ABSOLUTE INVARIANT OPERATORS ON DIFFERENTIABLE MANIFOLDS

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Abstract

The aim of this paper is to study some properties of the algebra of absolute invariant operators which produce trace decomposition of tensors. The emphasis is on the family of projections, which do provide insight of some problems of differential geometry, getting a decomposition of the space of tensors of type $(1,3)$ into three components, invariant under the group $GL(V)$, in infinitely many possibilities, generalizing in this way the Strichartz splitting of curvature tensors.

The extension of the algebra to the $\mathcal{F}(M)$-module of absolute invariant tensor fields connects to δ-decompositions of geometrical object fields. Infinitely many invariants geometrical object fields to some transformations are obtained.

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1 Introduction

The development of the theory of the decomposition of the curvature tensors under the action of some groups was initiated by Singer and Thorpe [33]. Since these results and their ideas are very useful in the studies of some problems in geometry and topology of manifolds, many mathematicians have worked using this algebraic treatment of curvature tensors. Nomizu [30] studied generalized curvature tensors.

It is possible to get some inequalities for the quadratic invariants and to characterize some manifolds. Interesting applications of the theory of decompositions are used in the study of submanifolds in conformal differential geometry [17], classification of almost complex manifolds [8], almost product manifolds [28], homogeneous structures [1], [41].

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In the Kähler geometry, decompositions of the K-curvature tensors were given by Johnson [14], Mori [27], Sitaramayya [34]. Tricerri and Vanhecke [39] gave the decomposition of a quaternionic Kähler manifold and also on almost Hermitian geometry [40]. Janssens, Vanhecke [13] and Matzeu [22] studied the case of contact geometry.

The results in Riemannian projective geometry are given by Bokan [2], [4], which solved the problem of decomposition under the action of $SO(n)$. The complete decomposition of the space of curvature tensors under the action of $GL(V)$ was obtained by Strichartz [36]. The problem of decomposition in holomorphically projective geometry was studied by Nikčević [29].

Krupka [18], [19] and Mikeš [25] investigated the trace decomposition problem. All these splittings are in principle consequences of the general theorems on group representations of Weyl [48].

A new direction to study the problem of decompositions is initiated introducing some (r,r)-tensor algebras of invariant operators under some groups [10], [42], [43]. The absolute invariant tensor algebra $Inv(r)$, having the elements interpreted like endomorphisms on the space $T^1_{r-1}(M)$, enables us to get a trace decomposition of this space, the results of Krupka [18] being special cases of our theory.

The focus is on the infinite subset of projections in $Inv(r)$ which do provide good insight in some problems of differentiable manifolds. Let us mention that the Weyl projective curvature tensor and the Thomas projective connection are produced by this type of operators. Also, using some absolute invariant operators which are projections one gets the splitting of the space of tensors of type $(1,3)$ into three components invariant under the group $GL(V)$ into infinitely many ways. We should remark that in a particular case, one gets the Strichartz decomposition of the curvature tensors.

The extension of absolute invariant operators to $\mathcal{T}^1_{r-1}(M)$ and to the $\mathcal{F}(M)$-module $\mathcal{A}^1_{r-1}(M)$, a new space which is required by our theory, leads us to the problem of decompositions of tensors and connections.

Finally, one gets invariants for some transformations of geometrical object fields, extending the Thomas-Weyl theory.

2 Absolute invariant tensors algebra

Let V be a real n-dimensional vector space, where $n > 2$. Let $T^r_r(V)$ be the vector space of all tensors of type (r,r) and δ^i_j be the symbol of Kronecker. In $T^r_r(V)$ we consider the vector subspace

$$Inv(r) = \left\{ \sum_{\sigma\in S_r} x_\sigma \delta^{i_{\sigma(1)}}_{j_1} ... \delta^{i_{\sigma(r)}}_{j_r} | x_\sigma \in \mathbf{R},\ \sigma \in S_r \right\},$$

where S_r is the group of permutations. Any element $P \in Inv(r)$, which is an absolute invariant tensor (i.e., $\forall A \in GL(V),\ A\circ P = P$), is interpreted like an endomorphism on $T^1_{r-1}(V)$, producing a trace decomposition of this space. So, $Inv(r)$ becomes an algebra of absolute invariant operators, the product PQ of two elements P, Q being given by the rule

$$P^{i_1...i_{r-1}\ i_r}_{j_1\qquad\ j_2...j_r} Q^{j_2...j_r\ j_{r+1}}_{i_r\qquad i_{r+1}...i_{2r-1}}.$$

Some absolute invariant operators which are projectors do provide insight in some problems of differential geometry. Let us mention that the projectors from $\mathcal{I}nv(4)$ give the Weyl projective curvature tensor and the affine transformations of elements from $\mathcal{I}nv(3)$ produce the Thomas projective connection. Moreover, properties of the projections enable us to get splittings of the space of tensors of type $(1,3)$. So, for geometrical reasons, we study the subset of projections of $\mathcal{I}nv(r)$.

Important geometrical meanings have the cases $r = 3,\ 4$, which appear in the study of connections, torsion tensors, curvature tensors, Weyl projective curvature tensors etc.

Let us consider $r = 3$. Then

$$Inv(3) = \big\{P_a^{bc} : {}^r_{st} = x_1\delta_a^r\delta_s^b\delta_t^c + x_2\delta_a^b\delta_s^r\delta_t^c + x_3\delta_a^r\delta_s^c\delta_t^b +$$

$$+ x_4\delta_a^c\delta_s^b\delta_t^r + x_5\delta_a^c\delta_s^r\delta_t^b + x_6\delta_a^b\delta_s^c\delta_t^r | x_1, ..., x_6 \in \mathbf{R}\big\}\,.$$

Considering the basis $B = \{I_1, ..., I_6\}$, any element $P = \{P_a^{bc} : {}^r_{st}\}$ has the simplified expression $P = x_1I_1 + ... + x_6I_6$, being an endomorphism on $T_2^1(V)$. Hence the product PQ, $P_a^{bc} : {}^r_{st}\ Q_r^{st} : {}^i_{jk}$ of two elements from $Inv(3)$ is determined by the following multiplication table:

$*$	I_1	I_2	I_3	I_4	I_5	I_6
I_1	I_1	I_2	I_3	I_4	I_5	I_6
I_2	I_2	nI_2	I_6	I_6	I_2	nI_6
I_3	I_3	I_5	I_1	I_6	I_2	I_4
I_4	I_4	I_5	I_5	nI_4	nI_5	I_4
I_5	I_5	nI_5	I_4	I_4	I_5	nI_4
I_6	I_6	I_2	I_2	nI_6	nI_2	I_6

. (2.1)

The associated matrix of the algebra $Inv(3)$, with respect to the basis B, is:

$$A = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 0 & \alpha & 0 & 0 & 0 & \lambda \\ x_3 & x_5 & x_1 & x_6 & x_2 & x_4 \\ 0 & 0 & 0 & \gamma & \beta & 0 \\ 0 & 0 & 0 & \lambda & \alpha & 0 \\ 0 & \beta & 0 & 0 & 0 & \gamma \end{pmatrix},$$

where $\alpha = x_1 + nx_2 + x_5$, $\beta = x_2 + x_3 + nx_5$, $\gamma = x_1 + nx_4 + x_6$, $\lambda = x_3 + x_4 + nx_6$ and $\det A = (x_1^2 - x_3^2)(\alpha\gamma - \lambda\beta)$.

Theorem 2.1 *Let $T = \{T_{bc}^a\}$ be an arbitrary (1,2)-tensor. There is an infinite set of endomorphisms $P = \{P_a^{bc} : {}^r_{st}\}$ in $Inv(3)$, such that $\Omega = PT$ is a traceless tensor, where*

$$P_a^{bc} : {}^r_{st}T_{bc}^a = x_1T_{st}^r + x_3T_{ts}^r + \delta_s^r(x_2T_{at}^a + x_5T_{ta}^a) + \delta_t^r(x_6T_{as}^a + x_4T_{sa}^a).$$

The endomorphism $P=\sum\limits_{i=1}^{6} x_i I_i$ is a projective projection on the space $T_2^1(V)$ iff

$$\begin{aligned}
&x_1^2+x_3^2=x_1,\\
&2x_1x_2+nx_2^2+x_2x_5+x_3x_5+x_2x_6+x_3x_6+nx_5x_6=x_2,\\
&2x_1x_3=x_3,\\
&2x_1x_4+x_3x_6+nx_4^2+x_4x_6+x_3x_5+x_4x_5+nx_5x_6=x_4,\\
&2x_1x_5+x_2x_3+x_2x_4+x_3x_4+nx_4x_5+nx_2x_5+x_5^2=x_5,\\
&2x_1x_6+x_2x_3+x_2x_4+nx_2x_6+x_3x_4+nx_4x_6+x_6^2=x_6.
\end{aligned} \tag{2.2}$$

Theorem 2.2 [10]. *The quadratic system (2.2) has an infinite set of solutions.*

Let us consider the case $r=4$. Then

$$\begin{aligned}
&Inv(4)=\{\mathcal{P}_a^{bcd}{}_{stp}^{r}=y_1\delta_a^r\delta_s^b\delta_t^c\delta_p^d+y_2\delta_a^r\delta_s^b\delta_t^d\delta_p^c+y_3\delta_a^r\delta_s^c\delta_t^b\delta_p^d+\\
&+y_4\delta_a^r\delta_s^c\delta_t^d\delta_p^b+y_5\delta_a^r\delta_s^d\delta_t^b\delta_p^c+y_6\delta_a^r\delta_s^d\delta_t^c\delta_p^b+y_7\delta_a^d\delta_s^b\delta_t^c\delta_p^r+\\
&+y_8\delta_a^d\delta_s^b\delta_t^r\delta_p^c+y_9\delta_a^d\delta_s^c\delta_t^b\delta_p^r+y_{10}\delta_a^d\delta_s^c\delta_t^r\delta_p^b+y_{11}\delta_a^d\delta_s^r\delta_t^b\delta_p^c+\\
&+y_{12}\delta_a^d\delta_s^r\delta_t^c\delta_p^b+y_{13}\delta_a^c\delta_s^b\delta_t^d\delta_p^r+y_{14}\delta_a^c\delta_s^b\delta_t^r\delta_p^d+y_{15}\delta_a^c\delta_s^d\delta_t^b\delta_p^r+\\
&+y_{16}\delta_a^c\delta_s^d\delta_t^r\delta_p^b+y_{17}\delta_a^c\delta_s^r\delta_t^b\delta_p^d+y_{18}\delta_a^c\delta_s^r\delta_t^d\delta_p^b+y_{19}\delta_a^b\delta_s^c\delta_t^d\delta_p^r+\\
&+y_{20}\delta_a^b\delta_s^c\delta_t^r\delta_p^d+y_{21}\delta_a^b\delta_s^d\delta_t^c\delta_p^r+y_{22}\delta_a^b\delta_s^d\delta_t^r\delta_p^c+y_{23}\delta_a^b\delta_s^r\delta_t^d\delta_p^c+\\
&+y_{24}\delta_a^b\delta_s^r\delta_t^c\delta_p^d|y_i\in\mathbf{R},\quad i\in\{1,...,24\}\}.
\end{aligned}$$

With respect to the basis $B=\{\mathcal{I}_1,...,\mathcal{I}_{24}\}$, any element $\mathcal{P}=\{\mathcal{P}_a^{bcd}:{}_{stp}^{r}\}$ has the simplified expression $\mathcal{P}=\sum\limits_{i=1}^{24} y_i\mathcal{I}_i$ and can be written as:

$$\mathcal{P}_i^{jkl}:{}_{stp}^{r}=\delta_i^r P_1:{}_s^{jk}:{}_{tp}^{l}+\delta_i^l P_2:{}_s^{jk}:{}_{tp}^{r}+\delta_i^k P_3:{}_s^{kl}:{}_{tp}^{r}+\delta_i^j P_4:{}_s^{kl}:{}_{tp}^{r},$$

where $P_1,P_2,P_3,P_4\in Inv(3)$. So, it is determined by operators from the algebra $Inv(3)$.

Theorem 2.3 *Let $T=\{T_{jkl}^i\}$ be an arbitrary (1,3)-tensor. There is an infinite set of endomorphisms $\mathcal{P}=\{\mathcal{P}_i^{jkl}:{}_{stp}^{r}\}$ of $Inv(4)$, such that $\Omega=\mathcal{P}T$ is a traceless tensor, where:*

$$\begin{aligned}
&\mathcal{P}_i^{jkl}:{}_{stp}^{r}T_{jkl}^i=y_1T_{stp}^r+y_2T_{spt}^r+y_3T_{tsp}^r+y_4T_{pst}^r+y_5T_{tps}^r+y_6T_{pts}^r+\\
&+\delta_p^r\left(y_7T_{sti}^i+y_9T_{tsi}^i+y_{13}T_{sit}^i+y_{15}T_{tis}^i+y_{19}T_{ist}^i+y_{21}T_{its}^i\right)+\\
&+\delta_t^r\left(y_8T_{spi}^i+y_{10}T_{psi}^i+y_{14}T_{sip}^i+y_{16}T_{pis}^i+y_{20}T_{isp}^i+y_{22}T_{ips}^i\right)+\\
&+\delta_s^r\left(y_{11}T_{tpi}^i+y_{12}T_{pti}^i+y_{17}T_{tip}^i+y_{18}T_{pit}^i+y_{23}T_{ipt}^i+y_{24}T_{itp}^i\right).
\end{aligned}$$

For geometrical reasons, we study the subalgebra $Span\{\mathcal{I}_1, \mathcal{I}_7, ..., \mathcal{I}_{12}\} \subset Inv(4)$, having the following multiplication table:

$*$	$\mathcal{I}_1$	$\mathcal{I}_7$	$\mathcal{I}_8$	$\mathcal{I}_9$	$\mathcal{I}_{10}$	$\mathcal{I}_{11}$	$\mathcal{I}_{12}$
$\mathcal{I}_1$	$\mathcal{I}_1$	$\mathcal{I}_7$	$\mathcal{I}_8$	$\mathcal{I}_9$	$\mathcal{I}_{10}$	$\mathcal{I}_{11}$	$\mathcal{I}_{12}$
$\mathcal{I}_7$	$\mathcal{I}_7$	$n\mathcal{I}_7$	$n\mathcal{I}_8$	$n\mathcal{I}_9$	$n\mathcal{I}_{10}$	$n\mathcal{I}_{11}$	$n\mathcal{I}_{12}$
$\mathcal{I}_8$	$\mathcal{I}_8$	$\mathcal{I}_7$	$\mathcal{I}_8$	$\mathcal{I}_9$	$\mathcal{I}_{10}$	$\mathcal{I}_{11}$	$\mathcal{I}_{12}$
$\mathcal{I}_9$	$\mathcal{I}_9$	$n\mathcal{I}_9$	$n\mathcal{I}_{10}$	$n\mathcal{I}_7$	$n\mathcal{I}_8$	$n\mathcal{I}_{12}$	$n\mathcal{I}_{11}$
$\mathcal{I}_{10}$	$\mathcal{I}_{10}$	$\mathcal{I}_9$	$\mathcal{I}_{10}$	$\mathcal{I}_7$	$\mathcal{I}_8$	$\mathcal{I}_{12}$	$\mathcal{I}_{11}$
$\mathcal{I}_{11}$	$\mathcal{I}_{11}$	$\mathcal{I}_9$	$\mathcal{I}_{10}$	$\mathcal{I}_7$	$\mathcal{I}_8$	$\mathcal{I}_{12}$	$\mathcal{I}_{11}$
$\mathcal{I}_{12}$	$\mathcal{I}_{12}$	$\mathcal{I}_7$	$\mathcal{I}_8$	$\mathcal{I}_9$	$\mathcal{I}_{10}$	$\mathcal{I}_{11}$	$\mathcal{I}_{12}$

.

Indeed, $\mathcal{P} = y_1\mathcal{I}_1 + \sum_{k=7}^{12} y_k\mathcal{I}_k$ produces the Weyl projective curvature tensor for some values of the coefficients $y_1, y_7, ..., y_{12}$ in particular cases and it is a projective projection on $T_3^1(V)$ if and only if:

$$\begin{cases} y_1^2 = y_1, \\ y_8\tau + y_{10}\rho = 0, \\ y_8\tau + y_{10}\rho = 0, \\ y_{10}\tau + y_8\rho = 0, \\ y_{11}\tau + y_{12}\rho = 0, \\ y_{12}\tau + y_{11}\rho = 0, \end{cases} \tag{2.3}$$

where $\tau = 2y_1 + ny_7 + y_8 + y_{12} - 1$, $\rho = ny_9 + y_{10} + y_{11}$.

The system (2.3) is equivalent to:

$$\begin{cases} y_1^2 = y_1, \\ (\rho - \tau)(y_7 - y_9) = 0, \\ (\rho - \tau)(y_8 - y_{10}) = 0, \\ (\rho - \tau)(y_{11} - y_{12}) = 0. \end{cases} \tag{2.4}$$

I. $\rho = \tau$. Then (3.1) becomes:

$$\begin{cases} y_1^2 = y_1, \\ \rho(y_7 + y_9) = 0, \\ \rho(y_8 + y_{10}) = 0, \\ \rho(y_{11} + y_{12}) = 0. \end{cases} \tag{2.5}$$

a). $\tau = 2y_1 + ny_7 + y_8 + y_{12} - 1 = 0$, $\rho = ny_9 + y_{10} + y_{11} = 0$, $y_1 \in \{0, 1\}$.

b). $y_7 = -y_9$, $y_8 = -y_{10}$, $y_{11} = -y_{12}$. One gets:

$y_1 + ny_7 + y_8 + y_{12} = \frac{1}{2}, y_1 \in \{0, 1\}$.

II. $\rho \neq \tau \Rightarrow y_7 = y_9, y_8 = y_{10}, y_{11} = y_{12}$. The system (2.3) leads to:

$$\begin{cases} y_1^2 = y_1, \\ y_7(\rho + \tau) = 0, \\ y_8(\rho + \tau) = 0, \\ y_{11}(\rho + \tau) = 0. \end{cases}$$

a). For $\rho+\tau=0$ we obtain $y_1+ny_7+y_8+y_{12}=\dfrac{1}{2}, y_1\in\{0,1\}$.
b). $y_7=...=y_{12}, y_1\in\{0,1\}$.

3 The generalization of the Strichartz decomposition

Strichartz [36] found a decomposition of $\mathcal{K}(V)$, the space of curvature tensors, into irreducible components under the action of $GL(V)$, using properties of the theory of representations. The projective projections from $Span\{\mathcal{I}_1,\mathcal{I}_7,...,\mathcal{I}_{12}\}$ acting on $\mathcal{K}(V)$ produce Strichartz decomposition, but we extend this result getting infinitely many possibilities to split into three components the space of tensors of type $(1,3)$.

Proposition 3.1 *Let $\mathcal{K}(V)$ be the space of tensors R of type (1,3), verifying $R^r_{stl}+R^r_{slt}=0$ and the first Bianchi identity. The endomorphisms $\mathcal{P}\in Span\{\mathcal{I}_1,\mathcal{I}_7,...,\mathcal{I}_{12}\}$ of the algebra $\mathcal{I}nv(4)$ which apply $\mathcal{K}(V)$ into the same subspace have the form:*

$$\mathcal{P}=y_1\mathcal{I}_1+y_7\mathcal{I}_7-y_7\mathcal{I}_8-y_{10}\mathcal{I}_9+y_{10}\mathcal{I}_{10}-(y_7+y_{10})\mathcal{I}_{11}+(y_7+y_{10})\mathcal{I}_{12}.$$

Proof. Let $R\in\mathcal{K}(V)$ and $\mathcal{P}R=\bar{R}$.

We have $\bar{R}^r_{stl}+\bar{R}^r_{slt}=0$ if and only if

$$[(y_7+y_8)\mathcal{I}_7+(y_7+y_8)\mathcal{I}_8+(y_9+y_{10})\mathcal{I}_9+$$

$$+(y_9+y_{10})\mathcal{I}_{10}+(y_{11}+y_{12})\mathcal{I}_{11}+(y_{11}+y_{12})\mathcal{I}_{12}]R=0.$$

Hence:

$$\begin{cases} y_7+y_8=0,\\ y_9+y_{10}=0,\\ y_{11}+y_{12}=0.\end{cases}$$

Then $\bar{R}$ verifies the first Bianchi identity if and only if

$$\delta^r_l\,[(y_7+y_{10}+y_{11})R_{st}+(y_8+y_9+y_{12})R_{ts}]+\delta^r_t\,[(y_7+y_{10}+y_{11})R_{ls}+$$

$$+(y_8+y_9+y_{12})R_{sl}]+\delta^r_s\,[(y_7+y_{10}+y_{11})R_{tl}+(y_8+y_9+y_{12})R_{lt}]=0.$$

We find:

$$\begin{cases} y_7+y_{10}+y_{11}=0,\\ y_8+y_9+y_{12}=0.\end{cases}$$

□

Remark 3.1 *$Dim\ Ker(\mathcal{P}|_{\mathcal{K}(V)})=16$.*

Proof. The properties of the tensors from $\mathcal{K}(V)$ imply that

$$\{\mathcal{I}_1+\mathcal{I}_2,\ \mathcal{I}_3+\mathcal{I}_5,\ \mathcal{I}_4+\mathcal{I}_6,\ \mathcal{I}_7+\mathcal{I}_{13},\ \mathcal{I}_8+\mathcal{I}_{14},\ \mathcal{I}_9+\mathcal{I}_{15},$$

$$\mathcal{I}_{10}+\mathcal{I}_{16},\ \mathcal{I}_{11}+\mathcal{I}_{17},\ \mathcal{I}_{12}+\mathcal{I}_{18},\ \mathcal{I}_{19}+\mathcal{I}_{21},\ \mathcal{I}_{20}+\mathcal{I}_{22},\ \mathcal{I}_{23}+\mathcal{I}_{24},$$

$$\mathcal{I}_1 + \mathcal{I}_5 + \mathcal{I}_4,\ \mathcal{I}_7 + \mathcal{I}_{15} + \mathcal{I}_{19},\ \mathcal{I}_8 + \mathcal{I}_{16} + \mathcal{I}_{20},\ \mathcal{I}_{11} + \mathcal{I}_{18} + \mathcal{I}_{24}\}$$

is a basis in $Ker(\mathcal{P}|_{\mathcal{K}(V)})$.□

Proposition 3.2 *The projective projections $\mathcal{P} \in Span\{\mathcal{I}_1, \mathcal{I}_7, ..., \mathcal{I}_{12}\}$ on $\mathcal{K}(V)$ belong to the following set:*

$$\Big\{\mathcal{I}_1 - \frac{1}{2(n-1)}(\mathcal{I}_7 - \mathcal{I}_8 + \mathcal{I}_9 - \mathcal{I}_{10}), \frac{1}{2(n-1)}(\mathcal{I}_7 - \mathcal{I}_8 + \mathcal{I}_9 - \mathcal{I}_{10})\ ;$$

$$\mathcal{I}_1 + \frac{1}{2(n+1)}(-\mathcal{I}_7 + \mathcal{I}_8 + \mathcal{I}_9 - \mathcal{I}_{10} + 2\mathcal{I}_{11} - 2\mathcal{I}_{12}),$$

$$\frac{1}{2(n+1)}(\mathcal{I}_7 - \mathcal{I}_8 - \mathcal{I}_9 + \mathcal{I}_{10} - 2\mathcal{I}_{11} + 2\mathcal{I}_{12});$$

$$\mathcal{I}_1 - \frac{n}{n^2-1}\mathcal{I}_7 + \frac{n}{n^2-1}\mathcal{I}_8 - \frac{1}{n^2-1}\mathcal{I}_9 + \frac{1}{n^2-1}\mathcal{I}_{10} + \frac{1}{n+1}\mathcal{I}_{11} - \frac{1}{n+1}\mathcal{I}_{12},$$

$$\frac{n}{n^2-1}\mathcal{I}_7 - \frac{n}{n^2-1}\mathcal{I}_8 + \frac{1}{n^2-1}\mathcal{I}_9 - \frac{1}{n^2-1}\mathcal{I}_{10} - \frac{1}{n+1}\mathcal{I}_{11} + \frac{1}{n+1}\mathcal{I}_{12}\Big\}.$$

In the set of infinite family of projective projections from $Span\{\mathcal{I}_1, \mathcal{I}_7, ..., \mathcal{I}_{12}\}$, this finite subset with particular properties has geometrical meanings.

Theorem 3.1 *There is an infinite set of nonvanishing projective projections $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in Span\{\mathcal{I}_1, \mathcal{I}_7, ..., \mathcal{I}_{12}\}$, such that*

$$\mathcal{P} + \mathcal{Q} + R = \mathcal{I}_1, \mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = \mathcal{R}\mathcal{Q} = \mathcal{Q}\mathcal{R} = 0$$

and

$$T^1_3(V) = Im\mathcal{P} \oplus Im\mathcal{Q} \oplus Im\mathcal{R}$$

hold.

Proof. Let

$$\mathcal{P} = \mathcal{I}_1 + \sum_{i=7}^{12} y_i\mathcal{I}_i, \mathcal{Q} = \sum_{j=7}^{12} y'_j\mathcal{I}_j, \mathcal{R} = \sum_{k=7}^{12} y''_k\mathcal{I}_k$$

be such that $\mathcal{P} + \mathcal{Q} + R = \mathcal{I}_1, \mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = \mathcal{R}\mathcal{Q} = \mathcal{Q}\mathcal{R} = 0$.

I. a).

$$\begin{cases} 1 + ny_7 + y_8 + y_{12} = 0 \\ ny_9 + y_{10} + y_{11} = 0. \end{cases}$$

We get $\mathcal{P}\mathcal{Q} = \mathcal{P}\mathcal{R} = 0$.

1.

$$\begin{cases} ny'_7 + y'_8 + y'_{12} = 1 \\ ny'_9 + y'_{10} + y'_{11} = 0. \end{cases}$$

$\mathcal{Q}\mathcal{R} = 0$ implies $\sum_{k=7}^{12} y''_k\mathcal{I}_k = 0$. So, $\mathcal{R} = 0$ and $\mathcal{Q} = \mathcal{I}_1 - \mathcal{P}$.

2.
$$\begin{cases} ny'_7+y'_8+y'_{12}=\dfrac{1}{2} \\ y'_7=-y'_9, y'_8=-y'_{10}, y'_{11}=-y'_{12}. \end{cases}$$

Also $ny'_9+y'_{10}+y'_{11}=-\frac{1}{2}$.

$\mathcal{QR}=0$ implies:

$$(y''_7-y''_9)\mathcal{I}_7+(y''_8-y''_{10})\mathcal{I}_8+(y''_9-y''_7)\mathcal{I}_9+$$

$$+(y''_{10}-y''_8)\mathcal{I}_{10}+(y''_{11}-y''_{12})\mathcal{I}_{11}+(y''_{12}-y''_{11})\mathcal{I}_{12}=0.$$

Hence $y''_7=y''_9, y''_8=y''_{10}, y''_{11}=y''_{12}$. $\mathcal{R}$ being projective projection, one gets $\mathcal{R}=0$ or $ny''_7+y''_8+y''_{12}=\frac{1}{2}$.

Finally, $T^1_3(V)=Im\mathcal{P}\oplus Im\mathcal{Q}\oplus Im\mathcal{R}$, where:

$$\begin{cases} 1+ny_7+y_8+y_{12}=0 \\ ny_9+y_{10}+y_{11}=0, \end{cases}$$

$$\begin{cases} ny'_7+y'_8+y'_{12}=\dfrac{1}{2} \\ y'_7=-y'_9, y'_8=-y'_{10}, y'_{11}=-y'_{12}, \end{cases}$$

$$\begin{cases} y''_7=y''_9, y''_8=y''_{10}, y''_{11}=y''_{12} \\ ny''_7+y''_8+y''_{12}=\dfrac{1}{2}. \end{cases}$$

3. $y'_7=y'_9, y'_8=y'_{10}, y'_{11}=y'_{12}$. $\mathcal{Q}$ is a projective projection. We have $\mathcal{Q}=0(\Longrightarrow \mathcal{R}=\mathcal{I}_1-\mathcal{P})$ or $ny'_7+y'_8+y'_{12}=\frac{1}{2}$. In the second case we obtain $ny'_9+y'_{10}+y'_{11}=\frac{1}{2}$. $\mathcal{QR}=0$ leads to $y''_7=-y''_9, y''_8=-y''_{10}, y''_{11}=-y''_{12}$. So, $ny''_7+y''_8+y''_{12}=\frac{1}{2}$.

We get $T^1_3(V)=Im\mathcal{P}\oplus Im\mathcal{Q}\oplus Im\mathcal{R}$.

4. If $\mathcal{Q}=0$, then $\mathcal{R}=\mathcal{I}_1-\mathcal{P}$.

I. b).
$$\begin{cases} ny_7+y_8+y_{12}=-\dfrac{1}{2} \\ y_7=-y_9, y_8=-y_{10}, y_{11}=-y_{12}. \end{cases}$$

We find $ny_9+y_{10}+y_{11}=\frac{1}{2}$.

From $\mathcal{PQ}=0$ we get:

$$(y'_7-y'_9)\mathcal{I}_7+(y'_8-y'_{10})\mathcal{I}_8+(y'_9-y'_7)\mathcal{I}_9+$$

$$+(y'_{10}-y'_8)\mathcal{I}_{10}+(y'_{11}-y'_{12})\mathcal{I}_{11}+(y'_{12}-y'_{11})\mathcal{I}_{12}=0.$$

Hence $y'_7=y'_9, y'_8=y'_{10}, y'_{11}=y'_{12}$. In a similar way $\mathcal{PR}=0$ leads to

$y''_7=y''_9, y''_8=y''_{10}, y''_{11}=y''_{12}$. If $\mathcal{P}+\mathcal{Q}+R=\mathcal{I}_1$, then $y_7=y_9, y_8=y_{10}, y_{11}=y_{12}$. So, $\mathcal{P}=\mathcal{I}_1, \mathcal{Q}=\mathcal{R}=0$.

II. $y_7=y_9, y_8=y_{10}, y_{11}=y_{12}$.

a). $1+ny_7+y_8+y_{12}=\frac{1}{2}$. We have $ny_9+y_{10}+y_{11}=-\frac{1}{2}$. $\mathcal{PQ}=0$ implies $y'_7=y'_9, y'_8=y'_{10}, y'_{11}=y'_{12}$.

From $\mathcal{P}\mathcal{R} = 0$ one gets $y''_7 = y''_9, y''_8 = y''_{10}, y''_{11} = y''_{12}$.
We have:
$\mathcal{R} = 0$ or $ny'_7 + y'_8 + y'_{12} = \frac{1}{2}$,
$\mathcal{Q} = 0$ or $ny''_7 + y''_8 + y''_{12} = \frac{1}{2}$.
In all these cases at least one projection is vanishing.
b). $\mathcal{P} = \mathcal{I}_1, \mathcal{Q} = \mathcal{R} = 0$. □

Proposition 3.3 *Any tensor* $R \in T^1_3(V)$ *splits into infinitely many ways* $R = R' + R'' + R'''$, *such that* $R'_{ij} = 0, R''_{ij}$ *is a symmetric tensor and* R'''_{ij} *is a skew-symmetric tensor.*

Proof. Let $R \in T^1_3(V)$ and $\mathcal{P} \in Span\{\mathcal{I}_1, \mathcal{I}_7, ..., \mathcal{I}_{12}\}$ be a projective projection.
$(\mathcal{P}\mathcal{R})^a_{sta} = (y_1 + ny_7 + y_8 + y_{12})R_{st} + (ny_9 + y_{10} + y_{11})R_{ts}$.
Using the previous proposition, $T^1_3(V) = Im\mathcal{P} \oplus Im\mathcal{Q} \oplus Im\mathcal{R}$, where:

$$\begin{cases} 1 + ny_7 + y_8 + y_{12} = 0 \\ ny_9 + y_{10} + y_{11} = 0, \end{cases}$$

$$\begin{cases} ny'_7 + y'_8 + y'_{12} = \frac{1}{2} \\ y'_7 = -y'_9, y'_8 = -y'_{10}, y'_{11} = -y'_{12}, \end{cases}$$

$$\begin{cases} y''_7 = y''_9, y''_8 = y''_{10}, y''_{11} = y''_{12} \\ ny''_7 + y''_8 + y''_{12} = \frac{1}{2}, \end{cases}$$

one gets $(\mathcal{P}R)_{st} = 0, (\mathcal{Q}R)_{st} = \frac{1}{2}(R_{st} - R_{ts}), (\mathcal{R}R)_{st} = \frac{1}{2}(R_{st} + R_{ts})$. □

Remark 3.2 In the particular case

$$\mathcal{P} = \mathcal{I}_1 - \frac{n}{n^2-1}\mathcal{I}_7 + \frac{n}{n^2-1}\mathcal{I}_8 - \frac{1}{n^2-1}\mathcal{I}_9 + \frac{1}{n^2-1}\mathcal{I}_{10} + \frac{1}{n+1}\mathcal{I}_{11} - \frac{1}{n+1}\mathcal{I}_{12},$$

$$\mathcal{Q} = \frac{1}{2(n+1)}(\mathcal{I}_7 - \mathcal{I}_8 - \mathcal{I}_9 + \mathcal{I}_{10} - 2\mathcal{I}_{11} + 2\mathcal{I}_{12}),$$

$$\mathcal{R} = \frac{1}{2(n-1)}(\mathcal{I}_7 - \mathcal{I}_8 + \mathcal{I}_9 - \mathcal{I}_{10})$$

we find again the Strichartz decomposition [36] of curvature tensors into three irreducible components under the action of the group $GL(V)$:

$$\mathcal{K}(V) = Im\mathcal{P} \oplus Im\mathcal{Q} \oplus Im\mathcal{R},$$

where $\dim \mathcal{K}(V) = \frac{n^2(n^2-1)}{3}$, $\dim \mathrm{Im}\, \mathcal{R} = \frac{n(n+1)}{2}$,
$\dim Im\mathcal{Q} = \frac{n(n-1)}{2}$, :: $\dim Im\mathcal{P} = \frac{n^2(n^2-4)}{3}$.
If the Ricci tensor associated to $R \in \mathcal{K}(V)$ is symmetric, then $\mathcal{P}R = W$ is the Weyl projective curvature tensor.
This decomposition is not irreducible under the action of $O(n)$ or $SO(n)$.

4 δ-decompositions of some geometrical object fields on differentiable manifolds

Let M be a differentiable n-dimensional manifold and $\mathcal{T}_r^r(M)$ be the bundle of (r,r)-tensor fields of M. Then

$$\mathcal{I}nv(r) = \left\{ P^{i_1\dots i_r}_{j_1\dots j_r} = \sum_{\sigma\in S_r} f_\sigma \delta^{i_{\sigma(1)}}_{j_1}\dots\delta^{i_{\sigma(r)}}_{j_r} \mid f_\sigma \in \mathcal{F}(M),\ \sigma\in S_r \right\}$$

is the $\mathcal{F}(M)$-module of absolute invariant tensor fields.

We define the $\mathcal{F}(M)$-module $\mathcal{A}^1_{r-1}(M)$, the union of parallel affine spaces of geometrical object fields of type $(1, r-1)$ on M ([26]), whose difference or whose skew-symmetric part with respect to a pair of indices belongs to $\mathcal{T}^1_{r-1}(M)$. Obviously, for $r = 3$, the space $\mathcal{C}$ of all affine connections on M and $\mathcal{T}^1_2(M)$ are examples of such parallel affine spaces.

Each element P of $\mathcal{I}nv(r)$ acts like an endomorphisms on $\mathcal{T}^1_{r-1}(M)$ and induces an affine transformation on $\mathcal{A}^1_{r-1}(M)$, producing trace decomposition of this space.

Remark 4.1 The image of the family of the projective projections on $\mathcal{C}$ produces infinitely many parallel affine spaces.

If the projective projections act on the symmetric affine connections, then in the cases:

a). $x_1 = 1, x_3 = 0, x_2 = x_4 = \dfrac{n}{1-n^2}, x_5 = x_6 = \dfrac{1}{n^2-1}$,

b). $x_1 = 1, x_3 = 0, x_5 = x_4 = -\dfrac{1}{1+n} - \lambda, x_6 = x_2 = \lambda \in \mathcal{F}(M)$,

c). $x_1 = x_3 = \dfrac{1}{2}, x_2 = x_4 = x_5 = x_6 = -\dfrac{1}{2(n+1)}$,

d). $x_1 = x_3 = \dfrac{1}{2}, x_2 = x_4 = \dfrac{1}{n^2-1}, x_5 = x_6 = -\dfrac{n}{1-n^2}$,

we find the Thomas projective connection:

$$\Pi^r_{st} = \Gamma^r_{st} - \frac{1}{n+1}(\delta^r_s\Gamma_t + \delta^r_t\Gamma_s),\ \Gamma_t = \Gamma^a_{at}.$$

These particular cases and the splitting like a direct sum motivate the general construction of the projective projections.

Remark 4.2 Let $\Gamma \in \mathcal{A}^1_2(M)$. If $A \in \mathcal{A}^1_3(M)$ is a geometrical object field defined by:

$$A^r_{stl} = \frac{\partial \Gamma^r_{sl}}{\partial x^t} + \Gamma^r_{mt}\Gamma^m_{st}$$

and $\mathcal{R} = \mathcal{I}_1 - \mathcal{I}_2$ is an affine transformation on $\mathcal{A}^1_3(M)$, then $\mathcal{R}A = R \in \mathcal{A}^1_3(M)$ is the exotic curvature tensor associated to Γ.

Moreover, in the particular case when Γ is an affine connection, then $\mathcal{R}A = R$ is the curvature tensor field.

The results obtained in the case of a vector space lead to the following result:

Theorem 4.1 *There is an infinite set of affine transformations on $\mathcal{A}_3^1(M)$ of nonvanishing projective projections $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in Span_{\mathcal{F}(M)}\{\mathcal{I}_1, \mathcal{I}_7, \ldots, \mathcal{I}_{12}\}$ such that*

$$\mathcal{P} + \mathcal{Q} + R = \mathcal{I}_1, \mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = \mathcal{R}\mathcal{Q} = \mathcal{Q}\mathcal{R} = 0$$

and

$$\mathcal{A}_3^1(M) = Im\mathcal{P} \oplus Im\mathcal{Q} \oplus Im\mathcal{R}.$$

Theorem 4.2 *Let $P = x_1 I_1 + x_2 I_2 + x_6 I_6$, $\mathcal{P} = y_1\mathcal{I}_1 + y_7\mathcal{I}_7 + y_8\mathcal{I}_8$, $\mathcal{T} = z_1\mathcal{I}_1 + z_7\mathcal{I}_7 + z_8\mathcal{I}_8$ be transformations on $\mathcal{A}_2^1(M)$ and $\mathcal{A}_3^1(M)$ respectively, of projective projections from $\mathcal{I}nv(3)$ and $\mathcal{I}nv(4)$ respectively.*

There is an infinite set of invariants

$$W = \mathcal{P}R = \mathcal{T}\overline{R}$$

of the transformations of geometrical objects

$$\Gamma \xrightarrow{P} \overline{\Gamma},$$

where :

Γ *is an affine symmetric connection, having a symmetric Ricci tensor,*
$\overline{\Gamma} = P\Gamma \in \mathcal{A}_2^1(M)$ *is the exotic Thomas projective connection,*
R *is the curvature tensor field associated to* Γ*,*
$\overline{R} \in \mathcal{A}_3^1(M)$ *is the exotic curvature tensor associated to the* $\overline{\Gamma}$*.*

Taking into account only nontrivial possibilities for $P, \mathcal{P}, \mathcal{T}$, one gets that the exotic Weyl projective tensor field $W = \mathcal{P}R = \mathcal{T}\overline{R}$ is invariant to the following transformations:

a). $\Gamma \xrightarrow{P} \overline{\Gamma}$, where $P = I_1 - \dfrac{1}{n} I_2$,
$\mathcal{T} = \mathcal{P} \in \{y_7\mathcal{I}_7 + (1 - ny_7)\mathcal{I}_8,\ \mathcal{I}_1 - y_7\mathcal{I}_7 + (ny_7 - 1)\mathcal{I}_8\},\ y_7 \in \mathcal{F}(M)$;

b). $\Gamma \xrightarrow{P} \overline{\Gamma}$, where $P = I_1 + x_2 I_2 - (1 + nx_2) I_6,\ x_2 \in \mathcal{F}(M) \setminus \left\{-\dfrac{1}{n}\right\}$, $\mathcal{T} = \mathcal{P} \in \left\{\mathcal{I}_1 - \dfrac{1}{n-1}\mathcal{I}_7 + \dfrac{1}{n-1}\mathcal{I}_8\right\}$.

We should remark that in this case $W = \mathcal{T}R = \mathcal{P}\overline{R}$ is the Weyl projective curvature tensor and if $x_2 = -\dfrac{1}{n+1}$ then $\overline{\Gamma}$ is the Thomas projective connection.

Hence, using this method, we generalize the known properties of this tensor, extending the Thomas-Weyl theory.

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References

[1] E. Abbena, S. Garbiero, *Almost Hermitian homogeneous structures*, Proc. Eding. Math. Soc., 31 (1988), 375-395.

[2] N. Bokan, S. Nikčević, *A characterization of the projective and holomorphic projective structures,* Arch. Math., 62 (1994), 368-377.

[3] N. Bokan, *The decomposition theory and its applications,* In: The math. heritage of C.F. Gauss, G.M. Rassias, ed., Singapore, (1991), 66-99.

[4] N. Bokan, *On the complete decomposition of curvature tensor of Riemannian manifolds with symmetric connection*, Rendiconti del Circ. Mat. di Palermo, II, 39 (1990), 331-380.

[5] D. Catto, M. Francaviglia, *On certain decompositions of tensor fields of order four*, Atti Accad. Sci. Torino Cl. Fis. Mat. Natur., 114, 3-4 (1980-81), 243-248.

[6] J. Chollet, *Equality of symmetrized decomposable tensors*, Ph. D. Thesis, University of California, Santa Barbara, (1979).

[7] A. Gray, *Invariants of curvature operators of four dimensional Riemannian manifolds,* Proc. Thirteen Bienal Semin. Canad. Math. Cong., Halifax, 2 (1971), 42-65.

[8] A. Gray, L. Helvella, *The sixteen classes of almost Hermitian manifolds and their linear invariants* , Ann. Mat. Pura Appl., 123 (1980), 325-358.

[9] A. Gray, L. Vanhecke, *Decomposition of the space of covariant derivatives of curvature tensors* (preprint), (1991).

[10] I.E. Hirică, C. Udrişte, *Decompositions of tensor fields and connections*, Hadronic Journal Supplement, USA, (1998) - Proceedings of International Workshop on Fundamental Open Problems in Mathematics, Physics and other Sciences, Chinese Academy of Sciences, Beijing, August 28, (1997).

[11] N. Iwahori, *Some remarks on tensor invariants of $O(n)$, $U(n)$, $Sp(n)$,* J. Math. Soc. Japan, 10 (1958), 145-160.

[12] D. Janssens, *Krommingstensoren, Decompositietheorie en toepassingen*, Doctoratsverhandeling, Kath. Univ. Leuven, (1977).

[13] D. Janssens, L. Vanhecke, *Almost contact structures and curvature tensors*, Kodai Math. J., 4 (1981), 1-27.

[14] D.L. Johnson, *Sectional curvature and curvature normal forms,* Michigan Math., 27 (1980), 275-294.

[15] T. Kashiwada, *Notes on Tricerri-Vanhecke's decomposition of curvature tensors*, Geometry of Manifolds (Matsumoto, 1988), 95-10 , Perspectives in Math.,8. Academic Press, Boston, M.A., (1989).

[16] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Interscience Publishers, New-York-London, I (1963), II (1969).

[17] O. Kowalski, *Partial curvature structures and conformal geometry of submanifolds*, J. Diff. Geom., 8 (1973), 53-70.

[18] D. Krupka, *The trace decomposition of tensors of type (1,2) and (1,3)*, Eds. L. Tamassy, J. Szenthe, New Developments in Diff. Geom., Proc. of Colloquium on Differential Geometry, Debrecen, July 26-30, (1994); Mathematics and Its Applications, 350, Kluwer Academic Publishers, (1996), 243-253.

[19] D. Krupka, *The trace decomposition problem*, Beiträge zür Algebra und Geometrie, 36, 2 (1995), 303-315.

[20] D. Krupka and J. Janyska, *Lectures on Differential Invariants*, Univerzita J.E. Purkyne Brne, (1990).

[21] R.S. Kulkarni, *On the Bianchi identities*, Math. Ann., 199 (1972), 93-103.

[22] P. Matzeu, *Decomposition of curvature tensors on almost contact manifolds,* Rend. Sem. Mat. Univ. Politec. Torino, 47, 2 (1989), 179-210.

[23] P. Matzeu, S. Nikčević, *Linear algebra of curvature tensor on Hermitian manifold,* An. Ştiinţ. Univ. "Al.I. Cuza", Iaşi Secţ. Ia Math. (N.S.), 37 (1991), 71-86.

[24] R. Merris, *Multilinear Algebra*, Algebra, Logic and Applications Series, volume 8, Gordon and Breach Science Publishers, (1997).

[25] J. Mikeš, *On general trace decomposition problem*, Diff. Geom. and Its Appl., Brno 1995, Masaryk, Univ. Brno (1996), 45-50.

[26] R. Miron, M. Anastasiei, *Vector bundles and Lagrange spaces with applications to relativity,* Geometry Balkan Press, Bucharest, (1997).

[27] H. Mori, *On the decomposition of generalized K-curvature tensor fields*, Tôhoku Math. J., 25 (1973), 225-235.

[28] A.M. Naveira, *A classification of Riemannian almost-product manifolds*, Rendiconti di Matematica, 3, VII (1983), 577-592.

[29] S. Nikčević, *On the decomposition of curvature tensor fields on Hermitian manifolds,* Colloq. Math. Soc. Janos Bolyai 56. Diff. Geom., Eger (Hungary), (1989), 555-568.

[30] K. Nomizu, *On the decomposition of generalized curvature tensor fields*, Differential Geometry, in honor of K.Yano, Kinokuniya, Tokyo, (1972), 335-345.

[31] G.N. de Oliveira, J.A. Dias da Silva, *Conditions for equality of decomposable tensors II*, Linear Algebra and Appl., 28 (1979), 161-176.

[32] E. Omachi, *Orthogonal decompositions of ∇R, ∇Ric and ${}^c\nabla R$ in Riemannian and Kähler manifolds,* Tensor, N.S., 50 (1991), 213-218.

[33] I.M. Singer, J.A. Thorpe, *The curvature of 4-dimensional Einstein spaces*, Global Analysis (Paper in honor of K. Kodaira), University of Tokyo Press, Tokyo, (1969), 355-365.

[34] M. Sitaramayya, *Curvature tensors in Kähler manifolds*, Trans. Amer. Math. Soc., 183 (1973), 341-383.

[35] G. Stanilov, *Geometric interpretation of some transformations of curvature like tensors,* Tensor N.S., 37 (1982), 155-161.

[36] R.S. Strichartz, *Linear algebra of curvature tensors and their covariant derivatives*, Can. J. Math., 15 (1988), 1105-1143.

[37] V. de Smedt, *Decomposition of the curvature tensor of hyper-Kähler manifolds,* Lett. Math. Physics, 30 (1994), 105-117.

[38] T.Y. Thomas, *A projective theory of affinely connected manifolds,* Math. Zeitschrift, 25 (1926), 723-733.

[39] F. Tricceri, L. Vanhecke, *Decomposition of a space of curvature tensors on a quaternionic Kähler manifold and spectrum theory*, Simon Stevin, 53 (1979), 163-173.

[40] F. Tricerri, L. Vanhecke, *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc., 267 (1981), 356-398.

[41] F. Tricceri, L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc., Lect. Notes 83, Cambridge Univ. Press. (1983).

[42] C. Udrişte, I.E. Hirică, *Family of projective projections on tensors and connections*, Balkan Journal of Geometry and Its Applications, 2, 2 (1997), 139-156.

[43] C. Udrişte, I.E. Hirică, *Conformal invariant operators on Riemannian manifolds,* Rend. del Sem. Mat. di Messina (to appear).

[44] L. Vanhecke, *Curvature tensors*, J. Korean Math. Soc., 14 (1977), 143-151.

[45] L. Vanhecke, *On the decomposition of curvature tensor fields on almost Hermitian manifolds*, Proc. Conf. on Diff. Geom., Michigan State Univ., (1976), 16-33.

[46] L. Vanhecke, *Decomposition theory for curvature tensors*, Diff. Geom. Meeting, Contact group on Diff. Geometry and Alg. Topol. N.F.W.O., Liège, (1976), 99-124.

[47] G. Vrănceanu, *Leçons de Géométrie différentielle, I,II* Ed.Acad.Române, Bucureşti, (1957).

[48] H. Weyl, *The Classical Groups, their invariants and representations*, Princeton, (1946).

[49] Y. Yoshimatsu, *On the decomposition of generalized S-curvature-like tensor fields*, Tôhoku Math. J., 29 (1977), 235-254.

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