\right]+\frac{1-\lambda}{4}\xi\cdot S..
\end{aligned}\tag{14}$$

Adding the equations (13), (14) and using the relations (5) we obtain

$$\eta(\nabla_X QX) + \eta\nabla_{\varphi X} Q\varphi X = \frac{1}{2\lambda}\left[\eta(QX)X \cdot \lambda + \eta(Q\varphi X)\varphi X \cdot \lambda + \right. \\ \left. + 2\eta(QX)\eta(Q\varphi X)\right]. \quad (15)$$

On the other hand using the Proposition 2.1 and the relations (3), (5) and (7) we have:

$$\eta(\nabla_X QX) + \eta\nabla_{\varphi X} Q\varphi X = \frac{1}{2\lambda}\left[\eta(QX)X \cdot \lambda + \eta(Q\varphi X)\varphi X \cdot \lambda + \right. \\ \left. + 2\eta(QX)\eta(Q\varphi X)\right] + \frac{1}{2}\xi \cdot S. \quad (16)$$

Comparing (15) and (16) we get $\xi \cdot S = 0$. □

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