HERMITIAN STRUCTURES AND COMPATIBLE CONNECTIONS ON $\mathbb{A}$-BUNDLES*

M. Papatriantafillou

Abstract

Our aim here is to investigate the conditions under which an $\mathbb{A}$-bundle is provided with generalized (: $\mathbb{A}$-valued) hermitian structures and compatible connections, in the general case when $\mathbb{A}$ is a *commutative locally m-convex* *-*algebra with unit*.

We prove that an $\mathbb{A}$-hermitian structure exists if the fibre type of the $\mathbb{A}$-bundle has an $\mathbb{A}$-hermitian inner product and the base space admits just one $\mathbb{A}$-valued partition of unity, or, if the structural group of the bundle reduces to the "$\mathbb{A}$-hermitian product preserving" automorphisms. Next, we endow an $\mathbb{A}$-bundle with a connection, assuming the existence of one $\mathbb{A}$-partition of unity, and we prove that this connection and the previous $\mathbb{A}$-hermitian structure are compatible.

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1 Introduction

Arbitrary locally convex spaces have a very poor geometric structure and this fact is reflected to the manifolds and vector bundles modelled on them (cf., for instance, [9, 11]). In particular, they do not have inner products, thus a vector bundle modelled on a locally convex space is not endowed with a Riemannian structure.

But some locally convex spaces, arising in pure mathematics and in theoretical physics, have the additional algebraic structure of a (projective finitely generated) module over a topological algebra $\mathbb{A}$ and a number of questions have been answered by the extension of the usual ring $\mathbb{R}$ or $\mathbb{C}$ of coefficients to the aforementioned algebra (see [1] in operator theory, [22] in theoretical physics, [10] in differential topology). In the case that $\mathbb{A}$ is a *-algebra, the modules are provided with $\mathbb{A}$-valued inner products

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and norms, defining their topology [17], and they behave like finite dimensional vector spaces, although, in general, they lack both bases and metric topologies.

On the other hand, manifolds and vector bundles modelled on such modules are found in various areas (see, for example, [10] in differential topology, [21] in partial differential equations, [2] in global analysis, [23] in the theory of jets), $\mathbb{A}$ usually being $\mathcal{C}(X)$ or $\mathcal{C}^\infty(X)$. For brevity, we call them $\mathbb{A}$-manifolds and $\mathbb{A}$-bundles, respectively.

Continuous $\mathbb{A}$-bundles have been extensively studied (see [5, 6, 7, 8, 14, 16]), while some differential aspects have appeared in [12, 15, 17], among which the existence of $\mathbb{A}$-valued Finsler structures. Our aim here is to investigate the conditions under which an $\mathbb{A}$-bundle is provided with generalized (: $\mathbb{A}$-valued) hermitian structures and compatible connections, in the general case when $\mathbb{A}$ is a *commutative locally m-convex *-algebra with unit*. In this investigation two obstacles appear: first, $\mathbb{A}$-manifolds never admit partitions of unity in the classical sense, and, secondly, the existence of a hermitian structure is not equivalent to the reduction of the structural group of the bundle to a special subgroup. However, we prove that an $\mathbb{A}$-hermitian structure exists if the fibre type of the $\mathbb{A}$-bundle has an $\mathbb{A}$-hermitian inner product and the base space admits just one $\mathbb{A}$-valued partition of unity (Theorem 4.5), or, if the structural group of the bundle reduces to the "$\mathbb{A}$-hermitian product preserving" automorphisms (Theorem 4.7). Next, we endow an $\mathbb{A}$-bundle with a connection, assuming the existence of one $\mathbb{A}$-partition of unity (Theorem 5.3), and we prove that this connection and the $\mathbb{A}$-hermitian structure of Theorem 4.7 are compatible (Theorem 5.4).

2 Preliminaries

We recall that a complex algebra $\mathbb{A}$ is a **-algebra*, if it is endowed with a map $^* : \mathbb{A} \to \mathbb{A}$, so that (i) $(za+b)^* = \bar{z}a^* + b^*$, (ii) $(ab)^* = b^*a^*$ and (iii) $(a^*)^* = a$, for every $a, b \in \mathbb{A}$, $z \in \mathbb{C}$. A *-algebra $\mathbb{A}$ is called a *locally m-convex* (abr. *lmc*) **-algebra*, if it is topologized by a family of seminorms that satisfy (i) $p(xy) \le p(x)p(y)$ and (ii) $p(x^*) = p(x)$, for every $x, y \in \mathbb{A}$ (for details, see [4]).

Throughout the paper, $\mathbb{A}$ denotes a *commutative lmc*) **-algebra with unit*.

Let $\mathcal{P}(\mathbb{A})$ be the category of projective finitely generated $\mathbb{A}$-modules. By definition, for any $M \in \mathcal{P}(\mathbb{A})$, there exist $M_1 \in \mathcal{P}(\mathbb{A})$ and $m \in \mathbb{N}$, so that $M \oplus M_1 \cong \mathbb{A}^m$. Let τ_M denote the *canonical topology* of M, i.e., the relative topology induced on M by the product topology of $\mathbb{A}^m$. Then: (i) τ_M is independent of either M_1 or m; (ii) (M, τ_M) is a topological $\mathbb{A}$-module (namely, the $\mathbb{A}$-module operations are jointly continuous); (iii) τ_M makes every $\mathbb{A}$-linear map $f : M \to N$ continuous, for any topological $\mathbb{A}$-module N (for details, see [13]).

In the sequel, every $M \in \mathcal{P}(\mathbb{A})$ is a topological $\mathbb{A}$-module provided with the canonical topology. For any $M, N \in \mathcal{P}(\mathbb{A})$ and $x \in M$, we denote by 0_M the zero element of M, by $\mathcal{N}(x)$ the set of open neighbourhoods of x and by $L_{\mathbb{A}}(M,N)$ (resp. $S_{\mathbb{A}}(M,N)$) the set of $\mathbb{A}$-linear (resp. skew-linear) maps $f : M \to N$; we recall that a map $f : M \to N$ is called *skew-linear*, if f is additive and $f(ax) = a^*f(x)$, for every $x \in M$ and $a \in \mathbb{A}$.

In $\mathcal{P}(\mathbb{A})$ we consider the following differentiation method: Let $M, N \in \mathcal{P}(\mathbb{A})$, $x \in M$, $U \in \mathcal{N}(x)$ and $f : U \to N$. We say that f is $\mathbb{A}$-*differentiable at* x, if there exist $Lf(x) \in L_{\mathbb{A}}(M, N)$ and $Sf(x) \in S_{\mathbb{A}}(M, N)$, such that the map

$$\phi(h) := f(x+h) - f(x) - Lf(x)(h) - Sf(x)(h)$$

satisfies the following condition:

$$\forall V \in \mathcal{N}(0_N)\, \exists U \in \mathcal{N}(0_M) : \forall B \in \mathcal{N}(0_{\mathbb{A}})\, \exists A \in \mathcal{N}(0_{\mathbb{A}}) :$$

$$\phi(aU + a^*U) \subseteq aBV + a^*BV, \quad \forall a \in \mathbb{A}.$$

We call $Df(x) := Lf(x) + Sf(x)$ *the differential of* f at x. If $Sf(x) = 0$ (resp. $Lf(x) = 0$), f is called $\mathbb{A}$-*holomorphic* (resp. $\mathbb{A}$-*antiholomorphic*) *at* x.

Let f be $\mathbb{A}$-differentiable at every $x \in U$. Since $L_{\mathbb{A}}(M, N) \oplus S_{\mathbb{A}}(M, N) \in \mathcal{P}(\mathbb{A})$, $\mathbb{A}$-differentiation may apply to

$$Df : U \to L_{\mathbb{A}}(M, N) \oplus S_{\mathbb{A}}(M, N),$$

inducing *the second differential* $D^2 f = D(Df)$ *of* f, and, successively, *the* n*-th differential* $D^n f$ *of* f, for any $n \in \mathbb{N}$. We will say that f is an $\mathbb{A}^\infty$-*differentiable* (resp. $\mathbb{A}^\infty$-*holomorphic*) *map on* U, if $D^k f$ exists (resp. $D^k f$ exists and $S^k f = 0$), for every $k \in \mathbb{N}$ (for details, we refer the reader to [15]).

3 $\mathbb{A}$-manifolds and $\mathbb{A}$-bundles

Let X be a Hausdorff topological manifold modelled on $M \in \mathcal{P}(\mathbb{A})$. We say that X is an $\mathbb{A}$-*manifold* (resp. $\mathbb{A}_h$-*manifold*), if its transition functions are $\mathbb{A}^\infty$-differentiable (resp. A^∞-holomorphic). If X, Y are $\mathbb{A}$-manifolds (resp. $\mathbb{A}_h$-manifolds), we say that $f : X \to Y$ is an $\mathbb{A}$-*map* (resp. $\mathbb{A}_h$-*map*), if its local representatives are $\mathbb{A}^\infty$-differentiable (resp. A^∞-holomorphic). The category of $\mathbb{A}$-manifolds (resp. $\mathbb{A}_h$-manifolds) and $\mathbb{A}$-maps (resp. $\mathbb{A}_h$-maps) will be denoted by $Man(\mathbb{A})$ (resp. $Man_h(\mathbb{A})$).

Let $X \in Man_h(\mathbb{A})$ modelled on M. We obtain tangent spaces, by considering classes of equivalent "curves" in the following way: an $\mathbb{A}$-*curve on* X is an $\mathbb{A}$-map $\alpha : A \to X$, with $A \in \mathcal{N}(0_{\mathbb{A}})$. The $\mathbb{A}$-curves α, β *are tangent at* $x \in X$, if $\alpha(0) = \beta(0) = x$ and there exists a chart (U, ϕ) at x with $D(\phi \circ \alpha)(0) = D(\phi \circ \beta)(0)$. We denote by $[(\alpha, x)]$ the induced equivalence class of α and by $T(X, x)$ the set of such quivalence classes. If M_* denotes the abelian group $(M, +)$ provided with the scalar multiplication

$$\mathbb{A} \times M_* \to M_* : (a, x) \to a^* x$$

and $M \oplus M_1 = \mathbb{A}^m$, then

$$M_* \oplus (M_1)_* = (M \oplus M_1)_* = (\mathbb{A}^m)_* = \mathbb{A}^m,$$

within $\mathbb{A}$-module isomorphisms, that is, $M_* \in \mathcal{P}(\mathbb{A})$. Let $x \in X$ and (U, ϕ) a chart at x. The map

$$\bar{\phi} : T(X, x) \to M \times M_* : [(\alpha, x)] \to (L(\phi \circ \alpha)(0), S(\phi \circ \alpha)(0)) \tag{1}$$

is a bijection establishing an $\mathbb{A}$-module structure on $T(X,x)$. We call $T(X,x)$ *the tangent space of* X *at* x. The *tangent bundle* $T(X)$ *of* X, i.e., the discrete union of all tangent spaces is an $\mathbb{A}$-manifold.

We note here, that if $\mathbb{A} = \mathbb{C}$, the tangent bundle introduced above coincides with the complexified tangent bundle of complex manifolds.

If $f : X \to Y$ is an $\mathbb{A}$-map, the *differential of* f

$$Tf : T(X) \to T(Y) : [(\alpha, x)] \to [(f \circ \alpha, f(x))]$$

is an $\mathbb{A}$-map and, for any $x \in X$, the restriction

$$T_x f : T(X,x) \to T(Y, f(x)) : [(\alpha, x)] \to [(f \circ \alpha, f(x))]$$

is an $\mathbb{A}$-linear map.

Let now $X \in Man_h(\mathbb{A})$, $E \in Man(\mathbb{A})$, $\pi : E \to X$ an $\mathbb{A}$-map and $M \in \mathcal{P}(\mathbb{A})$. We say that the triplet $\ell = (E, \pi, X)$ is an $\mathbb{A}$-*differentiable* $\mathbb{A}$-*bundle over* X *of fibre type* M, or, simply, an $\mathbb{A}$-*bundle*, if the following conditions hold:

i) $E_x := \pi^{-1}(x) \in \mathcal{P}(\mathbb{A})$, for every $x \in X$.

ii) There exists a *trivializing covering* $\{(U_i, \tau_i)\}_{i \in I}$, where $\{U_i\}_{i \in I}$ is an open covering of X and every $\tau_i : \pi^{-1}(U_i) \to U_i \times M$ is an isomorphism in $Man(\mathbb{A})$, such that $pr_1 \circ \tau_i = \pi$ and, for every $x \in U_i$, the restriction

$$\tau_{ix} := \tau_i|_{E_x} : E_x \to \{x\} \times M$$

is an $\mathbb{A}$-module isomorphism.

One would note here that in the Banach context, for the definition of vector bundles, one more condition is required, namely (VB 3) of [3]. However, in our framework, the properties of the canonical topology and the A^∞-differentiation imply this condition, making $\mathbb{A}$-bundles look like bundles of finite rank (cf. the analogous results for continuous R-bundles, where R is a topological ring [14] and for differentiable $\mathbb{A}$-bundles, where $\mathbb{A}$ is a commutative unital lmc algebra over $\mathbb{R}$ [17, 20]).

4 $\mathbb{A}$-hermitian structures

The involution of the algebra $\mathbb{A}$ endows the objects of $\mathcal{P}(\mathbb{A})$ with a structure generalizing hermitian inner products on complex vector spaces. In this section we investigate the conditions under which these generalized inner products provide a hermitian structure on an $\mathbb{A}$-bundle.

We recall that a *positive definite* $\mathbb{A}$-*hermitian inner product* on an $\mathbb{A}$-module M is a map $\alpha : M \times M \to \mathbb{A}$, satisfying the following conditions:

(i) α is $\mathbb{A}$-linear with respect to the first variable.

(ii) $\alpha(y,x) = (\alpha(x,y))^*$, for any $x, y \in M$.

(iii) For every $x \in M$, $\alpha(x,x)$ is positive in $\mathbb{A}$, that is,

$$sp_{\mathbb{A}}(\alpha(x,x)) := \{\lambda \in \mathbb{C} : \lambda \cdot 1_{\mathbb{A}} - \alpha(x,x) \text{ not invertible }\} \subseteq [0, +\infty).$$

(iv) The mapping $M \to L_{\mathbb{A}}(M, \mathbb{A})_* : x \to \alpha_x$, where $\alpha_x(y) := \alpha(y, x)$, for every $y \in M$, is an isomorphism of $\mathbb{A}$-modules.

For brevity, the pair (M, α) is called a *hermitian form*.

For every hermitian form (M, α), with $M \in \mathcal{P}(\mathbb{A})$, α is an $\mathbb{A}$-map and

$$L\alpha(x, y)(h, k) = \alpha(h, y), \qquad S\alpha(x, y)(h, k) = \alpha(x, k),$$

for every $(x, y), (h, k) \in M \times M$.

Let us recall that a *lmc C*-algebra* is a lmc *-algebra, whose seminorms satisfy the relation $p(x^*x) = (p(x))^2$, for every $x \in \mathbb{A}$. Regarding the existence of hermitian forms, we have

Theorem 4.1 [6] *Let $\mathbb{A}$ be a complete lmc C^*-algebra with unit and $M \in \mathcal{P}(\mathbb{A})$. Then M admits a positive definite $\mathbb{A}$-hermitian inner product α, which is unique up to an isomorphism, that is, if (M, β) is a hermitian form, then there exists an $\mathbb{A}$-automorphism f of M, such that $\beta \circ (f \times f) = \alpha$.* □

Corrolary 4.2 *Let $\mathbb{A}$ be a complete lmc C^*-algebra with unit and M a free finitely generated $\mathbb{A}$-module. If (M, β) is a hermitian form, Then M has an orthonormal basis with respect to β.*

Proof. By definition, M coincides with $\mathbb{A}^m$, for some $m \in \mathbb{N}$. The canonical basis $\{e_i\}_{i=1,\ldots,m}$ of $\mathbb{A}^M$ is orthonormal with respect to the posotive definite $\mathbb{A}$-hermitian inner product

$$\alpha : \mathbb{A}^m \times \mathbb{A}^m \to \mathbb{A} : ((x_i), (y_i)) \mapsto \sum_i x_i(y_i)^*.$$

If f is the $\mathbb{A}$-automorphism of $\mathbb{A}^m$ with $\beta \circ (f \times f) = \alpha$, then $\{f(e_i)\}_{i=1,\ldots,m}$ is the required basis of $\mathbb{A}^m$. □

Definition 4.3 An $\mathbb{A}$-*hermitian structure* on the $\mathbb{A}$-bundle $\ell = (E, \pi, X)$ is an $\mathbb{A}$-map $g : E \oplus E \to \mathbb{A}$, such that, $(E_x, g_x := g|_{E_x \oplus E_x})$ is a hermitian form, for every $x \in X$.

In the subsequent theorems we give sufficient conditions for the existence of $\mathbb{A}$-hermitian structures. But, we need first the following

Definition 4.4 Let $(X, \mathcal{A}) \in Man(\mathbb{A})$. An $\mathbb{A}$-*partition of unity* on X is a family $\{(U_i, \psi_i)\}_{i \in I}$, where $\{U_i\}_{i \in I}$ is a locally finite open covering of X and $\{\psi_i\}_{i \in I}$ is a family of $\mathbb{A}$-maps $\psi_i : X \to \mathbb{A}$, such that

(i) $supp(\psi_i) \subseteq U_i$, for any $i \in I$.

(ii) For any $x \in X$ and any $i \in I$, $\psi_i(x)$ is positive in $\mathbb{A}$.

(iii) $\sum_i \psi_i(x) = 1$, for any $x \in X$.

In the ordinary (finite dimensional or Banach) case, one assumes that every open covering of X admits a locally finite refinement with a subordinate partition of unity. However, such an assumption is too strong. We only need that the bundle has (at least) one locally finite trivializing covering $\{(U_i, \tau_i)\}_{i\in I}$, so that $\{U_i\}_{i\in I}$ has a subordinate $\mathbb{A}$-partition of unity. This last condition is proved to hold for a class of $\mathbb{A}$-manifolds, if $\mathbb{A}$ is the algebra $\mathcal{C}(X)$ of continuous complex valued functions on a Hausdorff completely regular topological space X, or its subalgebra $\mathcal{C}^\infty(X)$ of smooth functions, in the case that X is a compact smooth manifold (see [18], [19]).

Following the classical arguments, one has:

Theorem 4.5 *Let ℓ be an $\mathbb{A}$-bundle of fibre type $M \in \mathcal{P}(\mathbb{A})$. If (M, α) is an $\mathbb{A}$-hermitian form and ℓ has a localy finite trivializing covering with a subordinate $\mathbb{A}$-partition of unity, then ℓ is provided with an $\mathbb{A}$-hermitian structure.* □

In the case of a unital commutative complete lmc C*-algebra, for every $M \in \mathcal{P}(\mathbb{A})$ there is a hermitian form (M, α) (Theorem 4.1). Besides, if the base space X has a locally finite atlas consisting of charts whose image is a sphere with respect to the $\mathbb{A}$-valued norm defined by α, then X has an $\mathbb{A}$-partition of unity subordinate to this atlas (see [19] in conjunction with [18]). As a result, we obtain

Theorem 4.6 *Let $\mathbb{A}$ be a unital commutative complete lmc C*-algebra. If the $\mathbb{A}$-bundle $\ell = (E, \pi, X)$ has a localy finite trivializing covering, so that the respective charts of X are sent to spheres, then ℓ is provided with an $\mathbb{A}$-hermitian structure.* □

If (M, α) is a hermitian form, we denote by $GL(M, \alpha)$ the group of $\mathbb{A}$-linear automorphisms f of M satisfying $\alpha \circ (f \times f) = \alpha$. As usually, if M is the fibre type of ℓ, we say that *the structural group of ℓ reduces to $GL(M, \alpha)$*, if ℓ has a trivializing covering $\{(U_i, \tau_i)\}_{i\in I}$, with

$$\tau_{jx} \circ \tau_{ix}^{-1} \in GL(M, \alpha), \quad \forall i, j \in I,\ x \in U_i \cap U_j. \tag{2}$$

If (2) holds, it is clear that the formula

$$s(x) := \alpha \circ (\tau_{ix} \times \tau_{ix}). \tag{3}$$

defines a hermitian form $(E_x, s(x))$ independent of the choice of i, and that the induced map $s : E \oplus E \to \mathbb{A}$ locally coincides with $\alpha \circ (\tau_i \times \tau_i)$, hence it is an $\mathbb{A}$-map. As a result, we have

Theorem 4.7 *Let ℓ be an $\mathbb{A}$-bundle of fibre type M and (M, α) a hermitian form. If the structural group of ℓ reduces to $GL(M, \alpha)$, then ℓ has an $\mathbb{A}$-hermitian structure.* □

If $\mathbb{A}$ is a complete C*-algebra, the converse of Theorem 4.7 is true in the topological case ([16]). Besides, the reduction of the structural group of a bundle does not depend on the choice of α (ibid.).

5 $\mathbb{A}$-connections

In this section, we define $\mathbb{A}$-connections as operators between the sections of certain $\mathbb{A}$-bundles and we construct such a connection on a bundle having a trivializing covering with a subordinate $\mathbb{A}$-partition of unity (Theorem 5.3). This connection is compatible with the hermitian structure obtained in Theorem 4.7 (Theorem 5.4).

Proposition 5.1 *Let $(X, \mathcal{A}) \in Man_h(\mathbb{A})$ modelled on M and let $\ell = (E, \pi, X)$ be an $\mathbb{A}$-bundle of fibre type N. If $\tilde{\ell} := (L(TX, E), \tilde{\pi}, X)$, where*

$$L(TX, E) := \bigcup_{x \in X} L_{\mathbb{A}}(T(X, x), E_x)$$

and $\tilde{\pi} : L(TX, E) \to X$ is the natural projection, then $\tilde{\ell}$ admits the structure of an $\mathbb{A}$-bundle of fibre type $P := L_{\mathbb{A}}(M, N)$.

Proof. If $\{(U_i, \phi_i)\}_{i \in I}$ is an atlas of X and $\{(U_i, \tau_i)\}_{i \in I}$ a trivializing covering of ℓ, let

$$\tilde{\tau}_i : \tilde{\pi}^{-1}(U_i) \to U_i \times P : f \mapsto (\tilde{\pi}(f), \tau_{ix} \circ f \circ \bar{\phi}_i^{-1}),$$

where $x = \tilde{\pi}(f)$ and $\bar{\phi}_i$ is given by (1). Then $\{(U_i, \tilde{\tau}_i)\}_{i \in I}$ is a trivializing covering of $\tilde{\ell}$. □

Definition 5.2 Let $\ell = (E, \pi, X)$ be an $\mathbb{A}$-bundle. We denote by $\Gamma(X, E)$ and $\Gamma(X, L(TX, E))$ the $\mathbb{A}$-modules of the $\mathbb{A}$-differentiable sections of ℓ and of $\tilde{\ell}$, respectively. We say that a mapping

$$D : \Gamma(X, E) \to \Gamma(X, L(TX, E)),$$

is an $\mathbb{A}$-*connection on* ℓ, if it is $\mathbb{A}$-linear and it satisfies the *Leibniz condition:*

$$Df\xi = Tf \cdot \xi + f \cdot D\xi,$$

for any $\xi \in \Gamma(X, E)$ and any $\mathbb{A}$-map $f : X \to \mathbb{A}$.

Besides, we say that D is *compatible* with a hermitian structure g of ℓ, if

$$g_x(D\xi(x)(v), \eta(x)) + g_x(\xi(x), D_\eta(x)(v)) = T_x(g \circ (\xi, \eta))(v),$$

for every $x \in X$, $\xi, \eta \in \Gamma(X, E)$ and $v \in T(X, x)$.

Theorem 5.3 *Let ℓ be an $\mathbb{A}$-bundle having a locally finite trivializing covering with a subordinate $\mathbb{A}$-partition of unity. Then ℓ has an $\mathbb{A}$-connection.*

Proof. (i) Assume first that the fibre type of ℓ is a free finitely generated $\mathbb{A}$-module $\mathbb{A}^m$. Let $\{(U_i, \tau_i)\}_{i \in I}$ be the locally finite trivializing covering of ℓ and $\{(U_i, \psi_i\}_{i \in I}$ the subordinate $\mathbb{A}$-partition of unity. For every $i \in I$, we set

$$\varepsilon_{ij} : U_i \to E_i := \pi^{-1}(U_i) : x \mapsto \tau_i^{-1}(x, b_j) \tag{4}$$

where $\{b_j\}_{1\leq j\leq m}$ is an arbitrary basis of $\mathbb{A}^m$. Then $\{\varepsilon_{ij}\}_{1\leq j\leq m}$ is an $\mathbb{A}$-differentiable local frame of ℓ and every $\xi \in \Gamma(U_i, E_i)$ is written as $\xi = \sum_j \xi_{ij} \cdot \varepsilon_{ij}$, where $\xi_{ij} : U_i \to \mathbb{A}$ is an $\mathbb{A}$-map, for every $j = 1, ..., m$. We set

$$D_i : \Gamma(U_i, E_i) \to \Gamma(U_i, L(TU_i, E_i)) : \xi \mapsto D_i(\xi) := \sum_j (T\xi_{ij}) \cdot \varepsilon_{ij}.$$

It is straightforward that D_i is a local $\mathbb{A}$-connection and that

$$D : \Gamma(X, E) \to \Gamma(X, L(TX, E)) : \xi \mapsto D\xi := \sum_i \psi_i \cdot D_i(\xi)$$

is an $\mathbb{A}$-connection on ℓ.

(ii) Suppose now that the fibre type of ℓ is $M \in \mathcal{P}(\mathbb{A})$ and let $N \in \mathcal{P}(\mathbb{A})$ and $m \in \mathbb{N}$ with $M \oplus N = \mathbb{A}^m$. We consider the trivial $\mathbb{A}$-bundle $\ell_o := (X \times N, pr_1, X)$ and the Whitney sum $\ell \oplus \ell_0 = (F, \bar{\pi}, X)$. Let $\ell \oplus \ell_o$ be endowed with the $\mathbb{A}$-connection $\tilde{D} : \Gamma(X, F) \to \Gamma(X, L(TX, F))$, obtained in (i). If $I : E \to F$ denotes the canonical injection and $Pr : F \to E$ the canonical projection, we obtain an $\mathbb{A}$-connection D on ℓ, setting

$$D\xi : X \to L(TX, E) : x \mapsto D\xi(x) := Pr \circ \tilde{D}(I \circ \xi)(x). \qquad \square$$

If $\mathbb{A}$ is a complete C^*-algebra, the $\mathbb{A}$-hermitian structures obtained in Theorem 4.7 and the $\mathbb{A}$-connections of the above Theorem 5.3 are compatible. In fact, we have

Theorem 5.4 *Let $\mathbb{A}$ be a commutative complete lmc C^*-algebra with unit and ℓ an $\mathbb{A}$-bundle of fibre type M. Suppose that ℓ has a locally finite trivializing covering $\{(U_i, \tau_i)\}_{i\in I}$ admitting a subordinate $\mathbb{A}$-partition of unity and satisfying (2), for a hermitian form (M, α). Then ℓ has an $\mathbb{A}$-hermitian structure and a compatible $\mathbb{A}$-connection.*

Proof. (i) Assume first that the fibre type of ℓ is $\mathbb{A}^m$. Let $\{x_j\}_{1\leq j\leq m}$ be an orthonormal basis of $\mathbb{A}$ with respect to α (cf. Cor. 4.2) and, for every $i \in I$, let $\{\varepsilon_{ij}\}_{1\leq j\leq m}$ be the induced local frame of ℓ (see (4)). Let now g be the $\mathbb{A}$-hermitian structure of ℓ, constructed in Theorem 4.7, and D the $\mathbb{A}$-connection obtained in Theorem 5.3. Then, for every $i \in I$,

$$g_x(D_i\xi(x)(v), \eta(x)) + g_x(\xi(x), D_i\eta(x)(v)) =$$

$$= \sum_j (T_x\xi_{ij}(v) \cdot (\eta(x))^* + \xi_{ij}(x)(T_x\eta_{ij}(v)^*),$$

where $\xi|_{U_i} = \sum_j \xi_{ij} e_{ij}$, $\eta|_{U_i} = \sum_j \eta_{ij} e_{ij}$, $x \in U_i$ and $v \in T(X, x)$. Thus,

$$g_x(D\xi(x)(v), \eta(x)) + g_x(\xi(x), D\eta(x)(v)) =$$

$$= \sum_i \psi_i(x) \cdot \sum_j T_x(\xi_{ij} \cdot (\eta_{ij})^*)(v). \qquad (5)$$

On the other hand,

$$g \circ (\xi, \eta)(x) = g(\sum_j \xi_{ij}(x)\varepsilon_{ij}(x), \sum_j \eta_{ij}(x)\varepsilon_{ij}(x)) = \sum_j \xi_{ij}(x) \cdot \eta_{ij}(x)^*,$$

for every (U_i, τ_i) containing x, consequently,

$$T_x(g \circ (\xi, \eta))(v) = \sum_j T_x(\xi_{ij} \cdot (\eta_{ij})^*)(v), \tag{6}$$

implying the required equality.

(ii) Suppose now that the fibre type of ℓ is $M \in \mathcal{P}(\mathbb{A})$. Let $N \in \mathcal{P}(\mathbb{A})$ and $m \in \mathbb{N}$ with $M \oplus N \equiv \mathbb{A}^m$. We consider the trivial $\mathbb{A}$-bundle $\ell_o = (X \times N, pr_1, X)$ and the sum $\ell \oplus \ell_o$. The latter is an $\mathbb{A}$-bundle of fibre type $\mathbb{A}^m$, the structural group of which reduces to $GL(M \times N, \alpha \oplus \alpha_o)$, for any positive definite $\mathbb{A}$-hermitian inner product α_o on N. Consider ℓ and $\ell \oplus \ell_o$ provided with the $\mathbb{A}$-hermitian structures g and $\widetilde{g}$, where

$$g(x) := \alpha \circ (\tau_{ix} \times \tau_{ix}) \ ; \ i \in I, \ x \in U_i,$$
$$\widetilde{g}(x) := g(x) \oplus \alpha_o \ , \ x \in X.$$

Then, if $\{x_j\}_{1 \le j \le n}$ is again an orthonormal basis of $\mathbb{A}$ with respect to $\alpha \oplus \alpha_o$, and, for every $i \in I$, $(U_i, \Phi_i := \tau_i \times id_N)$ is the trivializing pair of $\ell \oplus \ell_o$ induced by (U_i, τ_i) of ℓ, the sections $\varepsilon_{ij}(x) := \Phi_i^{-1}(x, x_j)$, $(x \in U_i, j = 1, ..., n)$ form an orthonormal $\mathbb{A}$-differentiable frame on U_i. As in (i), we construct an $\mathbb{A}$-connection $\widetilde{D}$ on $\ell \oplus \ell_o$ which is compatible with $\widetilde{g}$. Now, setting

$$D : \Gamma(X, E) \to \Gamma(X, L(TX, E)) : \xi \mapsto Pr(\widetilde{D}(I \circ \xi))$$

(see the proof of Theorem 3.3), we obtain:

$$\begin{aligned}
g_x(D\xi(x)(v), \eta(x)) \quad + \quad & g_x(\xi(x), D\eta(x)(v)) = \\
= \quad & g_x(Pr \circ \widetilde{D}(I \circ \xi)(x)(v), I \circ \eta(x)) + \\
& \quad + g_x(I \circ \xi(x), Pr \circ \widetilde{D}(I \circ \eta)(x)(v)) = \\
= \quad & g_x(Pr \circ \widetilde{D}(I \circ \xi)(x)(v), \eta(x)) + \\
& \quad + \alpha_o((1 - Pr) \circ \widetilde{D}(I \circ \xi)(x)(v), 0) + \\
& \quad + g_x(\xi(x), Pr \circ \widetilde{D}(I \circ \eta)(x)(v)) + \\
& \quad + \alpha_o(0, (1 - Pr) \circ (\widetilde{D}(I \circ \eta)(x)(v)) = \\
= \quad & \widetilde{g}_x(\widetilde{D}(I \circ \xi)(x)(v), I \circ \eta(x)) + \\
& \quad + \widetilde{g}_x(I \circ \xi(x), \widetilde{D}(I \circ \eta)(x)(v)) = \\
= \quad & T_x(\widetilde{g} \circ (I \circ \xi, I \circ \eta))(v) = \\
= \quad & T_x(g \circ (\xi, \eta))(v),
\end{aligned}$$

which completes the proof. □

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Author's address:

M. Papatriantafillou
Department of Mathematics
University of Athens
Panepistimiopolis
Athens 157 84, Greece
E-mail address: mpapatr@atlas.uoa.gr