LAGRANGE SPACES OF GRAVITY AND GAUGE FIELD THEORIES

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Abstract

In this paper a model of unification of gravity and gauge field theories is proposed. This model, based on the properties of the Lagrange spaces as in [1], can be proposed as an alternative to the Kaluza-Klein theories. The major difference between our approach and that of Kaluza-Klein lies in the use of different geometries. In particular, the Kaluza-Klein theories are formulated over the total space of a principal fiber bundle, whose fibers are isomorphic copies of the symmetry group, whereas our approach is formulated over the total space of the tangent bundle TM, which is endowed with the structure of the Lagrange space. In terms of this space the gauge transformations are acquire a space-time geometrical meaning, since they correspond to the Caratheodory transformations of the proposed Lagrangian, whereas the Yang-Mills equations are derived from the study of the properties of the h-deflection tensor. Furthermore the conventional Lagrangian $R - \frac{1}{4}F_{A\alpha\beta}F_A^{\alpha\beta}$, R being the scalar curvature of the gravitational field and $F_{A\alpha\beta}$ the components of the gauge field strength tensor, is seen to be equal to the scalar curvature of the tangent bundle TM.

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1 Introduction

Our purpose is to present a theory that unifies gravity and gauge field theories on the classical level. In our approach, this unification will be formulated over the tangent bundle TM of the space-time manifold M. This model has been worked out by some geometers and theoretical physicists, see for example [1, 2, 4, 5, 6, 7, 8], for the case of electrodynamics. Our work extends this effort to gauge theories of any symmetry group.

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In particular, we enrich TM with the structure of a Lagrange space and examine the conditions that enable us to conceive an equivalence between the gauge transformations of the gauge potentials and the Caretheodory transformations of the given regular Lagrangian. In other words, we conceive the gauge transformations as transformations that establish an equivalence relation between the possible (physically equivalent) Lagrange structures over TM. It is remarkable, that such an approach leads to the formulation of the gauge potential as given in Penrose and Rindler [3]. After that, we give specific forms to the non-linear connection on TM, the d-connection and the metric structure and subsequently calculate all the torsions and curvatures of the space. It is important to note that our approach guaranties a scalar curvature of the form $R - \frac{1}{4}F_{Aae}F^{ae}_{A}$, with R the Ricci scalar of the gravitational field. In this way, the deduced scalar curvature takes the form of the conventional Lagrangian generally used to describe the interaction of a gravitational with a gauge field.

In section 2, we establish the geometrical framework, define its basic notions and properties and give a short description of the geometrical structures to be used. In section 3, all the previously defined structures are given a specific form. In particular, first we define our symbolism and explain how we conceive a gauge transformation in the present geometrical framework. Next, we give specific forms to the non-linear connection, the metric structure and the d-connections. We complete section 3 with the calculation of the torsions and curvatures. Finally, in section 4 we derive the gauge field equations from the properties of the h-deflection tensor and present the Einstein equations.

2 The geometrical framework

Consider a real n-dimensional differentiable manifold M and let $\xi = (TM, p, M)$ be its tangent bundle. The total space TM will be given the local coordinates (x^i, y^a),where $1 \leq i, a \leq n$. (Throughout the rest of this presentation we will use small lattin indices $i, j, \ldots$ to denote components of tensors on the horizontal subspace and Greek indices $\alpha, \beta, \ldots$ to denote components of tensors on the vertical subspace.) We are particularly interested in the geometry of the tangent space TTM of TM. At any point (x^i, y^a) of TM, one can choose a coordinate basis of the form $\partial_i \equiv \partial/\partial x^i, \partial_a \equiv \partial/\partial y^a$ on the tangent space TTM, that is, any vector $X \in TTM$ will be written locally as $X = X^i\partial_i + X^a\partial_a$.

In the following, we give a short introductory presentation of the geometry of the tangent bundle TM. This includes the introduction of the notion of the non-linear connection, the notion of d-connections, its torsion and finally its curvature. These concepts and the terminology are taken from the book of R. Miron and M. Anastasei.

There is a number of equivalent ways to define the notion of the non-linear connection, see Miron, Anastasiei [1], but here we choose the following.

A non-linear connection N on the total space TM is characterized by the existence of a subbundle HTM of the bundle TTM, such that the Whitney sum

$$TTM = HTM \oplus VTM \tag{1}$$

holds. The subspace HTM is called horizontal and the subspace VTM is called vertical.

A local basis which contains bases both for the horizontal and vertical subspace of TTM is:

$$\{\delta_i \equiv \frac{\delta}{\delta x^i} = \partial_i - N_i^a(x,y)\partial_a, \partial_\beta\}, \tag{2}$$

where the symbols N_i^a give the coefficients of the non-linear connection N.

Before going on to define the weak torsion and curvature of the non-linear connection, let us define the notion of the d-tensor fields.

▷ A tensor field W of type $(p+r, q+s)$ on TM is said to be a d-tensor field or an M-tensor field of type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ on TM if

$$\begin{gathered} W(\omega_1,\ldots,\omega_p,X_1,\ldots,X_q,\omega_{p+1,\ldots,}\omega_{p+r},X_{q+1},\ldots,X_{q+s}) = \\ W(h\omega_1,\ldots,h\omega_p,hX_1,\ldots,hX_q,v\omega_{p+1},\ldots,v\omega_{p+r},vX_{q+1},\ldots,vX_{q+s}), \end{gathered} \tag{3}$$

where h is the horizontal projection and v the vertical one.

Locally, a d-tensor field of type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ will be written in the form :

$$W = W^{i_1\ldots i_p a_1 \ldots a_r}_{j_1\ldots j_q \beta_1\ldots\beta_s}\delta_{i_1}\otimes\cdots\otimes\partial_{a_1}\otimes\cdots\otimes dx^{j_1}\cdots\otimes\delta y^{\beta_1}\otimes\cdots\otimes\delta y^{\beta_s}, \tag{4}$$

where

$$\delta y^a = dy^a + N_i^a(x,y)dx^i \tag{5}$$

is the adapted vertical co-basis.

Associated to the non-linear connection N are two important d-tensor fields, with the components:

$$t^a_{jk} \equiv \partial_k N^a_j - \partial_j N^a_k,\ R^a_{jk} \equiv \delta_k N^a_j - \delta_j N^a_k. \tag{6}$$

The first is called the weak torsion and the second the curvature of the non-linear connection N.

In order to enrich the geometrical structure of the total space TM, we consider the possibility of a linear connection on TM, which respects the already imposed structure on TTM due to the existence of a non-linear connection. Thus, we introduce the notion of the distinguished or d-connection (cf. [1]).

A linear connection ∇ on TM will be called *distinguished* iff it preserves under parallel transportation the horizontal and the vertical distributions HTM and VTM respectively.

With respect to the adapted basis (δ_i, ∂_a), the local representation of the d-connections is given by:

$$\nabla_{\delta_k}\delta_j = L^i_{jk}(x,y)\delta_i, \tag{7}$$

$$\nabla_{\delta_k}\partial_\beta = \tilde{L}^a_{\beta k}(x,y)\partial_a, \tag{8}$$

$$\nabla_{\partial_\gamma}\delta_j = \tilde{C}^i_{j\gamma}(x,y)\delta_i, \tag{9}$$

$$\nabla_{\partial_\gamma}\partial_\beta = C^a_{\beta\gamma}(x,y)\partial_a, \tag{10}$$

and

$$\nabla_{\delta_k} f = \frac{\delta f}{\delta x^k}, \tag{11}$$

$$\nabla_{\partial_\gamma} f = \frac{\partial f}{\partial y^\gamma}, \tag{12}$$

where f is a scalar field over TM. Based on the above symbolism, from now on we shall denote this d-connection as $D\Gamma = (L^i_{jk}, \tilde{L}^a_{\beta k}, \tilde{C}^i_{j\gamma}, C^a_{\beta\gamma})$. Note that although this is a local form, it constitutes a uniquely global object, if the local coefficients satisfy the following transformation rules on intersections:

$$L^{i'}_{j'k'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}} L^i_{jk} + \frac{\partial x^{i'}}{\partial x^k} \frac{\partial^2 x^k}{\partial x^{j'} \partial x^{k'}}, \tag{13}$$

$$\tilde{L}^{a'}_{\beta' k'} = M^{a'}_a M^{\beta'}_\beta \frac{\partial x^k}{\partial x^{k'}} \tilde{L}^a_{\beta k} + M^{a'}_\gamma \frac{\partial M^\gamma_{\beta'}}{\partial x^{k'}}, \tag{14}$$

$$\tilde{C}^{i'}_{j'\gamma'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial xj'} M^\gamma_{\gamma'} \tilde{C}^i_{j\gamma}, \tag{15}$$

$$C^{a'}_{\beta'\gamma'} = M^{a'}_a M^\beta_{\beta'} M^\gamma_{\gamma'} C^\alpha_{\beta\gamma}\ , \tag{16}$$

where $\partial_{a'} = M^a_{a'} \partial_a$ and $M^a_{a'} = \frac{\partial y^a}{\partial y^{a'}}$.

Before completing this short introduction to d- connections, we add a little more symbolism on h- and v-covariant differentiation of d-tensors. For example, let W be a d-tensor field of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ given by

$$W = W^{ia}_{j\beta} \delta_i \otimes \partial_a \otimes dx^j \otimes \delta y^\beta. \tag{17}$$

For the vector field $X = X^k \delta_k + Y^c \partial_c$ the h- and v-covariant derivatives are given respectively as

$$\nabla_{hx} W = X^k W^{ia}_{j\beta \backslash k} \delta_i \otimes \partial_a \otimes dx^j \otimes \delta y^\beta, \tag{18}$$

where:

$$W^{ia}_{j\beta|k} = \delta_k W^{ia}_{j\beta} + L^i_{hk} W^{ha}_{j\beta} + \tilde{L}^a_{\gamma k} W^{i\gamma}_{j\beta} - L^h_{jk} W^{ia}_{h\beta} - \tilde{L}^\gamma_{\beta k} W^{ia}_{j\gamma} \tag{19}$$

and

$$\nabla_{vX} W = Y^\gamma W^{ia}_{j\beta}|_\gamma \delta_i \otimes \partial_a \otimes dx^j \otimes \delta y^\beta, \tag{20}$$

where:

$$W^{ia}_{j\beta}|_\gamma = \partial_\gamma W^{ia}_{j\beta} + \tilde{C}^i_{h\gamma} W^{ha}_{j\beta} + C^a_{\delta\gamma} W^{i\delta}_{j\beta} - \tilde{C}^h_{j\gamma} W^{ia}_{h\beta} - C^\delta_{\beta\gamma} W^{ia}_{j\delta}. \tag{21}$$

The above notations can be generalized in an obvious way for a d-tensor field of any order.

In this section we shall also introduce and calculate the torsion and curvature tensors of d-connections. Let us start with the torsion tensor T. This is given by the relation:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \forall X,Y \in \mathcal{X}(TM). \tag{22}$$

It can be decomposed in the following way:

$$T(X,Y) = T(hX,hY) + T(hX,vY) + T(vX,hY) + T(vX,vY). \tag{23}$$

Taking into account the skew-symmetry of T and the equation $h[vX,vY] = 0$,we deduce that T can be known completely if the following five components are given. These give also the local form of the torsion:

$$hT(\delta_k,\delta_j) = T^i_{jk}\delta_i, \tag{24}$$

$$vT(\delta_k,\delta_j) = \tilde{T}^a_{jk}\partial_a, \tag{25}$$

$$hT(\partial_\beta,\delta_j) = \tilde{P}^i_{j\beta}\delta_i, \tag{26}$$

$$vT(\partial_\beta,\delta_j) = P^a_{j\beta}\partial_a, \tag{27}$$

$$vT(\partial_\gamma,\partial_\beta) = S^a_{\beta\gamma}\partial_a. \tag{28}$$

The following theorem gives the local components of the torsion tensor:

Theorem. **[1]** *The local components in the frame* (δ_i,∂_a) *of the torsion of a d-connection* $D\Gamma = (L^i_{jk},\tilde{L}^a_{\beta k},\tilde{C}^i_{ja},C^a_{\beta\gamma})$ *are:*

$$T^i_{jk} = L^i_{jk} - L^i_{kj}, \tag{29}$$

$$\tilde{T}^a_{jk} = R^a_{jk} \qquad \textit{(given in (6))}, \tag{30}$$

$$\tilde{P}^i_{j\beta} = \tilde{C}^i_{j\beta}, \tag{31}$$

$$P^a_{j\beta} = \partial_\beta N^a_j - \tilde{L}^a_{\beta j}, \tag{32}$$

$$S^a_{\beta\gamma} = C^a_{\beta\gamma} - C^a_{\gamma\beta}. \tag{33}$$

Next we introduce the curvature tensor R of the d-connection ∇. This is given by:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \forall X,Y,Z \in \mathcal{X}(TM). \tag{34}$$

This tensor is completely known when the following six components, determining its local form, are given:

$$R(\delta_k,\delta_j)\delta_h = R^i_{hjk}\delta_i, \tag{35}$$

$$R(\delta_k,\delta_j)\partial_\beta = \tilde{R}^a_{\beta jk}\partial_a, \tag{36}$$

$$R(\partial_\gamma,\delta_k)\delta_j = \tilde{P}^i_{jk\gamma}\delta_i, \tag{37}$$

$$R(\partial_\gamma,\delta_k)\partial_\beta = P^a_{\beta k\gamma}\partial_a, \tag{38}$$

$$R(\partial_\gamma,\partial_\beta)\delta_j = \tilde{S}^i_{j\beta\gamma}\delta_i, \tag{39}$$

$$R(\partial_\delta,\partial_\gamma)\partial_\beta = S^a_{\beta\gamma\delta}\partial_a. \tag{40}$$

Denoting the local components of the curvature of a d-connection ∇ on TM as $D\Gamma = (L^i_{jk},\tilde{L}^a_{bk},\tilde{C}^i_{jb},C^a_{bc})$, a direct computation yields to the following formulas obtained in [1]:

$$R^i_{hjk} = \delta_k L^i_{hj} - \delta_j L^i_{hk} + L^m_{hj}L^i_{mk} - L^m_{hk}L^i_{mj} + \tilde{C}^i_{ha}R^a_{jk}, \tag{41}$$

$$\tilde{R}^a_{\beta jk} = \delta_k \tilde{L}^a_{\beta j} - \delta_j \tilde{L}^a_{\beta k} + \tilde{L}^\gamma_{\beta j}\tilde{L}^a_{\gamma k} - \tilde{L}^\gamma_{\beta k}\tilde{L}^a_{\gamma j} + C^a_{\beta\gamma}R^\gamma_{jk}, \tag{42}$$

$$\tilde{P}^i_{jka} = \partial_a L^i_{jk} - \tilde{C}^i_{ja\backslash k} + \tilde{C}^i_{j\beta}P^\beta_{ka}, \tag{43}$$

$$P^a_{\beta k\gamma} = \partial_\gamma \tilde{L}^a_{\beta k} - C^a_{\beta\gamma\backslash k} + C^a_{\beta\delta}P^\delta_{k\gamma}, \tag{44}$$

$$\tilde{S}^i_{j\beta\gamma} = \partial_\gamma \tilde{C}^i_{j\beta} - \partial_\beta \tilde{C}^i_{j\gamma} + \tilde{C}^h_{j\beta}\tilde{C}^i_{h\gamma} - \tilde{C}^h_{j\gamma}\tilde{C}^i_{h\beta}, \tag{45}$$

$$S^a_{\beta\gamma\delta} = \partial_\delta C^a_{\beta\gamma} - \partial_\gamma C^a_{\beta\delta} + C^\varepsilon_{\beta\gamma}C^a_{\varepsilon\delta} - C^\varepsilon_{\beta\delta}C^a_{\varepsilon\gamma}. \tag{46}$$

For a complete presentation of the local form of the Bianchi and Ricci identities see [1].

2.1 Metric structures on TM

The tangent bundle will be endowed with a metric structure. The metric tensor will be denoted as G. This tensor is chosen in such a way that the horizontal and vertical subspaces are vertical to each other. That is, we demand that

$$G(hX, vY) = 0, \forall X, Y \in \mathcal{X}(TM), \tag{47}$$

where h and v are the horizontal and vertical projectors associated to the non-linear connection N. Thus the local form of G can be written as:

$$G(x, y) = g_{ij}(x, y)dx^i \otimes dx^j + h_{\alpha\beta}(x, y)\delta y^a \otimes \delta y^\beta, \tag{48}$$

where

$$g_{ij} = G(\delta_i, \delta_j) \qquad \text{and} \qquad h_{\alpha\beta} = G(\partial_\alpha, \partial_\beta). \tag{49}$$

Remark In section 2 the above metric tensor will be given a more specific form. In fact, we shall associate g_{ij} with the gravitational metric and $h_{\alpha\beta}$ with both the gauge field and the gravitational metric.

2.2 The notion of Lagrange spaces

From [1] we quote the following definition of a Lagrange space.

$\triangleright$ *A regular Lagrangian* is a function $L : TM \to R$ which fulfills the following conditions:

a) L is of class C^∞ on $TM \setminus \{0\}$ and continuous on the image of the null section in the tangent bundle.

b) The matrix with the entries

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} \tag{50}$$

has rank $n = \dim M$ on $TM - \{0\}$.

$\triangleright$ A *Lagrange space* is a pair $L^n = (M, L(x, y))$, which consists of a smooth manifold M and a regular Lagrangian L, for which the quadratic form $\Phi(\xi) = g_{ij}\xi^i\xi^j$ has constant signature on $TM - \{0\}$.

2.3 Caratheodory transformations

Let $\gamma : [0,1] \to M$ be a parameterized curve on the manifold M. Then the action integral associated with L along γ is defined as

$$I(\gamma) = \int_0^1 L(x, \dot{x}) dt. \tag{51}$$

Using the calculus of variations we find that a necessary condition for γ to be an extremum of the action integral is that it satisfies the Euler-Lagrange equations

$$\frac{d}{dt}(\frac{\partial L}{\partial y^i}) - \frac{\partial L}{\partial x^i} = 0, \qquad y^i = \frac{dx^i}{dt}. \tag{52}$$

This set of equations can be satisfied simultaneously by different regular Lagrangians. That is, for different L and L' we have:

$$\frac{d}{dt}(\frac{\partial L}{\partial y^i}) - \frac{\partial L}{\partial x^i} = \frac{d}{dt}(\frac{\partial L'}{\partial y^i}) - \frac{\partial L'}{\partial x^i}, \qquad y^i = \frac{dx^i}{dt}. \tag{53}$$

These two Lagrangians will be called equivalent. The relation of equivalence is given by the Caratheodory transformation, see [1], by which two regular Lagrangians L and L' are equivalent iff

$$L'(x,y) = L(x,y) + \frac{\partial \phi}{\partial x^i} y^i + a, \tag{54}$$

where ϕ is an arbitrary function on M and a any constant.

In the next section we shall demand that this transformation becomes a gauge transformation of the gauge potentials that enter L.

3 Geometric Unification of Gravity and Gauge Field Theories

In this section we propose a model of unification of gravity and gauge field theories based in the above geometrical framework. Thus, all the above geometrical objects will be given a specific form leading eventually to the gauge field equations and the Einstein equations.

In our approach we will consider a family of Lagrangians $L^{\mathcal{E}}$,parameterized by the elements of the set of real-valued linear operators $\mathcal{E} : g \to R$ acting on the Lie algebra g of the symmetry group G.

In order to define these Lagrangians let us consider a basis of generators $\{t^B\}, 1 \leqslant B \leqslant m = \dim G$, on the Lie algebra g^2. Thus, the gauge potential, being a $g-$valued 1-form A, can be written as

$$A = \sum_{B,i} A_{Bi} dx^i t^B, \tag{55}$$

where A_{Bi} are the components of the gauge potential in the above basis. Of course, this is the form taken by the gauge potential A, after the choice of a gauge. A choice of gauge is the choice of a local basis for the fiber (being a vector space in a representation of the group G) of the vector bundle $E(\pi, M, G)$, that describes geometrically the gauge theory. We denote this basis as $\{a_{AB}e_B\}$, where $\{e_B\}$ is the standard basis on G. Thus, a_{CB} is nothing more than the $_B-$component, relative to the standard basis, of the $_C-$vector of the basis on the fiber. Notice that the gauge indices in these symbols can be raised via the relations:

$$a_{AB}a^{CB} = \delta^C_A \qquad \text{and} \qquad a_{AB}a^{AC} = \delta^C_B. \tag{56}$$

In virtue of the above notation we formulate our family of regular Lagrangians as:

$$L^{\mathcal{E}}(x,y) = g_{ij}(x)y^i y^j + \mathcal{E}(a_{AB}(x)A_{Bi}(x)t^A)y^i, \tag{57}$$

where g_{ij} is the gravitational metric. Thus, we have constructed a family of Lagrange spaces, the members of which we denote as $(M, L^{\mathcal{E}})$. In the following, these spaces will take a specific structure. In particular, we will define their metric structure, non-linear connection and d-connection. We will also calculate the torsion and curvature d-tensors.

However, before proceeding to these definitions, let us first look deeper into the implications of the choice (57) for the gauge-covariant derivatives D, in connection with the Caratheodory transformations.

In order to do this, let us define the notion of the gauge-covariant derivative D. This derivative is defined to act over the algebra of G- and $g-$valued tensors. For a $g-$valued p-form $u_{A\mathcal{M}}$, where $_{\mathcal{M}}$ is a space-time multi-index of the form $ij\ldots a\beta\ldots$, we define the gauge-covariant derivatives of $u_{A\mathcal{M}}$ as usual:

$$D_i u_{A\mathcal{M}} = u_{A\mathcal{M}|i} + f_{ABC}A_{Bi}u_{C\mathcal{M}} \tag{58}$$

and

$$D_a u_{A\mathcal{M}} = u_{A\mathcal{M}|a} + f_{ABC}A_{Ba}u_{C\mathcal{M}}, \tag{59}$$

where f_{ABC} are the structure constants of the Lie group G and $u_{A\mathcal{M}|i}, u_{A\mathcal{M}|a}$ means respectively the h- and v-covariant derivatives of $u_{A\mathcal{M}}$. Physically, more interesting till now are the forms with multi-index of the form $ij\ldots$, and the covariant derivative of relation (58). Thus, in the following we will be referring to these only.

Our next step is to show under which conditions a Caratheodory transformation of the Lagrangian (57) yields a gauge transformation of the potential A. We will reveal that the necessary and sufficient condition relates the gauge potential with the choice of a gauge and the gauge-covariant derivative D.

For the Caratheodory transformation we get from (54) and (57)

$$\tilde{L}^{\mathcal{E}}(x,y) = L^{\mathcal{E}}(x,y) + y^i\partial_i\phi. \tag{60}$$

We demand this transformation to gauge transform the potential A. That is:

$$\tilde{L}^{\mathcal{E}}(x,y) = g_{ij}y^i y^j + \mathcal{E}(a_{AB}\tilde{A}_{Bi}t^A y^i),$$

where

$$\tilde{A}_{Bi} = A_{Bi} + D_i\phi_B \tag{61}$$

and ϕ_B is a function on M.

Choosing

$$\phi(x) = \mathcal{E}(a_{AB}(x)\phi_B(x)t^A) \tag{62}$$

in (60) we find:

$$g_{ij}y^iy^j + \mathcal{E}(a_{AB}\tilde{A}_{Bi}t^Ay^i) = g_{ij}y^iy^j + \mathcal{E}(a_{AB}A_{Bi}t^Ay^i + t^Ay^i\partial_i(a_{AB}\phi_B)) \Longleftrightarrow$$

$$\mathcal{E}(a_{AB}\tilde{A}_{Bi}t^Ay^i) = \mathcal{E}(t^Ay^i(a_{AB}A_{Bi} + \partial_i(a_{AB}\phi_B))).$$

Since the operator $\mathcal{E}$ and y^i are arbitrary and t^A is linearly independent to each other, we have:

$$a_{AB}\tilde{A}_{Bi} = a_{AB}A_{Bi} + \partial_i(a_{AB}\phi_B) \Longleftrightarrow \tilde{A}_{Bi} = A_{Bi} + a^{AB}\partial_i(a_{AC}\phi_C) \Longleftrightarrow$$

$$D_i\phi_B = a^{AB}\partial_i(a_{AC}\phi_C) \Longleftrightarrow$$

$$a_{AB}D_i\phi_B = \partial_i(a_{AC}\phi_C). \tag{63}$$

Setting $u_A = a_{AB}\phi_B$ in (37) we find:

$$a_{AB}D_i(a^{DB}u_D) = \partial_i(a_{AC}a^{DC}u_D) \Longleftrightarrow$$

$$\partial_i u_A = a_{AB}D_i(a^{DB}u_D). \tag{64}$$

This relation also gives:

$$D_iu_A = \partial_iu_A - a_{AB}D_i(a^{DB}u_D) + D_iu_A \Longleftrightarrow$$

$$D_iu_A = \partial_iu_A - a_{AB}D_i(a^{DB}u_D) + a_{AB}a^{CB}D_iu_C \Longleftrightarrow$$

$$D_iu_A = \partial_iu_A - a_{AB}(D_ia^{DB})u_D \Longleftrightarrow$$

$$f_{ABC}A_{Bi} = -a_{AD}D_ia^{CD}. \tag{65}$$

This is the desired condition. It can also be found in Penrose and Rindler [3], pp.347. There, the relation 5.5.12 can be seen to be identical to ours (39), after choosing the adjoint representation of G. However, in [3] this relation is taken ad hoc, whereas in our approach it is deduced by our demand that the Caratheodory transformation of (31) gauge-transforms the gauge potential.

Before completing this short parenthesis notice also that (39) gives in general

$$\nabla_iu_{Ajk\ldots} = a_{AB}D_i(a^{CB}u_{Cjk\ldots}). \tag{66}$$

This relation will be very useful in section 3, where we derive the gauge field equations of motion.

Now, we may proceed to determine the structure of the tangent bundle. Let us start with the definition of the non-linear connection.

3.1 The non-linear connection

In our approach, the non-linear connection N on TM will be given in ad hoc form, which will eventually prove to be suitable for the derivation of the gauge field equations. Thus, we set:

$$N_i^a(x,y) = \Gamma_{i\gamma}^a(x)y^\gamma - \mathcal{E}(a_{AB}t^A g^{\alpha\beta}F_{B\beta i}), \tag{67}$$

where $g_{a\beta}$ is the gravitational metric, Γ_{jk}^a its Christoffel symbols and $F_{B\beta i}$ the field strength components. One can easily check that these coefficients for N_i^a have the following required transformation under a change of coordinates $(x^i, y^a) \to (x'^i, y'^a)$. That is, N_i^a transform as:

$$N_{i'}^{a'} = M_a^{a'} \frac{\partial x^i}{\partial x^{i'}} N_i^a - \frac{\partial M_a^{a'}}{\partial x^i} y^a. \tag{68}$$

3.2 The metric structure

As mentioned in section 2, the metric tensor on TM has the form:

$$G(x,y) = \tilde{g}_{ij}(x,y)dx^i \otimes dx^j + h_{\alpha\beta}(x,y)\delta y^a \otimes \delta y^\beta. \tag{69}$$

In our approach, $\tilde{g}_{ij}$ will be given by (24) as:

$$\tilde{g}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^{\mathcal{E}}}{\partial y^i \partial y^j} = g_{ij}(x), \tag{70}$$

i.e. $\tilde{g}_{ij}$ becomes the y-independent gravitational metric. On the other hand, for dim $G \geqslant 2$ the metric of the vertical subspace can be considered in the form:

$$h_{\alpha\beta}(x,y) = \lambda_{Aa}(x,y)\lambda_{A\beta}(x,y), \tag{71}$$

where λ_{Aa} denotes the components of m tetrads on the vertical subspace. In the following, it will be seen that these quantities are not randomly chosen, but they are related to the field strength tensor F_{Bij} of the gauge field. If we also consider the quantities λ_A^a given by the relation:

$$\lambda_{Aa}\lambda_A^\beta = \delta_a^\beta, \tag{72}$$

then we have:

$$h^{a\beta} = \lambda_A^a \lambda_A^\beta. \tag{73}$$

3.3 The d-connection. Torsions and Curvatures

We choose the d-connection to have $\tilde{L}_{\beta k}^a = 0, \tilde{C}_{ja}^i = 0$ and $L_{jk}^i = \Gamma_{jk}^i$, where Γ_{jk}^i are the Christoffel symbols of the gravitational metric, i.e. to be of the local form $D\Gamma = (\Gamma_{jk}^i, 0, 0, C_{\beta\gamma}^a)$.In addition for the components $C_{\beta\gamma}^a$ we set:

$$C_{\beta\gamma}^a = \frac{1}{2} g^{\alpha\delta} F_{A\delta\gamma}\lambda_{A\beta}. \tag{74}$$

Notice that, since $F_{Aa\beta}$ and λ_{Aa} transform as d-tensors, the above defined coefficients $C^a_{\beta\gamma}$ have the right transformation (16) under a change of coordinates.

The torsions of this connection can be calculated from relations (29)-(33) and (67). Thus:

$$T^i_{jk} = 0, \tag{75}$$

$$\tilde{T}^a_{jk} = R^a_{jk} = R^a_{\gamma jk} y^\gamma + 2\mathcal{E}(t^A)(\partial_{[j|}(a_{AB} g^{\alpha\beta} F_{B\beta|k]}) + a_{AB} g^{\gamma\beta} F_{B\beta[k\backslash}\Gamma^a_{\backslash j]\gamma}, \tag{76}$$

$$\tilde{P}^i_{j\beta} = 0, \tag{77}$$

$$P^a_{j\beta} = \Gamma^a_{j\beta}, \tag{78}$$

$$S^a_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(F_{A\delta\gamma}\lambda_{A\beta} - F_{A\beta\delta}\lambda_{A\gamma}) = g^{\alpha\delta} F_{A\delta[\gamma|}\lambda_{A|\beta]}. \tag{79}$$

For the curvatures, the relations (41)-(46) and (74) yield:

$$R^i_{hjk} = \partial_k \Gamma^i_{hj} - \partial_j \Gamma^i_{hk} + \Gamma^m_{hj}\Gamma^i_{mk} - \Gamma^m_{hk}\Gamma^i_{mj}, \tag{80}$$

$$\tilde{R}^a_{\beta jk} = C^a_{\beta\gamma} R^\gamma_{jk} = \frac{1}{2} g^{\alpha\delta} F_{A\delta\gamma}\lambda_{A\beta}, \tag{81}$$

$$\tilde{P}^i_{jka} = 0, \tag{82}$$

$$P^a_{\beta\kappa\gamma} = -C^a_{\beta\gamma\backslash k} + C^a_{\beta\delta}\Gamma^\delta_{k\gamma} = -\frac{1}{2}\delta_k(g^{\alpha\delta} F_{A\delta\gamma}\lambda_{A\beta}) + \frac{1}{2} g^{a\varepsilon} F_{A\varepsilon\delta}\lambda_{A\beta}\Gamma^\delta_{k\gamma}, \tag{83}$$

$$\tilde{S}^i_{j\beta\gamma} = 0, \tag{84}$$

$$S^a_{\beta\gamma\delta} = \frac{1}{2} F^{\cdot a}_{A\cdot\gamma}\partial_\delta\lambda_{A\beta} - \frac{1}{2} F^{\cdot a}_{A\cdot\delta}\partial_\gamma\lambda_{A\beta} + \frac{1}{4} F^{\cdot\varepsilon}_{A\cdot\gamma} F^{\cdot a}_{B\cdot\delta}\lambda_{A\beta}\lambda_{B\varepsilon} - \frac{1}{4} F^{\cdot\varepsilon}_{A\cdot\delta} F^{\cdot a}_{B\cdot\gamma}\lambda_{A\beta}\lambda_{B\varepsilon}. \tag{85}$$

In this equation and the rest of the text, the indices $\alpha, \beta, ...$ in $F^{\cdot\cdot\alpha}_{A\cdot\cdot\beta}$ are lowered or raised via the gravitational metric $g_{\alpha\beta}$.

From (85) we can now calculate the Ricci scalar for this d-connection. It will be equal to $R + S$, where R is the gravitational scalar and S the scalar corresponding to the curvature $S^a_{\beta\gamma\delta}$. In order to calculate it, let us first determine the Ricci tensor $S_{\beta\gamma}$. We have:

$$S_{\beta\gamma} = S^a_{\beta\gamma\alpha} = \frac{1}{2} F^{\cdot a}_{A\cdot\gamma}\partial_a\lambda_{A\beta} - \frac{1}{2} F^{\cdot a}_{A\cdot a}\partial_\gamma\lambda_{A\beta} + \frac{1}{4} F^{\cdot\varepsilon}_{A\cdot\gamma} F^{\cdot a}_{B\cdot a}\lambda_{A\beta}\lambda_{B\varepsilon} - \frac{1}{4} F^{\cdot\varepsilon}_{A\cdot a} F^{\cdot a}_{B\cdot\gamma}\lambda_{A\beta}\lambda_{B\varepsilon}.$$

Since $F_{Aa\beta}$ is antisymmetric in its space-time indices we must have $F^{\cdot a}_{A\cdot a} = 0$. Thus:

$$S_{\beta\gamma} = \frac{1}{2} F^{\cdot a}_{A\cdot\gamma}\partial_a\lambda_{A\beta} - \frac{1}{4} F^{\cdot\varepsilon}_{A\cdot a} F^{\cdot a}_{B\cdot\gamma}\lambda_{A\beta}\lambda_{B\varepsilon} \overset{(79)}{=} \tag{86}$$

$$= \frac{1}{2} F^{\cdot a}_{A\cdot\gamma}\partial_a\lambda_{A\beta} - \frac{1}{4} F^{\cdot\varepsilon}_{A\cdot a} S^a_{\varepsilon\gamma}\lambda_{A\beta} + \frac{1}{4} F^{\cdot\varepsilon}_{A\cdot a} F^{\cdot a}_{B\cdot\varepsilon}\lambda_{A\beta}\lambda_{B\gamma}. \tag{87}$$

Therefore:

$$S = h^{\beta\gamma} S_{\beta\gamma} = \frac{1}{2} h^{\beta\gamma} F^{\cdot a}_{A\cdot\gamma}\partial_a\lambda_{A\beta} - \frac{1}{4} F^{\cdot\varepsilon}_{A\cdot a} S^a_{\varepsilon\gamma} h^{\beta\gamma}\lambda_{A\beta} - \frac{1}{4} F_{Aa\varepsilon} F^{a\varepsilon}_A = \tag{88}$$

$$= \frac{1}{2} F^{\cdot\varepsilon}_{A\cdot a}(h^{\beta a}\partial_\varepsilon\lambda_{A\beta} - \frac{1}{2} S^a_{\varepsilon\gamma} h^{\beta\gamma}\lambda_{A\beta}) - \frac{1}{4} F_{Aa\varepsilon} F^{a\varepsilon}_A. \tag{89}$$

As mentioned above, the scalar curvature $\Re$ of our space will be:

$$\Re = R + S. \tag{90}$$

In virtue of (88) we get:

$$\Re = R + \frac{1}{2}F^{\varepsilon}_{Aa}(h^{\beta a}\partial_{\varepsilon}\lambda_{A\beta} - \frac{1}{2}S^{a}_{\varepsilon\gamma}h^{\beta\gamma}\lambda_{A\beta}) - \frac{1}{4}F_{Aa\varepsilon}F^{a\varepsilon}_{A}. \tag{91}$$

In the case where the first term of the right hand side of (88) vanishes, we have:

$$\partial_{\varepsilon}\lambda_{A\delta} - \frac{1}{2}h_{a\delta}h^{\beta\gamma}S^{a}_{\varepsilon\gamma}\lambda_{A\beta} = 0. \tag{92}$$

Thus:

$$S = -\frac{1}{4}F_{Aa\varepsilon}F^{a\varepsilon}_{A} \tag{93}$$

and the scalar $\Re$ of our d-connection becomes the scalar curvature of the conventional gauge field theory with the presence of a gravitational field

$$\Re = R - \frac{1}{4}F_{Aa\varepsilon}F^{a\varepsilon}_{A}. \tag{94}$$

At this point, we present the solution of equation (91). In order to do this, let us make the ansatz:

$$\lambda_{A\delta}(x,y) = f(x,y)k_{A\delta}(x), \tag{95}$$

where f is a function over TM to be determined and $k_{A\delta}$ a $g-$valued covector on TM satisfying the relation:

$$g^{a\beta}k_{\Lambda a}k_{B\beta} = \delta_{\Lambda B}. \tag{96}$$

This implies that by setting

$$h_{a\beta}(x,y) = f^{2}(x,y)g_{a\beta}(x), \tag{97}$$

we obtain the required:

$$h^{a\beta}\lambda_{Aa}\lambda_{B\beta} = \delta_{AB}. \tag{98}$$

With the use of (95) and (97) in (92) we take:

$$\partial_{\varepsilon}\lambda_{A\delta} - \frac{1}{2}h_{a\delta}h^{\beta\gamma}S^{a}_{\varepsilon\gamma}\lambda_{A\beta} = 0 \Longleftrightarrow$$

$$\partial_{\varepsilon}\lambda_{A\delta} - \frac{1}{4}h_{a\delta}h^{\beta\gamma}F^{a}_{B\gamma}\lambda_{B\varepsilon}\lambda_{A\beta} + \frac{1}{4}h_{a\delta}h^{\beta\gamma}F^{a}_{B\varepsilon}\lambda_{B\gamma}\lambda_{A\beta} = 0 \overset{(95),(97)}{\Longleftrightarrow}$$

$$\partial_{\varepsilon}(fk_{A\delta}) - \frac{1}{4}g_{a\delta}f^{2}\frac{1}{f^{2}}g^{\beta\gamma}F^{a}_{B\gamma}f^{2}k_{B\varepsilon}k_{A\beta} + \frac{1}{4}g_{a\delta}f^{2}\frac{1}{f^{2}}g^{\beta\gamma}F^{a}_{B\varepsilon}f^{2}k_{B\gamma}k_{A\beta} = 0 \Longleftrightarrow$$

$$k_{A\delta}\partial_{\varepsilon}f - \frac{1}{4}f^{2}F^{\beta}_{B\delta}k_{B\varepsilon}k_{A\beta} + \frac{1}{4}f^{2}F_{B\delta\varepsilon}\delta_{AB} = 0 \Longleftrightarrow$$

$$k_{A\delta}\partial_\varepsilon f - \frac{1}{4}f^2F^\beta_{B\delta}k_{B\varepsilon}k_{A\beta} + \frac{1}{4}f^2F_{A\delta\varepsilon} = 0 \Longleftrightarrow$$
$$\partial_\varepsilon f - \frac{1}{4}f^2F^\beta_{B\delta}k^\delta_A k_{B\varepsilon}k_{A\beta} + \frac{1}{4}f^2F_{A\delta\varepsilon}k^\delta_A = 0 \Longleftrightarrow$$
$$\partial_\varepsilon f + \frac{1}{4}f^2F_{A\delta\varepsilon}k^\delta_A = 0 \Longleftrightarrow \frac{1}{f^2}\partial_\varepsilon f = -\frac{1}{4}F^\delta_{A\varepsilon}k_{A\delta} \Longleftrightarrow$$
$$\partial_\varepsilon(\frac{1}{f^2}) = \frac{1}{4}F^\delta_{A\varepsilon}k_{A\delta}.$$

Integrating with respect to y we get:

$$\frac{1}{f} = \frac{1}{4}F^\delta_{A\varepsilon}k_{A\delta}y^\varepsilon + g(x),$$

where g is a function to be determined by initial conditions at some global section of the tangent bundle TM. Thus, we have:

$$f(x,y) = \frac{1}{\frac{1}{4}F^\delta_{A\varepsilon}k_{A\delta}y^\varepsilon + g(x)},$$

i.e.e in virtue of (95):

$$\lambda_{A\delta}(x,y) = \frac{k_{A\delta}(x)}{\frac{1}{4}F^\delta_{A\varepsilon}k_{A\delta}y^\varepsilon + g(x)}. \tag{99}$$

Remark 1. We conclude that the solution of equation (92),i.e. equation (99), implies that λ_{Aa} is not a randomly chosen covariant vector. The ansatz (95) expresses it as a multiple of a covector field $k_{A\delta}$ over the space-time manifold M, which as in (71), can be given as a set of tetrads for the gravitational metric $g_{a\beta}$, assuming of course that $\dim G > 1$. On the other hand, the factor f is a scalar field over TM given in terms of the gauge field components $F_{Aa\varepsilon}$ and the tetrads $k_{A\delta}$.

Remark 2. Note that, when $\dim G = 1, h_{a\beta}$ cannot be chosen as in (71), since then it will not be invertible. Equivalently, neither $g_{a\beta}$ can be written as in (71) in terms of $k_{A\delta}$. In that case, in order to have equation (93), covector λ_a has to be of unit length with respect to $h_{a\beta}$, i.e. we must have $h^{a\beta}\lambda_a\lambda_\beta = 1$. Equivalently, for $k_{A\delta}$ we must have $g^{a\beta}k_a k_\beta = 1$.

4 The Field Equations

The field equations can be derived from the following action:

$$I = \int_M \Re d\mu. \tag{100}$$

by the use of the calculus of variations. In particular, the variative with respect to g_{ij} will give the Einstein equations of the gravitational field, while the variative with respect to the gauge potential A_{Bi} will give the gauge field equations. However, in this presentation we derive the gauge field equations from the properties of the h-deflection tensor. On the other hand, the Einstein equations for the gravitational field are not derived here explicitly, but they are presented at the end of this section.

4.1 Gauge Field Equations

First let us give the definition of the h-deflection tensor.

We call *h-deflection tensor* of the d-connection ∇ the d-tensor field D^i_j given by

$$D^i_j \equiv y^i_{|j}. \tag{101}$$

It is easy to see that the h-deflection tensor satisfies the equation:

$$D^i_j = \Gamma^i_{j\gamma} y^\gamma - N^i_j. \tag{102}$$

Let us take:

$$D^i_j = \mathcal{E}(a_{AB} t^A g^{ri} F_{Brj}) \qquad \text{i.e.} \qquad D_{ij} = \mathcal{E}(a_{AB} t^A F_{Aij}). \tag{103}$$

in equation (67). Writing down the Ricci identities for this tensor one have:

$$D_{ij|k} - D_{ik|j} = R_{hijk} y^h - R^r_{jk} y_{i\backslash r}. \tag{104}$$

In [2] it has been proved that (104) gives:

$$\sum_{(ijk)} D_{[ij]\backslash k} = 0, \tag{105}$$

$\sum_{(ijk)}$ meaning the cyclic summation over the indices i, j, k. From (105) and (103) we have:

$$\sum_{(ijk)} D_{[ij]\backslash k} = \sum_{(ijk)} \nabla_k [\mathcal{E}(a_{AB} t^A F_{Bij})] = 0.$$

Due to the arbitrariness of the choice of the operator $\mathcal{E}$, we deduce that:

$$\begin{aligned} \sum_{(ijk)} \nabla_k (a_{AB} F_{Bij}) &= \sum_{(ijk)} a_{AB} D_k (a^{CB} a_{CD} F_{Dij}) = \\ \sum_{(ijk)} a_{AB} D_k F_{Bij} &= 0. \end{aligned} \tag{106}$$

Therefore, we have:

$$\sum_{(ijk)} D_k F_{Aij} = 0. \tag{107}$$

The last equation gives the first group of gauge field equations, the Bianchi identities on the vector bundle $E(\pi, M, G)$, that describes geometrically the gauge field.

For the second group of gauge field equations, consider the 1-form hJ_{Bi}, which we will call the d-horizontal charge current density covector for matter fields, defined by the equation:

$$\mathcal{E}(a_{AB} t^A hJ_{Bi}) = \frac{1}{2} g^{jk} (y_{i\backslash jk} - y_{j\backslash ik}). \tag{108}$$

This gives:

$$\begin{aligned}\mathcal{E}(a_{AB}t^A hJ_{Bi}) &= \frac{1}{2}g^{jk}(y_{i\backslash jk}-y_{j\backslash ik})=\frac{1}{2}g^{jk}(D_{ij\backslash k}-D_{ji\backslash k})= \\ g^{jk}D_{ij\backslash k} &= g^{jk}\nabla_k((a_{AB}t^A F_{Bij}))= \\ g^{jk}\mathcal{E}(a_{AB}t^A D_k(a^{CB}a_{CD}F_{Dij})) &= g^{jk}\mathcal{E}(a_{AB}t^A D_k F_{Bij}).\end{aligned} \tag{109}$$

Thus, we take

$$hJ_{Ai}=D^j F_{Aij}. \tag{110}$$

In the present case of the pure Yang-Mills theory, this becomes the second group of gauge field equations and it can be seen that $hJ_{Ai}=0$, due to the vanishing of the torsion T^i_{jk}. Notice that (110) is also derived after applying the principle of least action in the action integral (100). Of course, in the presence of matter fields, hJ_{Ai} will obtain a non-zero contribution from their charge currents. It would be interesting to see whether this contribution can be expressed through a non-vanishing gravitational torsion T^i_{jk}.

Furthermore, note that even in the absence of matter fields the pure Yang-Mills field has a non-zero charge current density covector hj_{Ai}, not entering the gauge field equations, which is defined as:

$$hj_{Ai}=\nabla^j F_{Aij}. \tag{111}$$

Due to the field equations $D^j F_{Aij}=0$, this covector is also equal to - $f_{ABC}A^j_B F_{Cij.}$

In order to prove the conservation of the charge, note that:

$$hj^i_{B|i}=\nabla_i\nabla_k F^{ik}_B=\frac{1}{2}(F^{ij}_{B|j|i}-F^{ij}_{B|i|j})=$$

$$=(R_{ij}-R_{ij})F^{ij}_B=0.$$

Thus:

$$hj^i_{A|i}=0. \tag{112}$$

In this way, we have given the physically interesting gauge field equations and proved the charge conservation.

4.2 The Einstein equations

The Einstein equations associated to the d-connection D are as follows:

$$R_{ij}-\frac{1}{2}(R-\frac{1}{4}F_{Aa\varepsilon}F^{a\varepsilon}_A)g_{ij}=k\mathcal{T}_{ij}\ , \tag{113}$$

where $\mathcal{T}_{ij}$ are the components of the energy-momentum tensor field.

Remark 3. Note that there is an alternative way to elaborate the present geometrical framework. In particular, we may keep the relation (91) without imposing (92). This would mean that the metric $h_{a\beta}$ is no longer dependent on the gravitational metric and assuming that $h_{a\beta\backslash\gamma} = 0$, its dynamical behavior is determined by the Einstein equation:

$$S_{a\beta} - \frac{1}{2}(R+S)h_{a\beta} = k\mathcal{T}_{a\beta}\,, \tag{114}$$

which is derived from the action of (100) and the variative with respect to the metric $h_{\alpha\beta}$. However, this approach is physically undistinguished..

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