

CLIFFORD ALGEBRAS ON SOBOLEV SPACES

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Abstract

Let X be a compact (spin) manifold, E an Hermitian vector bundle over X and D a non-degenerate 1-st order selfadjoint (pseudo) differential operator acting on the smooth cross-sections of E . Fixing the L^2 -norm $\|f\|$ of a section f of E , we fix the Sobolev k -norm $\|f\|_k$ of f to be $\|D^k f\|$. Then we say that the virtual dimension of the Sobolev k -space $W^k(X)$ of sections of E is $\nu = \zeta_{|D|}(0)$. If ν is an integer, we can construct the Clifford algebra $C(W^{-k})[e^\infty]$ over $W^{-k}(X)$ with the infinite spinor e^∞ . Modifying e^∞ according to *mod.8* class of ν , we represent e^∞ using D and its Green operator G . The Grassmann map from $C(W^{-k})[e^\infty]$ to $\wedge W^{-k} + \wedge W^k, \wedge W^k$ is regarded to be the module of $(\infty - p)$ -forms over $W^k(X)$; it is also defined in [3] and [4]. Generalized Clifford algebra over $W^{-k}(X)$ is also constructed when ν is a fractional number (see [10], [11]).

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1 Introduction

In [3] we proposed a Hodge operator on a mapping space together with the definition and the calculations of the $(\infty - p)$ -forms on a Sobolev space $W^k(X)$ (see [4], [5]). Here X is a compact (spin) manifold and the Sobolev metric is fixed to be

$$\|f\|_k = \|D^k f\|, \quad \|f\| \text{ is the fixed } L^2\text{-metric of } f. \quad (1)$$

Consider D as a fixed non-degenerate 1-st order selfadjoint elliptic (pseudo) differential operator acting on the smooth sections of E , which is an Hermitian vector bundle over X . Using the Sobolev duality, a differential form (on an open set of) $W^k(X)$ takes the values in $\wedge W^{-k}$, the Grassmann algebra over $W^{-k}(X)$. So the $(\infty - p)$ -forms on $W(X)$ should take the values in $\wedge W^k$, the Grassmann algebra over $W^k(X)$. In order to proceed algebraic and analytic calculations involving $(\infty - p)$ -forms, we

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need to investigate what does it mean “dimension” and “volume” of $W^k(X)$, etc. For this purpose, we use the spectral eta-function $\eta_D(s)$ of D defined by:

$$\eta_D(s) = \sum_{\lambda \in \text{Spec}(D)} \text{sgn} \lambda |\lambda|^{-s}. \quad (2)$$

It is known that the analytic continuation of $\eta_D(s)$ is holomorphic at $s = 0$. We also introduce the following functions:

$$\begin{aligned} \zeta_{|D|}(s) &= \eta_{D^2}(s/2) = \sum_{\lambda \in \text{Spec}(D)} |\lambda|^{-s}, \\ \zeta_{D,\pm}(s) &= \frac{\zeta_{|D|}(s) \pm \eta_D(s)}{2}. \end{aligned}$$

Analytic continuations of these functions are also holomorphic at $s = 0$ (see [9]). We set

$$\begin{aligned} \nu &= \zeta_{|D|}(0), \quad \nu_{\pm} = \zeta_{D,\pm}(0), \\ \det |D| &= \exp(-\zeta'_{|D|}(0)), \quad \det D = \exp(\nu_- \pi i/2) \det |D|, \end{aligned}$$

where ν is the virtual dimension of $W^k(X)$. Several algebraic and analytic calculations involving $(\infty - p)$ -forms are accepted when ν is an integer ([3], [5]). Among them, we showed:

Theorem. *An exterior differentiable $(\infty - p)$ -form is (globally) exact.*

As a consequence, the exterior derivation d is not nilpotent on the space of $(\infty - p)$ -forms. Using this fact, we gave geometric models of differential modules on which the derivation d satisfies

$$d^N = 0, \quad d^{N-1} \neq 0, \quad N > 2, \quad (3)$$

(see [5]). Such module has been considered by Kerner in his generalized gauge theory (see [10], [11]). In Kerner’s generalized gauge theory, the corresponding generalized Grassmann algebra is based on the commutation relation

$$a_1 a_2 \cdots a_N = q a_2 \cdots a_N a_1, \quad q = \exp(2\pi i/N). \quad (4)$$

In this paper we construct the Clifford algebra $C(W^{-k})[e^\infty]$ over $W^{-k}(X)$, with an infinite spinor e^∞ . Here $C(W^{-k})$ means the usual Clifford algebra over $W^{-k}(X)$ (see [12]). It is generated by $\{e_\lambda\}$, the orthonormal basis of $L^2(X)$ consisting by the proper functions of D , using the relations:

$$e_\lambda e_\mu + e_\mu e_\lambda = -2 \text{sgn} \lambda |\lambda|^{-2k} \delta_{\lambda\mu}. \quad (5)$$

Since we thought e^∞ to be $\prod e_\lambda$, we impose the following commutation relations:

$$e_\lambda e^\infty = (-1)^{\nu-1} e^\infty e_\lambda, \quad e^\infty e^\infty = (-1)^{\nu-} (-1)^{\nu(\nu+1)/2} (\det |D|)^{-2k}. \quad (6)$$

In the global study $e^\infty e^\infty$ defines (the power of) the determinant bundle (see [4]). But since in this paper we concern only with the local problem, we replace e^∞ by

$$E^\infty = (-1)^{-\nu_-/2} (-1)^{-\nu(\nu+1)/4} (\det |D|)^{2k} e^\infty, \quad (7)$$

so that $E^\infty E^\infty$ is equal to 1. This E^∞ may be thought as the infinite product $c_\nu (\text{sgn} \lambda)^{-1/2} |\lambda|^k e_\lambda$, where c_ν depends on the *mod*.8 class of ν as follows:

$$\begin{aligned} c_\nu &= 1, \nu \equiv 0, 7; \quad c_\nu = -i, \nu \equiv 1; \quad c_\nu = -1, \nu \equiv 3, \\ c_\nu &= \exp(\pi i/4), \nu \equiv 2, 4, 6; \quad c_\nu = i, \nu \equiv 5, \end{aligned}$$

(see Section 3). But in order to do this modification, we need to define the symmetry group $G = G(D)$ related to $C(W^{-k})[e^\infty]$. Roughly speaking, this group is the extension of the group of those linear operators on $L^2(X)$ that almost fix the proper spaces of D and have determinants. Then we adjoin D , $|D|$ and $zP_+ + wP_-$, $z, w \in C^*$. Here P_\pm mean the projections on the positive and negative proper spaces of D . Since we know that

$$\det(t|D|) = t^\nu \det |D|, \quad \det(|D|^m) = (\det |D|)^m, \quad (8)$$

we may define $\det((zP_+ + wP_-)T)$, the determinant of $(zP_+ + wP_-)T$, etc., by:

$$\det((zP_+ + wP_-)T) = z^{\nu_+} w^{\nu_-} \det T. \quad (9)$$

Hence the determinant is defined on $G(D)$ if ν_+ and ν_- are both integers (see Section 2).

The representation $r : C(W^{-k}) \rightarrow B(\wedge W^{-k})$, the algebra of bounded linear operators on $\wedge W^{-k}$, is defined in [12]. Using this representation, we define a representation $R : C(W^{-k})[e^\infty] \rightarrow B(\wedge W^{-k} + \wedge W^k)$ such that

$$R(E^\infty) = \begin{bmatrix} 0 & D^{2k} \\ G^{2k} & 0 \end{bmatrix}. \quad (10)$$

The (module) isomorphism $s : C(W^{-k})[e^\infty] \simeq \wedge W^{-k} + \wedge W^k$ is also obtained. The Grassmann map $gr : C(W^{-k})[e^\infty] \rightarrow \wedge W^{-k} \oplus \wedge W^k$, which recover the multiplication in $W^{-k} \oplus \wedge W^k$, as the Grassmann algebra with $(\infty - p)$ -forms, is also defined in Section 4. But this map violate the associativity in $C(W^{-k})[e^\infty]$ (see [7]). It also shows that the multiplicative structure of the Grassmann algebra with $(\infty - p)$ -forms depends on the metric (see [3], see [13]).

When D is the Dirac operator, it is important to construct a Clifford algebra which has some half-infinite spinors (see [15]). We give its definition and we construct a representation in Section 5. Detailed study will be done in future.

If the virtual dimension of $W^{-k}(X)$ is a fractional number n/m , $(n, m) = 1$, $m > 1$, the m -direct sum $W^{-k}(X)(m) = W^{-k}(X) \oplus \cdots \oplus W^{-k}(X)$ with the metric fixed by $D(m) = D \oplus \cdots \oplus D$ has the virtual dimension n . Hence we may consider the m -th power of the infinite spinor e^∞ on $W^{-k}(X)$, which is be represented by:

$$R(e^\infty \cdots e^\infty) = \begin{bmatrix} 0 & D^{2k(m)} \\ G^{2k(m)} & 0 \end{bmatrix}. \quad (11)$$

According to the Kerner's generalized Grassmann algebra, we impose the following commutation relations:

$$\begin{aligned} e_\lambda e^\infty &= c(e^\infty)^{-1} e_\lambda, & (e^\infty)^{-1} &= (e^\infty)^{2m-1}, & c &= a = \exp(2\pi i/m), & m \text{ is odd,} \\ & & & & c &= b = \exp(\pi i/m), & m \text{ is even.} \end{aligned}$$

Then we can define the generalized Clifford algebra $C(W^{-k})[e^\infty]$ and its representation $R : C(W^{-k})[e^\infty] \rightarrow B(\wedge W^{-k}(m) \oplus \wedge W^k(m))$ together with the (module) isomorphism $s : C(W^{-k})[e^\infty] \simeq \wedge W^{-k}(m) \oplus W^k(m)$ (see Section 6). The construction of the corresponding generalized Grassmann algebra is not yet obtained.

2 Symmetry group of the Sobolev space with respect to D

Since D is an elliptic operator over a compact manifold, it admits the following spectral decomposition:

$$Df = \sum_{\lambda} (f, e_\lambda) e_\lambda, \text{ where } \{e_\lambda\} \text{ is an ortho-normal basis of } L^2(X). \quad (12)$$

Fixing the L^2 -norm $\|f\|$ of a section f of the Hermitian bundle E , by selecting a Riemannian metric of X and a Hermitian structure of E , the Sobolev k -norm $\|f\|_k$ of f is fixed to be

$$\|f\|_k = \|D^k f\|. \quad (13)$$

Similarly, the Sobolev $(-k)$ -norm of f is fixed to be

$$\|f\|_{-k} = \|Gf\|_{-k}, \text{ } G \text{ is the Green operator of } D. \quad (14)$$

Using the inner product (f, g) of $L^2(X)$ such that $(f, f) = \|f\|^2$, the Sobolev duality between $W^{-k}(X)$ and $W^k(X)$ is fixed to be:

$$\langle u, f \rangle = (G^k u, D^k f), \text{ } u \in W^k(X), \text{ } f \in W^{-k}(X). \quad (15)$$

As stated in Introduction, we define the virtual dimension of $W^k(X)$ to be $\nu = \zeta_{|D|}(0)$. It does not depend on k . Using the zeta-regularization, we define $\det D$ and $\det |D|$. These definitions are leading to the following definitions of the determinants of zI and $zP_+ + wP_-$:

$$\det(zI) = z^\nu, \det(zP_+ + wP_-) = z^{\nu_+} w^{\nu_-}. \quad (16)$$

They are uniquely defined if ν_+ and ν_- are integers.

Let E_λ be the proper space of D corresponding to λ . The algebra of bounded linear operators T on $L^2(X)$ such that

$$TE_\lambda \subset E_\lambda, \text{ } T^*E_\lambda \subset E_\lambda, \quad (17)$$

except a finite number of proper values of D , is denoted by $gl(D)$. The group of invertible elements in $gl(D)$ is denoted by $GL(D)$. If $T \in gl(D)$, there is a finite set $A = \{\lambda, \dots, \lambda_m\}$ of proper values of D such that setting $F = \sum_{\lambda \in A} E_\lambda$, then

$$TF \subset F, TE_\lambda \subset E_\lambda, \lambda \notin A. \quad (18)$$

Since F is a finite dimensional space and each E is finite dimensional, $\det(T|F)$ and $\det(T|E_\lambda)$, $\lambda \notin A$, are defined. Since we have

$$\det(T|(F \oplus E_\lambda)) = \det(T|F) \det(T|E_\lambda), \lambda \notin A, \quad (19)$$

the algebra and the group

$$\begin{aligned} gl(D)(f) &= \{T \in gl(D) \mid \prod_{\lambda \notin A} \det(T|E_\lambda) \text{ converges}\}, \\ GL(D)(f) &= gl(D)(f) \cap GL(D), \end{aligned}$$

are well defined. If $T \in gl(D)(f)$, we define its determinant $\det T$ by

$$\det T = \det(T|F) \prod_{\lambda \notin A} \det(T|E_\lambda). \quad (20)$$

This definition does not depend on the choice of F . If $z \neq 1$, then zI does not belong to $GL(D)(f)$. Under the integrity assumption on ν , we adjoin zI to $GL(D)(f)$ and we define

$$\det(zT) = z^\nu \det T. \quad (21)$$

Similarly, under the integrity assumptions on ν_\pm , we adjoin $zP_+ + wP_-$ to $GL(D)(f)$ and we define

$$\det((zP_+ + wP_-)T) = z^{\nu_+} w^{\nu_-} \det T. \quad (22)$$

Finally, we adjoin D and $|D|$ to this extended group. We regard G and $|G|$ to be the inverses of D and $|D|$ respectively. This extension of $GL(D)(f)$ is denoted by $G(D)$. Since the commutator $[D, T]$, $T \in GL(D)(f)$, is a finite rank operator, we can define $\det(DT)$, $T \in GL(D)(f)$, by

$$\det(DT) = \det D \det T. \quad (23)$$

Hence if ν_\pm are both integers, the determinant is defined on $G(D)$.

3 Clifford algebra over a Sobolev space with an infinite spinor

We consider the Clifford algebra $C(W^{-k})$ over $W^{-k}(X)$ to be the algebra over the complex number field \mathbf{C} generated by e using the relations:

$$e_\lambda e_\mu + e_\mu e_\lambda = -2\text{sgn}\lambda |\lambda|^{-2k} \delta_{\lambda\mu}. \quad (24)$$

Then $T \in G(D)$ acts on $C(W^{-k})$ as an isomorphism. It means that we may consider that $C(W^{-k})$ is generated by $\{Te_\lambda\}$.

The submodule of $C(W^{-k})$ generated by at most p -product of the generators, is denoted by $C(p)$. Then we have the Grassmann map:

$$gr : C(p)/C(p-2) \simeq \wedge^p W^{-k}. \quad (25)$$

Using this Grassmann map, the Poisson bracket $\{x, y\} \in \wedge^{p+q-2} W^{-k}$, $x \in \wedge^p W^{-k}$, $y \in \wedge^q W^{-k}$, is defined to be

$$\{x, y\} = [a, b] \text{ mod. } C(p+q-4), \quad gr(a) = x, \quad gr(b) = y, \quad [a, b] = ab - (-1)^{pq}ba. \quad (26)$$

Using this Poisson bracket, the representation $r : C(W^{-k}) \rightarrow B(\wedge W^{-k})$ is defined as follows:

$$r(a)(x) = gr(a) \wedge x + \{gr(a), x\}/2 \quad (27)$$

(see [12]). Here $\wedge W^{-k}$ is considered to be a Hilbert space according the norm

$$\left\| \sum u_p \right\|^2 = \sum \|u_p\|^2, \quad u_p \in \wedge^p W^{-k}, \quad (28)$$

where $\wedge^p W^{-k}$ is regarded to be the subspace of $W^{-*k}(X \times \cdots \times X)$ consisting of alternating functions. This r induces the module isomorphism $s : C(W^{-k}) \simeq \wedge W^{-k}$, such that $s(a) = r(a)1$, where 1 is the identity in $C = \wedge^0 W^{-k}$. We define the norm on $C(W^{-k})$ by the formula:

$$\|a\| = \|s(a)\|. \quad (29)$$

In the sequel we consider $C(W^{-k})$ to be the Hilbert space according this norm. It is isometric to $\wedge W^{-k}$.

We assume that the virtual dimension of $W^k(X)$ is an integer. Then the infinite spinor e^∞ , thought to be the infinite product $\prod e_\lambda$, must satisfy:

$$e_\lambda e_\mu = (-1)^{\nu-1} e^\infty e_\mu, \quad e_\infty e_\infty = (-1)^{\nu-} (-1)^{\nu(\nu+1)/2} (\det |D|)^{-2k}. \quad (30)$$

According to the *mod.8* class of ν , $(-1)^{\nu(\nu+1)/4}$ takes the following values:

$$\begin{aligned} (-1)^{\nu(\nu+1)/4} &= 1, \quad \nu \equiv 0, 7; \quad (-1)^{\nu(\nu+1)/4} = i, \quad \nu \equiv 1, 6, \\ (-1)^{\nu(\nu+1)/4} &= -1, \quad \nu \equiv 3, 4; \quad (-1)^{\nu(\nu+1)/4} = -i, \quad \nu \equiv 2, 5. \end{aligned}$$

Then the constants c_ν are chosen to be:

$$\begin{aligned} c_\nu &= 1, \quad \nu \equiv 0, 7; \quad c_\nu = -i, \quad \nu \equiv 1; \quad c_\nu = -1, \quad \nu \equiv 3, \\ c_\nu &= \exp(\pi i/4), \quad \nu \equiv 2, 4, 6; \quad c_\nu = i, \quad \nu \equiv 5, \end{aligned}$$

The operator T' defined by $T'e = c_\nu |\nu|^k e_\lambda$ belongs to $G(D)$. Hence we may thought that

$$\prod T'e_\lambda = \det T'e^\infty = (-1)^{-\nu(\nu+1)/4} \det |D|^k e^\infty. \quad (31)$$

The right hand side is denoted by $E^{\infty'}$. It satisfies:

$$e_\lambda E^{\infty'} = (-1)^{\nu-1} E^{\infty'} e_\lambda, \quad E^{\infty'} E^{\infty'} = (-1)^{\nu-}. \quad (32)$$

If ν_- is also an integer, we define a linear operator T by:

$$T e_\lambda = (\operatorname{sgn} \lambda)^{-1/2} T' e_\lambda, \quad \begin{aligned} (\operatorname{sgn} \lambda)^{-1/2} &= 1, \text{ if } \lambda \text{ is positive,} \\ (\operatorname{sgn} \lambda)^{-1/2} &= -i, \text{ if } \lambda \text{ is negative.} \end{aligned}$$

Then T belongs to $G(D)$, and in order to define E^∞ as $(-1)^{-\nu_-/2} E^{\infty'}$, we may thought that $E^\infty = \prod T e_\lambda$. This E^∞ satisfies

$$e_\lambda E^\infty = (-1)^{\nu-1} E^\infty e_\lambda, \quad E^\infty E^\infty = 1. \quad (33)$$

Definition 1. Let $B(\wedge W^{-k} \oplus \wedge W^k)$ be the algebra of bounded linear operators on $\wedge W^{-k} \oplus \wedge W^k$. We define a representation $R : C(W^{-k})[e^\infty] \rightarrow B(\wedge W^{-k} \oplus \wedge W^k)$ by:

$$\begin{aligned} R(E_\infty) &= \begin{bmatrix} 0 & D^{2k} \\ G^{2k} & 0 \end{bmatrix}, \\ R(e_\lambda) &= \begin{bmatrix} r(e_\lambda) & 0 \\ 0 & (-1)^{\nu-1} G^{2k} r(e_\lambda) D^{2k} \end{bmatrix}. \end{aligned} \quad (34)$$

Notice that on $\wedge W^{-k} \oplus \wedge W^k = C \oplus C$ we consider:

$$R(E^\infty) = \begin{bmatrix} 0 & (\det D)^{-(k+d)/2} \\ (\det D)^{(k+d)/2} & 0 \end{bmatrix}, \quad (35)$$

where d is the dimension of X .

By definition, we have:

$$\begin{aligned} R(E^{\infty'}) &= \begin{bmatrix} 0 & (-1)^{\nu-1} D^{2k} \\ (-1)^{\nu_-/2} G^{2k} & 0 \end{bmatrix}, \\ R(e^\infty) &= \\ & \begin{bmatrix} 0 & (-1)^{\nu_-/2} (-1)^{\nu(\nu+1)/4} (\det |D|)^k D \\ (-1)^{\nu_-/2} (-1)^{\nu(\nu+1)/4} (\det |D|)^{-k} G^{2k} & 0 \end{bmatrix}. \end{aligned} \quad (36)$$

When considered on a mapping space, they define cross-sections of the (direct sum of the power of the) determinant bundle (see [4] and [2]).

Since $E^\infty E^\infty$ is equal to 1, the proper values of E^∞ are $\{1, -1\}$. Their proper spaces are given by

$$S(+) = \{D^k f + G^k f | f \in \wedge L^2\}, \quad S(-) = \{D^k f - G^k f | f \in \wedge L^2\}. \quad (37)$$

Since $G^{2k} r(e_\lambda) D^{2k} G^k f$ is equal to $G^k (G^k r(e_\lambda) D^k) f$, we have:

$$\begin{aligned} r(e_\lambda) S(+) &= S(+), \quad r(e_\lambda) S(-) = S(-), \quad \text{if } \nu \text{ is odd,} \\ r(e_\lambda) S(+) &= S(-), \quad r(e_\lambda) S(-) = S(+), \quad \text{if } \nu \text{ is even.} \end{aligned}$$

Hence taking $S(+) \oplus S(-)$ as the direct sum decomposition of $\wedge W^{-k} \oplus \wedge W^k$, we get the following matrix type representations of e^∞ and e^∞ :

$$\begin{aligned} R(E^\infty) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ R(e_\lambda) &= \begin{bmatrix} r(e_\lambda)(+, +) & 0 \\ 0 & r(e_\lambda)(-, -) \end{bmatrix}, \text{ if } \nu \text{ is odd,} \\ R(e_\lambda) &= \begin{bmatrix} 0 & r(e_\lambda)(+, -) \\ r(e_\lambda)(-, +) & 0 \end{bmatrix} \text{ if } \nu \text{ is even.} \end{aligned} \quad (38)$$

Here $r(e_\lambda)(+, -)$ etc. are the maps:

$$r(e_\lambda)(+, -)(D^k f - G^k f) = r(e_\lambda)D^k f + G^{2k}r(e_\lambda)D^k f, \text{ etc.} \quad (39)$$

4 Grassmann map from $C(W^{-k})[e^\infty]$ to $\wedge W^{-k} \oplus \wedge W^k$

If x belongs to $C(W^{-k})$, we have:

$$\begin{aligned} R(x) &= \begin{bmatrix} r(x) & 0 \\ 0 & (-1)^{\nu-1}G^{2k}r(x)D^{2k} \end{bmatrix}, \\ R(xE^\infty) &= \begin{bmatrix} 0 & r(x)D^{2k} \\ (-1)^{\nu-1}G^{2k}r(x) & 0 \end{bmatrix}. \end{aligned} \quad (40)$$

Hence we get:

$$\begin{aligned} R(x) \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} r(x)1 \\ 0 \end{bmatrix}, \\ R(xE^\infty) \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 \\ (-1)^{\nu-1}G^{2k}r(x)1 \end{bmatrix}. \end{aligned} \quad (41)$$

Therefore, in order to define $s : C(W^{-k})[e^\infty] \rightarrow \wedge W^{-k} \oplus \wedge W^k$ by

$$s(y) = R(y) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (42)$$

we have

$$s(C(W^{-k})) = \wedge W^{-k}, \quad s(C(W^{-k})E^\infty) = G^{2k} \wedge W^{-k} = \wedge W^k, \quad (43)$$

as modules. We obtain the following equality of modules:

$$s(C(W^{-k})[e^\infty]) = \wedge W^{-k} \oplus \wedge W^k. \quad (44)$$

Precisely saying, we give a Hilbert space structure on $C(W^{-k})[e^\infty]$ by using the map s .

We set $C(W^{-k}) \oplus C(p)e^\infty$ by $C[e^\infty](p)$. The orthogonal complement of $C[e^\infty](p)$ in $C(W^{-k})[e^\infty]$ is denoted by $C[e^\infty](p)^\perp$. Since $C[e^\infty](p-2)^\perp$ is contained in $C[e^\infty](p)$, then $C[e^\infty](p-2)^\perp$ contains $C[e^\infty](p)^\perp$, and we have

$$C[e^\infty](p-2)^\perp / C[e^\infty](p)^\perp \simeq \wedge^p W^k(X). \quad (45)$$

Definition 2. Let $pr : C[e^\infty](p)^\perp \rightarrow C[e^\infty](p-2)^\perp$ be the projection. Then we define the Grassmann map $gr : C(W^{-k})e^\infty \rightarrow \wedge W^k(X)$ by:

$$gr(a) = pr(a) \text{ mod. } C[e^\infty](p)^\perp, \quad a \in C[e^\infty](p). \quad (46)$$

Let $x = gr(a)$ be an element of $\wedge^q W^{-k}$ and $y = gr(be^\infty)$ be an element of $\wedge^p W^k$. Then $pr(a(be^\infty))$ belongs to $C[e^\infty](p-q-2)$ and its class modulo $C[e^\infty](p-q)^\perp$ is determined by x and y . The Grassmann product $x \wedge y$ is obtained by

$$x \wedge y = gr(a(be^\infty)). \quad (47)$$

By definition, if $q > p$, then $x \wedge y$ is equal to 0, because $p-q$ and $p-q-2$ are both negative, so $C[e^\infty](p-q)^\perp$ coincides with $C[e^\infty](p-q-2)^\perp$. As a special case, we have $gr((ab)e^\infty) = 0$, if $gr(ab)$ is a non-zero element of $\wedge^{p+q} W^{-k}$. Therefore, in general, we have:

$$gr(a(be^\infty)) \neq gr((ab)e^\infty), \quad (48)$$

(see [7],[11]).

In conclusion, the multiplicative structure of the Grassmann algebra over $W^{-k}(X)$ with $(\infty - p)$ -forms depends on the metric of $W^{-k}(X)$ (and the Sobolev duality between $W^{-k}(X)$ and $W^k(X)$). The associative law of the Clifford multiplication in $C(W^{-k})[e^\infty]$ is violated by the Grassmann map.

Notice that the Sobolev duality was used in the definition of the Grassmann product ([3]). There is an alternative definition of the $(\infty - p)$ -forms. But it uses some filtration (by finite dimensional spaces) of an infinite dimensional space (see [13]).

5 Remarks on half infinite spinors

When D is the Dirac operator, it is an important problem (see [15]) to extend $C(W^{-k})$ by half infinite spinors $e^\infty(+)$ and $e^\infty(-)$, which are thought to be

$$e^\infty(+) = \prod_{\lambda > 0} e_\lambda, \quad e^\infty(-) = \prod_{\lambda < 0} e^\lambda. \quad (49)$$

In order to define such extension, we assume that ν_+ and ν_- are both integers. Then similar to e^∞ , we can regularize $e^\infty(+)$ and $e^\infty(-)$. Denoting the regularized elements by $E^\infty(+)$ and $E^\infty(-)$, they must satisfy:

$$E^\infty(+)E^\infty(-) = E^\infty, \quad E^\infty(-)E^\infty(+) = (-1)^{\nu_+\nu_-} E^\infty. \quad (50)$$

We notice that $\nu_+\nu_-$ is an even number if ν is odd, while $\nu_+\nu_-$ can be odd if ν is even.

We set $D(+) = DP_+$ and $D(-) = DP_-$. They are non-degenerate as operators on $P_+L^2(X)$ and $P_-L^2(X)$. So, they have Green operators $G(+)$ and $G(-)$. Since we have:

$$\begin{aligned} W^{-k}(X) &= D(+)^k P_+ L^2(X) \oplus D(-)^k P_- L^2(X), \\ W^{-k}(X) &= G(+)^k P_+ L^2(X) \oplus G(-)^k P_- L^2(X), \end{aligned}$$

and $D = D(+) + D(-)$, $G = G(+) + G(-)$, we may write:

$$R(E^\infty) = \begin{bmatrix} 0 & 0 & D(+)^{2k} & 0 \\ 0 & 0 & 0 & D(-)^{2k} \\ G(+)^{2k} & 0 & 0 & 0 \\ 0 & G(-)^{2k} & 0 & 0 \end{bmatrix}. \quad (51)$$

Hence taking $B(\wedge W^{-k}(+) \oplus W^{-k}(-) \oplus W^k(+) \oplus W^k(-))$, $\wedge W^{-k}(+) = D(+)^k \wedge P_+ L^2(X)$ etc., as the representation space, we can represent $E^\infty(+)$ and $E^\infty(-)$ as follows:

$$R(E^\infty(+)) = \begin{bmatrix} 0 & 0 & D(+)^{2k} & 0 \\ 0 & 1 & 0 & 0 \\ (-1)^{\nu_+ \nu_-} G(+)^{2k} & 0 & 0 & 0 \\ 0 & 0 & 0 & (-1)^{\nu_+ \nu_-} 1 \end{bmatrix}, \quad (52)$$

$$R(E^\infty(-)) = \begin{bmatrix} (-1)^{\nu_+ \nu_-} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & D(-)^{2k} \\ 0 & 0 & 1 & 0 \\ 0 & (-1)^{\nu_+ \nu_-} G(-)^{2k} & 0 & 0 \end{bmatrix}. \quad (53)$$

Notice that for the global definition of the Clifford bundle with half infinite spinors, associated with a $Map(X, G)$ -bundle B , we need the triviality of B at least as a loop group bundle (see [1], see [2]).

6 Clifford algebra over a Sobolev space with the m -th root of an infinite spinor

If the virtual dimension of $W^k(X)$ is a fractional number n/m , $(n, m) = 1$, $m > 1$, the m -direct sum $W^k(X)(m) = W^k(X) \oplus \cdots \oplus W^k(X)$, endowed with the metric $D^k(m) = D^k \oplus \cdots \oplus D^k$, has the virtual dimension n . Hence we can construct the Clifford algebra over $W^{-k}(X)(m)$ with the infinite spinor E^∞ . Since we have:

$$\sum_{p>0} \wedge^p W^{-k}(m) \subset \sum_{p>0} \wedge^p W^{-k} \oplus \cdots \oplus \sum_{p>0} \wedge^p W^{-k} \left(= \left(\sum_{p>0} \wedge^p W^{-k} \right)(m) \right), \quad (54)$$

we may write:

$$R(E^\infty) = \begin{bmatrix} D^{2k}(m) & 0 \\ 0 & G^{2k}(m) \end{bmatrix} = \begin{bmatrix} & & & \vdots & D^{2k} & & \\ & 0 & & \vdots & & \ddots & \\ & & & \vdots & & & D^{2k} \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ G^{2k} & & & \vdots & & & \\ & \ddots & & \vdots & & 0 & \\ & & G^{2k} & \vdots & & & \end{bmatrix}. \quad (55)$$

Hence we can use $B(\wedge W^{-k}(m) \oplus \wedge W^k(m))$ as a representation space E^∞ .

If there is a (regularized) infinite spinor e^∞ on $C(W^{-k})$, its m -th power must be equal to E^∞ , the regularized infinite spinor on $C(W^{-k}(m))$. Thus it must be:

$$(e^\infty)^m = e^\infty \cdots e^\infty = E^\infty, \text{ i.e. } (e^\infty)^{2m} = 1. \quad (56)$$

According to the Kerner's generalized Grassmann algebra([10],[11]), we impose the following commutation relations:

$$\begin{aligned} e_\lambda e^\infty &= c(e^\infty)^{-1} e_\lambda, & c &= a = \exp(2\pi i/m), & m \text{ is odd,} \\ & & c &= b = \exp(\pi i/m), & m \text{ is even.} \end{aligned} \quad (57)$$

Here $(e^\infty)^{-1}$ is equal to $(e^\infty)^{2m-1}$. We assume that e^∞ generates the cyclic group of order $2m$.

Definition 3. Using the $(2m, 2m)$ -matrix type representation of the elements of $B(\wedge W^{-k}(m) \oplus \wedge W^k(m))$, we define a representation $R : C(W^{-k})[e^\infty] \rightarrow B(\wedge W^{-k}(m) \oplus \wedge W^k(m))$ by:

$$R(e^\infty) = \begin{bmatrix} 0 & 1 & & \vdots & & & \\ & \ddots & \ddots & \vdots & & & 0 \\ & & \ddots & 1 & \vdots & 0 & \\ & & & 0 & \vdots & D^{2k} & 0 \\ \cdots & \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & \vdots & 0 & 1 & & & \\ & 0 & & & \vdots & & \ddots & \ddots & & \\ 0 & & & & \vdots & & & \ddots & 1 & \\ G^{2k} & 0 & & & \vdots & & & & & 0 \end{bmatrix},$$

$$R(e_\lambda) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \text{ where}$$

$$A = \begin{bmatrix} r(e_\lambda) & 0 \\ 0 & \end{bmatrix}, B = \begin{bmatrix} & & 0 \\ & & \cdot c^{2m-1}r(e_\lambda)D^{2k} \\ & \cdot & \cdot \\ 0 & c^{m\oplus 1}r(e_\lambda)D^{2k} & \end{bmatrix}, C = \begin{bmatrix} & & 0 \\ & & \cdot c^{m-1}G^{2k}r(e_\lambda) \\ & \cdot & \cdot \\ 0 & cG^{2k}r(e_\lambda) & \end{bmatrix} \text{ and } D = \begin{bmatrix} c^m G^{2k}r(e_\lambda)D^{2k} & \\ & 0 \end{bmatrix}$$

Notice that the duals of these matrices:

$$R(e^\infty) = \begin{bmatrix} 0 & & & & \vdots & & & & & D^{2k} \\ 1 & \ddots & & & \vdots & & & & & \\ & \ddots & \ddots & & \vdots & & & & & \\ & & 1 & 0 & \vdots & & & & & \\ \dots & \dots & \dots & \dots & \vdots & \dots & \dots & \dots & \dots & \\ & & & G^{2k} & \vdots & 0 & & & & \\ & & & & \vdots & 1 & \ddots & & & \\ & & & & \vdots & & \ddots & \ddots & & \\ & & & & \vdots & & & 1 & 0 & \end{bmatrix},$$

$$R(e_\lambda) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \text{ where } A = \begin{bmatrix} r(e_\lambda) & 0 \\ 0 & \end{bmatrix}, B = \begin{bmatrix} & & 0 \\ & & \cdot cr(e_\lambda)D^{2k} \\ & \cdot & \cdot \\ 0 & \cdot & \cdot \end{bmatrix},$$

$$C = \begin{bmatrix} & & 0 \\ & & \cdot \\ & \cdot & \cdot \\ 0 & c^{2m-1}G^{2k}r(e_\lambda) & \end{bmatrix} \text{ and } D = \begin{bmatrix} c^m G^{2k}r(e_\lambda)D^{2k} & \\ & \end{bmatrix} \text{ gives also}$$

a representation. But in the rest, we use only (58).

This selection of $R(e_\lambda)$ works only for odd elements of $C(W^{-k})$ because we have:

$$R(e_\lambda)R(e_\mu) = r(e_\lambda e_\mu)I, \text{ where } I \text{ is the identity.} \quad (58)$$

Hence if x is an even element of $C(W^{-k})$, we get

$$R(x) = r(x)I. \quad (59)$$

Let $u(i)$ and $v(j)$ be i -th and j -th unit vectors in the vector space representations of $W^{-k}(m)$ and $W^k(m)$ respectively. We set

$$S(c) = \{D^k fu(1) \oplus cD^k fu(2) \oplus \dots \oplus c^{m-1}D^k fu(m) \oplus c^m G^k fv(1) \oplus \dots \oplus c^{2m-1}G^k fv(m) | f \in \wedge L^2\}, c \in C.$$

We may regard $u(1)$ as an element of $\wedge^0 W^{-k}(m) = C^m$. Using the definition of R , if x belongs to $C(W^{-k})$, we have:

$$\begin{aligned} R(x(e^\infty)^i)u(1) &= c^i r(x)u(i), 0 \leq i < m, \\ r(x(e^\infty)^{m+i})u(1) &= -c^i G^k r(x)v(i), \\ c &= a \text{ if } m \text{ is odd, } c = b \text{ if } m \text{ is even.} \end{aligned}$$

Hence in order to define the map $s : C(W^{-k})[e^\infty] \rightarrow \wedge W^{-k}(m) \oplus \wedge W^k(m)$ by

$$s(y) = R(y)u(1), y \in C(W^{-k})[e^\infty], \quad (61)$$

we have:

$$s(C(W^{-k})[e^\infty]) \simeq \wedge W^{-k}(m) \oplus \wedge W^k(m). \quad (62)$$

The Hilbert space structure on $C(W^{-k})[e^\infty]$ is given via the map s .

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