ON GENERAL RANDERS-KROPINA FINSLERIAN METRICS

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Abstract

The fundamental tensor and angular metric of a generalized Randers-Kropina Finsler space are determined; the Cartan framework is developed, the C-reducibility equations are provided and considerations regarding the equations of geodesics and of their deviations are given.

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1 Introduction

The general Randers Finsler spaces were introduced by R.Miron ([10]); a particular case with evidentiated Riemannian part was considered in [19], where a series of considerations on the geodesics and on the physical relevance of the space were developed. The present approach provides a Finsler space which embraces the two cases. In particular, it provides the Randers metric - a good geometrical model for the unified electromagnetic and gravitational field theory ([9]) and the Kropina metric - used in the Lagrangian analytical dynamics ([17]). Its metric is a functional linear combination of Randers-type, Kropina and general fundamental functions. We shall present sufficient conditions for the metric to be a Finslerian one, then derive explicitly the Cartan connection coefficients, and discuss the equations of geodesics and of their deviations.

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2 Randers-Kropina Finslerian metrics

Let M be a smooth real n-dimensional manifold and $\tilde{T}M = TM\backslash\{0\}$. In the following we shall consider a Finslerian metric of the type

$$L(x,y) = \alpha a(x,y) + \beta b(x,y) + \gamma \frac{a^2(x,y)}{b(x,y)} + \Lambda(x,y), \tag{1}$$

where:

$a(x,y) = \sqrt{a_{ij}(x)y^i y^j}$, with $\{a_{ij}(x)\}$ Riemannian metric on M;
$b(x,y) = b_i(x)y^i$, with $\{b_i(x)\}$ covector field on M;
$\Lambda(x,y)$ is a Finsler fundamental function on TM, and $\alpha, \beta, \gamma \in \mathcal{F}(M)$.

Remarks. 1. The first two terms of L determine a Randers-type Finsler metric, while the third is a Kropina one.
2. For $\alpha = \gamma = 0$ and $\beta = 1$ the framework becomes the general Randers studied by Miron ([10]);
3. For $\alpha = 1$, $\gamma = \Lambda = 0$, and $\beta = \dfrac{e}{mc^2}$ (e being the electrical charge, m - the mass and c - the speed of light), we get the Randers-type metric useful in physical applications ([7]).
4. For $\gamma = 0$ and $\alpha = \beta = 1$, we have the case studied in [19].
5. For $\alpha = \Lambda = 0$, $\gamma = \beta = 1$, the Finsler metric becomes $L = \dfrac{a}{b} + b$, a case considered by M.Matsumoto in [9]. The more general Kropina metric

$$L_K = \frac{a}{b} + c_i(x)y^i = \frac{(a_{ij} + d_i d_j - b_i b_j - c_i c_j)y^i y^j}{b},$$

where $d_i = b_i + c_i$, will be investigated in a forecoming paper.

In the following we shall denote briefly $a(x,y)$ and $b(x,y)$ by a and b, respectively; also, throughout the paper, the indices "$,i$" and "$;a$" will represent, respectively, the partial differentiation with respect to x^i and y^a.
The space $F^n = (M, L(x,y))$ is a Finsler space provided that the fundamental tensor field $g_{ij} = \dfrac{1}{2}\dfrac{\partial^2 L^2(x,y)}{\partial y^i \partial y^j}$ is non-degenerate. By direct calculation, we obtain the following:

Theorem 2.1 *The fundamental tensor field g_{ij} associated to the metric $L(x,y)$ has the form*

$$g_{ij} = \mu a_{ij} + \lambda b_i b_j + \pi b_{\{i}y_{j\}} + \rho y_i y_j + \frac{1}{2}\Lambda^2_{;ij} + \lambda_{ij}, \tag{2}$$

where the coefficients are given by:

$$\begin{aligned}
\mu &= (\alpha^2 ab^2 + 2\gamma^2 a^3 + \alpha\beta b^3 + 2\beta\gamma ab^2 + 3\alpha\gamma a^2 b)/(ab^2),\\
\lambda &= (\beta^2 b^4 + 3\gamma^2 a^4 + 2\alpha\gamma a^3 b)/(b^4),
\end{aligned}$$

$$\begin{aligned}
\pi &= (-4\gamma^2 a^3 + \alpha\beta b^3 - 3\alpha\gamma a^2 b)/(ab^3),\\
\rho &= (4\gamma^2 a^3 - \alpha\beta b^3 + 3\alpha\gamma a^2 b)/(a^3 b^2), \quad \textit{and}\\
\lambda_{ij} &= \Lambda_{;ij}\frac{\alpha ab + \beta b^2 + \gamma a^2}{b} +\\
&+ \underset{ij}{S}\Lambda_{;i}\frac{\alpha b^2 y_j + \beta ab^2 b_j + \gamma a(2by_j - a^2 b_j)}{ab^2} +\\
&+ \Lambda\frac{\alpha b^3(a^2 a_{ij} - y_i y_j) + \gamma b^3(2b^2 a_{ij} - 2bb_{\{i}y_{j\}} + 2b_i b_j a^2)}{a^3 b^3},
\end{aligned}$$

and where we denoted $\tau_{\{ij\}} \equiv \underset{ij}{S}\tau_{ij} = \tau_{ij} + \tau_{ji}$ *and* $y_i = a_{ij}y^i$.

Remark. If $\Lambda = 0$, then in the expression of the fundamental tensor g_{ij}, the coefficients of α^2 and γ^2 multiply exactly the Riemannian and the Kropina fundamental tensor fields, respectively; for $\gamma = 0$, g_{ij} becomes the Randers metric tensor field. The proof of the theorem is based on the following general result.

Lemma 2.1 *Let* $\overset{k}{F} : TM \to \mathbf{R}$, $k = \overline{1,m}$ *be* m *Finsler metrics,* $\overset{k}{\alpha} \in \mathcal{F}(M)$ *and* $\overset{k}{g}_{ij}$ *be their associated Finsler fundamental tensor fields. Then the Finsler metric* $F = \sum_{k=\overline{1,m}} \overset{k}{\alpha}\overset{k}{F}$ *has the associated fundamental tensor*

$$g_{ij} = \sum_{k=\overline{1,m}} \overset{k}{\alpha}{}^2 \overset{k}{g}_{ij} + \sum_{k<l;k,l=\overline{1,m}} \overset{k}{\alpha}\overset{l}{\alpha}\left(\overset{\{k}{F}H(\overset{l\}}{F}) + \nabla\overset{\{k}{F}\otimes\nabla\overset{l\}}{F}\right),$$

where $H(\phi)$ *denotes the* y*-Hessian of the function* $\phi(x,y)$*, and* $\nabla\phi$*, its* y*-gradient.*

In order to study if the fundamental tensor field g_{ij} is non-degenerate, we must consider the case $\Lambda \equiv 0$. In this case, it admits the convenient form

$$g_{ij} = \mu a_{ij} + \frac{1}{\lambda}l_i l_j + \theta y_i y_j,$$

where $l_i = \lambda b_i + \pi y_i$, $\theta = \rho - \frac{\pi^2}{\lambda}$, and the main result is based on the following lemma.

Lemma 2.2 *Let* $(a_{ij}) \in GL(n,\mathbf{R})$, $\alpha,\beta \in \mathbf{R}^*$; $v_i \in \mathbf{R}$, $i = \overline{1,n}$, *and consider the matrix* $(b_{ij}) \in \mathcal{M}_n(\mathbf{R})$, *of coefficients*

$$b_{ij} = \alpha a_{ij} + \beta v_i v_j,$$

such that $\alpha + \beta v^2 \neq 0$, *where* $v^2 = a^{ij}v_i v_j$. *Then we have*
a) det $(b_{ij}) = \alpha^{n-1}A(\alpha + \beta v^2)$, *where* $A = det(a_{ij})$;
b) $(b_{ij}) \in GL(n,\mathbf{R})$ *and the coefficients of its inverse* (b^{ij}) *are*

$$b^{ij} = \frac{1}{\alpha}\left(a^{ij} - \beta\frac{v^i v^j}{\alpha + \beta v^2}\right),$$

where $v^i = a^{ij}v_j$ *and* (a^{ij}) *is the inverse matrix of* (a_{ij}).

Remark. For $\alpha = 1,\ \beta = \{-1, 1\}$, we obtain the technical lemmas from [10].

Theorem 2.2 *The fundamental tensor* $g_{ij}(x,y)$ *in (2) is non-degenerate, provided that the following conditions are satisfied*

$$\begin{cases} \mu \neq 0;\ \ \lambda(\mu + \lambda\tilde{b} + 2\pi b) + \pi^2 a^2 \neq 0 \\ \left[\lambda(\mu + \rho a^2) - \pi^2 a^2\right]\left[\lambda(\mu + \lambda\tilde{b} + 2\pi b) + \pi^2 a^2\right] - (\lambda\rho - \pi^2)(\lambda b + \pi a^2)^2 \neq 0, \end{cases}$$

where a^{ij} *is the reciprocal tensor field of the Riemannian metric* a_{ij}. *In this case, its inverse* $g^{ij}(x,y)$ *has the coefficients*

$$g^{ij} = \tilde{\mu}a^{ij} + \tilde{\lambda}b^i b^j + \tilde{\pi}b^{\{i}y^{j\}} + \tilde{\rho}y^i y^j, \tag{3}$$

where $b^i = a^{ij}b_j$, *and*

$$\begin{array}{rclrl} \tilde{\mu} & = & \dfrac{1}{\mu}, & \tilde{\lambda} = & (\lambda^2 p + v^2 q), \\ \tilde{\pi} & = & -(\lambda\pi p + uvq), & \tilde{\rho} = & -(\pi^2 p + u^2 q), \end{array}$$

and where we used the notations:

$$\begin{array}{rll} p = & (\mu w)^{-1}, & q = \theta w \cdot [w + \theta(\mu a^2 + 2\pi b a^2) + \theta\lambda(b^2 + a^2\tilde{b})]^{-1}, \\ \tilde{l} = & \lambda^2\tilde{b} + 2\pi\lambda b + \pi^2 a^2, & w = \mu\lambda + \tilde{l}, \\ u = & (\mu + \lambda\tilde{b} + \pi b)/p, & v = (\lambda b + \pi a^2)/q. \end{array}$$

Remarks. 1. The above coefficients have their homogeneity degrees described in the following table:

μ	λ	π	ρ	p	q	v	u	$\tilde{\mu}$	$\tilde{\lambda}$	$\tilde{\pi}$	$\tilde{\rho}$
0	0	−1	−2	0	−2	1	0	0	0	−1	−2

Using it, one can see easily that g^{ij} is 0-homogeneous.

2. The tensor field g^{ij} has also the convenient form

$$g^{ij} = \tilde{\mu}a^{ij} - pm^i m^j - qn^i n^j,$$

where $m^i = \lambda b^i + \pi y^i$ and $n^i = vb^i + uy^i$.

3. In particular, for $\alpha = \beta = 1,\ \gamma = 0$, and $\Lambda \neq 0$, the results for the general Randers space with outstanding Riemannian metric studied in [19] are obtained easily, replacing the notations of coefficients as described below

a_{ij}	g_{ij}	b_i	Λ	a
g_{ij}	f_{ij}	$\frac{q}{mc^2}A_i$	$g^{ij}A_iA_j\Lambda$	σ.

3 The Cartan connection

The Cartan connection of the space F^n is $C\Gamma = (N^i_j, F^i_{jk}, C^i_{jk})$ with

$$\left\{\begin{array}{l} N^i_j = \gamma^i_{ja}y^a - C^i_{jk}\gamma^k_{ab}y^a y^b \\ F^i_{jk} = g^{is}(\delta_{\{j}g_{sk\}} - \delta_s g_{jk})/2 \\ C^i_{jk} = g^{is}(g_{\{js;k\}} - g_{jk;s})/2, \end{array}\right. \tag{4}$$

where $\delta_k = \partial_k - N^a_k\dot{\partial}_a,\ \partial_k = \dfrac{\partial}{\partial x_k},\ \dot{\partial}_a = \dfrac{\partial}{\partial y^a}$ and

$$\gamma^i_{jk}(x,y) = \frac{1}{2}g^{is}(\partial_{\{j}g_{sk\}} - \partial_s g_{jk})$$

are the Finslerian Christoffel symbols of the metric (2).

Nevertheless, for computing the coefficients, we use the relations [9], [12]

$$\left\{\begin{array}{l} F^i_{jk} = \gamma^i_{jk} - C^i_{\{js}N^s_{k\}} - g^{il}C_{jka}N^a_l \\ C^i_{jk} = g^{is}C_{jsk},\ C_{jsk} = \dfrac{1}{2}\dfrac{\partial g_{js}}{\partial y^k} \end{array}\right.$$

Theorem 3.1 *The Cartan vertical covariant connection coefficients C_{ijk} have the expressions*

$$\begin{array}{rcl} C_{ijk} & = & \alpha_0 \underset{ijk}{S}\, a_{ij}b_k + \alpha_1 \underset{ijk}{S}\, a_{ij}y_k + \\ & + & \beta_0 b_i b_j b_k + \beta_1 \underset{ijk}{S}\, y_i b_j b_k + \beta_2 \underset{ijk}{S}\, y_i y_j b_k + \beta_3 y_i y_j y_k, \end{array} \tag{5}$$

where

$$\begin{array}{rclrl} \alpha_0 & = & \dfrac{\alpha\beta b^3 - 3\alpha\gamma a^b - 4a^3\gamma^2}{a^3 b}, & \beta_1 = & \dfrac{6\gamma a(2\gamma a + \alpha\beta)}{b^4}, \\ \alpha_1 & = & \dfrac{4\gamma^2 a^3 - 3\alpha\gamma a^2 b - \alpha\beta b^3}{a^3 b^2}, & \beta_2 = & -ovv8\gamma^2 a^3 + 3\alpha\gamma a^b + \alpha\beta b^3 a^3 b^3, \\ \beta_0 & = & -\dfrac{6a^3\gamma(2\gamma a + \alpha b)}{b^5}, & \beta_3 = & \dfrac{3\alpha\beta b}{a^5}. \end{array}$$

Corollary 3.1 *The Cartan vertical connection coefficients C^i_{jk} of the space F^n are*

$$\begin{array}{rcl} C^i_{jk} & = & \tilde{\mu}(\alpha_0\delta^i_{\{j}b_{k\}} + \alpha_1\delta^i_{\{j}y_{k\}}) + \\ & + & \lambda_0 b^i a_{jk} + \lambda_1 y^i a_{jk} + \mu_0 b^i b_j b_k + \mu_1 b^i b_{\{j}y_{k\}} + \\ & + & \mu_2 b^i y_j y_k + \nu_1 y^i b_j b_k + \nu_2 y^i b_{\{j}y_{k\}} + \nu_3 y^i y_j y_k, \end{array} \tag{6}$$

where

$$\begin{array}{lll} \lambda_0 = \tilde{\mu}\alpha_0 + \tilde{\lambda}u_0 + \tilde{\pi}u_1, & \lambda_1 = \tilde{\mu}\alpha_1 + \tilde{\pi}u_0 + \tilde{\rho}u_1, & \\ \mu_0 = \tilde{\mu}\beta_0 + \tilde{\lambda}v_0 + \tilde{\pi}w_0, & \mu_1 = \tilde{\mu}\beta_1 + \tilde{\lambda}v_1 + \tilde{\pi}w_1, & \mu_2 = \tilde{\mu}\beta_2 + \tilde{\lambda}v_2 + \tilde{\pi}w_2, \\ \nu_1 = \tilde{\mu}\beta_0 + \tilde{\pi}v_0 + \tilde{\rho}w_0, & \nu_2 = \tilde{\mu}\beta_1 + \tilde{\pi}v_1 + \tilde{\rho}w_1, & \nu_3 = \tilde{\mu}\beta_2 + \tilde{\pi}v_2 + \tilde{\rho}w_2, \\ u_0 = \alpha_1 b + \alpha_0\tilde{b}, & u_1 = \alpha_1 a^2 + \alpha_0 b, & \\ v_0 = 2\alpha_0 + \beta_0\tilde{b} + \beta_1 b, & w_0 = \beta_0 b + \beta_1 a^2, & \\ v_1 = \alpha_1 + \beta_1\tilde{b} + \beta_2 b, & w_1 = \alpha_0 + \beta_1 b + \beta_2 a^2, & \\ v_2 = \beta_2\tilde{b} + \beta_3 b, & w_2 = \beta_2 b + \beta_3 a^2. & \end{array}$$

Also, the Finslerian Christoffel symbols of I-st kind are provided by

$$\gamma_{jik} = \overset{\circ}{\gamma}_{jik} + \left(\pi b_i a_{jk} + \frac{1}{2}\lambda b_i b_{\{jk\}} + \rho a_{jk} y_i \right) /2 + \tilde{t}_{jik}, \tag{7}$$

where $b_{jk} = \partial_k b_j,\ y_i = a_{ij}y^j,\ \overset{\circ}{\gamma}_{jik} = (a_{\{ji,k\}} - a_{jk,i})/2$ are the Riemannian Christoffel symbols of I-st kind of $a_{ij}(x)$, and

$$\tilde{t}_{jik} = (t_{\{kij\}} - t_{ijk})/2, \qquad t_{kij} = \mu_{,k} a_{ij} + \lambda_{,k} b_i b_j + \pi_{,k} b_{\{i} y_{j\}} + \rho_{,k} y_i y_j.$$

Then the Finslerian Christoffel symbols of II-nd kind are

$$\gamma^i_{jk} = g^{is}\gamma_{jsk}. \tag{8}$$

Using the relations (4),(5), these provide the Cartan non-linear connection (N^i_j) and the horizontal connection coefficients (F^i_{jk}). The h- and v-covariant derivations are given, as usually, by

$$\begin{cases} X^i_{\ |k} = \delta_k X^i + F^i_{sk} X^s, \\ X^i \mid_a = \dot{\partial}_a X^i + C^i_{sa} X^s. \end{cases}$$

The curvature tensor fields are given by

$$\begin{cases} R_{j\ kl}^{\ i} = \delta_{(l} F^i_{jk)} + F^h_{j(k} F^i_{hl)} + C^i_{ja} R^a_{kl}, \\ P_{j\ kc}^{\ i} = F^i_{jk;c} - C^i_{jc|k} + C^i_{jb} P^b_{kc}, \\ S_{b\ cd}^{\ a} = F^2 C^a_{e(c} C^e_{bd)}, \end{cases} \tag{9}$$

where we denoted $\tau_{(ij)} = \tau_{ij} - \tau_{ji}$, and

$$\begin{cases} R^a_{kl} = \delta_{(l} N^a_{k)}, \\ P^a_{kc} = N^a_{k;c} - F^a_{ck}, \end{cases} \tag{10}$$

are the *v-hh* and *mixed torsions* of the Cartan connection, respectively.

We remark that for $n = 5,\ x = (x^1, x^2, x^3, x^4, x^5 \equiv x^0)$ and $a_{55} = 1,\ a_{5i} = a_{5i} = 0,\ \forall i = \overline{1,4}$, the d-tensor field R^a_{kl} (the curvature of the non-linear connection) represents - in Kaluza-Klein approach, the electromagnetic field tensor, and the non-linear connection plays the role of electromagnetic potential ([8]).

4 Stationary curves in F^n

The *(horizontal) Finslerian geodesics* $c : I \subset \mathbf{R} \to M, c(s) = x(s)$, where s is the arc-length parameter, are regarded as the extremals of the Lagrangian

$$L_h = \sqrt{g_{ij}(x, \mathcal{V})\mathcal{V}^i\mathcal{V}^j}, \ \text{ with } \ \mathcal{V}^i = \dot{x}^i,$$

where we denoted by dot the derivative with respect to s. Their equations have the general form ([9],[12],[14]):

$$\ddot{x}^i + \gamma^i_{jk}(x,y)\dot{x}^j\dot{x}^k = 0. \tag{11}$$

For our space F^n, these equations become

$$g_{ts}\ddot{x}^s + \gamma_{jtk}\dot{x}^j\dot{x}^k = 0,$$

or, after contraction with $\mu^{-1}a^{it}$, in the general case $\Lambda \neq 0$,

$$\ddot{x}^i + \frac{1}{\mu}a^{it}\phi_{ts}\ddot{x}^s + \frac{1}{\mu}\left(\overset{\circ}{\gamma}{}^i_{jk} + t^i_{jk} + \pi b^i a_{jk} + \frac{\lambda}{2}b^i b_{\{jk\}} + \rho y^i a_{jk} + \Lambda^i_{jk}\right)\dot{x}^j\dot{x}^k = 0,$$

where

$$\begin{cases} \overset{\circ}{\gamma}{}^{\ i}_{j\ k} = a^{is}\overset{\circ}{\gamma}_{jsk}, \ \phi_{ij} = g_{ij} - \mu a_{ij} \\ \Lambda^{\ i}_{j\ k} = \frac{1}{2}g^{is}(\Lambda_{\{js,k\}} - \Lambda_{jk,s}), \ \Lambda_{jk} = \frac{1}{2}\Lambda^2_{;ij} + \lambda_{ij}. \end{cases} \tag{12}$$

Remark. For $\alpha = \beta = 0$ and $\gamma = 1$ we obtain the equations of [19].

Also, the *vertical Finslerian geodesics*

$$c : I \subset \mathbf{R} \to T_{x_0}M, c(s) = (x_0, y(s)), \ x_0 \in M,$$

are regarded as the extremals of the Lagrangian

$$L_v = \sqrt{g_{ab}(x_0, \bar{\mathcal{V}})\bar{\mathcal{V}}^a\bar{\mathcal{V}}^b},$$

where $\bar{\mathcal{V}}^a = \frac{\delta y^a}{dt} = \dot{y}^a + N^a_j\dot{x}^j = \dot{y}^a$, since $x = x_0$.

They satisfy the equations ([1], [17]):

$$\ddot{y}^s + C^s_{ef}(x_0, y(t))\dot{y}^e\dot{y}^f = 0. \tag{13}$$

The types of geodesics described above can be regarded as particular cases of *stationary curves*

$$c : I \subset \mathbf{R} \to (x(s), y(s)) \in TM,$$

which minimize the general length-Lagrangian

$$L_m = \sqrt{g_{ij}\mathcal{V}^i\mathcal{V}^j + g_{ab}\bar{\mathcal{V}}^a\bar{\mathcal{V}}^b}, \qquad \mathcal{V}^i = \dot{x}^i, \ \bar{\mathcal{V}}^a = \dot{y}^a + N^a_i\dot{x}^i.$$

With respect to a metrical h- and v-symmetrical N-connection ∇ on TM, these curves satisfy the system

$$\begin{cases} \mathcal{F}^i \equiv \frac{\nabla\mathcal{V}^i}{ds} = g^{ij}(P_{bak}\bar{\mathcal{V}}^a\bar{\mathcal{V}}^b - C_{ija}\mathcal{V}^i\bar{\mathcal{V}}^a + R_{jak}\bar{\mathcal{V}}^a\mathcal{V}^k), \\ \mathcal{F}^a \equiv \frac{\nabla\mathcal{V}^a}{ds} = g^{ab}(C_{jib}\mathcal{V}^i\mathcal{V}^j - P_{bek}\bar{\mathcal{V}}^e\mathcal{V}^k). \end{cases} \tag{14}$$

Remarks. 1. For a symmetrical connection ∇, the first equation writes also

$$\frac{d^2x^i}{dt^2} + F^i_{jk}\mathcal{V}^j\mathcal{V}^k + C^i_{ja}\mathcal{V}^j\bar{\mathcal{V}}^a = 0, \tag{15}$$

providing thus the case presented in [17].
2. If ∇ is the N-Cartan linear connection on TM and $\bar{\mathcal{V}}^a = 0,\ a = \overline{1,n}$ (i.e., c is an *h-curve*), then it satisfies the equations $\dfrac{\delta y^a}{dt} = 0$, equivalent to (13) and (14). Thus, the h-curves of the Cartan connection are the Finslerian geodesics of F^n.
3. If ∇ is the N-Cartan linear connection on TM and $\mathcal{V}^i = 0,\ i = \overline{1,n}$ (i.e., c is a *v-curve*), then (13) and (14) are equivalent, so that the vertical geodesics ([16], [18]) coincide with the vertical stationary curves.

5 Deviations of stationary curves in F^n

Generally, let us consider a family of curves

$$c : I \times J \subset \mathbf{R}^2 \to TM,\ \ c(s,u) = ((x(s,u), y(s,u)),$$

where s is the arc-length parameter and u varies the curves in the family. Let also the *h- and v-covariant velocity vector-fields*, respectively be given by

$$\mathcal{V}^i = \frac{\partial x^i}{\partial s},\ \bar{\mathcal{V}}^a = \frac{\delta y^a}{\partial s} = \frac{\partial y^a}{\partial s} + N^a_j\frac{\partial x^j}{\partial s},$$

and the *h- and v-covariant deviation vector-fields*

$$\mathcal{Z}^i = \frac{\partial x^i}{\partial u},\ \bar{\mathcal{Z}}^a = \frac{\delta y^a}{\partial u} = \frac{\partial y^a}{\partial u} + N^a_j\frac{\partial x^j}{\partial u}.$$

Also, for any vector field $\mathcal{W} = \mathcal{W}^i\delta_i + \bar{\mathcal{W}}^a\dot{\partial}_a$, we introduce the covariant operators

$$\begin{cases} \delta_s\mathcal{W}^i = \partial_s\mathcal{W}^i + L^i_{jk}\mathcal{W}^i\mathcal{V}^k + C^i_{ja}\mathcal{V}^j\bar{\mathcal{V}}^a, \\ \delta_s\bar{\mathcal{W}}^a = \partial_s\bar{\mathcal{W}}^a + L^a_{bk}\bar{\mathcal{W}}^b\mathcal{V}^k + C^a_{bc}\bar{\mathcal{V}}^b\bar{\mathcal{V}}^c, \\ \delta_t\mathcal{W}^i = \partial_t\mathcal{W}^i + L^i_{jk}\mathcal{W}^i\mathcal{Z}^k + C^i_{ja}\mathcal{V}^j\bar{\mathcal{Z}}^a, \\ \delta_t\bar{\mathcal{W}}^a = \partial_t\bar{\mathcal{W}}^a + L^a_{bk}\bar{\mathcal{W}}^b\mathcal{Z}^k + C^a_{bc}\bar{\mathcal{V}}^b\bar{\mathcal{Z}}^c. \end{cases}$$

The equations of deviations of the stationary curves (14) have the form ([5],[6],[13]):

$$\begin{cases} \delta^2_s\mathcal{Z}^i & + & \delta_s[P_{b\ k}^{\ a}(\bar{\mathcal{V}}^b\mathcal{Z}^k - \bar{\mathcal{Z}}^b\mathcal{V}^k) = \\ & = & R_{j\ kl}^{\ i}\mathcal{V}^j\mathcal{Z}^k\mathcal{V}^l + P_{j\ kc}^{\ i}\mathcal{V}^j(\mathcal{Z}^k\bar{\mathcal{V}}^c - \mathcal{V}^k\bar{\mathcal{Z}}^c) + \delta_u\mathcal{F}^i, \\ \delta^2_s\mathcal{Z}^a & + & \delta_s[C^i_{ja}(\mathcal{V}^j\bar{\mathcal{Z}}^a - \mathcal{Z}^j\bar{\mathcal{V}}^a)] + R_{j\ k}^{\ a}\mathcal{V}^j\mathcal{Z}^k] = \\ & = & P_{b\ kc}^{\ a}\mathcal{V}^a(\mathcal{Z}^k\bar{\mathcal{V}}^c - \mathcal{V}^k\bar{\mathcal{Z}}^c) + S_{b\ cd}^{\ a}\bar{\mathcal{V}}^b\bar{\mathcal{Z}}^c\mathcal{V}^d + \delta_u\mathcal{F}^a. \end{cases} \tag{16}$$

As important particular cases, we get the equations of deviations of (horizontal) Finslerian geodesics

$$\delta^2_s\mathcal{Z}^i = R_{j\ kl}^{\ i}\mathcal{V}^j\mathcal{Z}^k\mathcal{V}^l \tag{17}$$

and also, the equations of deviations for vertical geodesics

$$\delta_s^2 \bar{\mathcal{Z}}^a + \frac{1}{L^2(x,y)} S_{b\ cd}^{\ a} \bar{\mathcal{V}}^b \bar{\mathcal{Z}}^c \bar{\mathcal{V}}^d = 0. \tag{18}$$

We remark that in this case $\bar{\mathcal{V}}^a = \dfrac{\delta y^a}{\partial s} = \dfrac{\partial y^a}{\partial s}$, and $\bar{\mathcal{Z}}^a = \frac{\delta y^a}{\partial u} = \frac{\partial y^a}{\partial u}$, since $x = x_0$.

6 The C-reducible case

Definition 6.1 *The space F^n is called to be C-reducible ([9]) iff the Cartan covariant vertical tensor field C_{ijk} satisfies the relation*

$$C_{ijk} = \frac{1}{2a^2} \underset{ijk}{S} (k_{ij} t_k), \tag{19}$$

where k_{ij} is the angular metric

$$k_{ij} = g_{ij} - l_i l_j, \qquad l_i = g_{is} y^s / L \tag{20}$$

and $t_i = b_i - b_s l^s l_i$, $l^i = y^i / L$.

We determine first the angular metric of the space F^n.

Theorem 6.1 *The angular metric k_{ij} has the coefficients*

$$k_{ij} = \mu a_{ij} + \gamma_0 b_i b_j + \gamma_1 y_{\{i} b_{j\}} + \gamma_2 y_i y_j, \tag{21}$$

where

$$\begin{cases} \gamma_0 = \lambda - (\lambda b + \pi a^2)^2 / L^2, \\ \gamma_1 = \pi - (\mu + \pi b + \rho a^2)(\lambda b + \pi a^2)/L^2, \\ \gamma_2 = \rho - (\mu + \pi b + \rho a^2)^2 / L^2. \end{cases}$$

Hint. The result comes out computing first the covector fields

$$l_i = g_{is} y^s / L = (\mu y_i + \lambda b b_i + \pi b y_i + \pi a^2 b_i + \rho a^2 y_i)/L.$$

Corollary 6.1 *The Finsler space F^n is C-reducible iff the following equations (in the unknowns $\alpha, \beta, \gamma, a_{ij}, b_i$) are satisfied*

$$\begin{array}{ll} \alpha_0 = 2\mu\omega a, & \alpha_1 = 2\mu\xi a, \\ \beta_0 = 2\gamma_0 \omega a, & \beta_1 = 2(\gamma_0 \xi + 2\gamma_1 \omega) a, \\ \beta_2 = 2(\gamma_2 \omega + 2\gamma_1 \xi) a, & \beta_3 = 2\gamma_2 \xi a, \end{array}$$

where we denoted

$$\omega = \frac{\left[L^2 - b(\lambda b + \pi a^2)\right]}{L^2}, \ \xi = -\frac{b(\mu + \pi b + \rho a^2)}{L^2},$$

and $\gamma_0, \gamma_1, \gamma_2$ are the coefficients of the angular metric k_{ij} (21).

Proof. Firstly we obtain $t_i = b_i\omega + y_i\xi$. Then, using (16), (19), (21), and identifying the coefficients of the homogeneous y-polynomials, the result comes out.

Corollary 6.2 *If the space F^n is C-reducible, then the equations of vertical geodesics write*

$$\ddot{y}^c + \frac{1}{\mu}a^{ce}\phi_{ef}\ddot{y}^f + \frac{1}{2\mu a^2}\quad\left[\quad \mu\delta^c_{\{e}t_{f\}} + b^c(\omega k_{ef} + \gamma_0 b_{\{e}t_{f\}} + \gamma_1 y_{\{e}t_{f\}}) + \right.$$
$$\left. + y^c(\xi k_{ef} + \gamma_1 b_{\{e}t_{f\}} + \gamma_2 y_{\{e}t_{f\}})\ \right] \dot{y}^e\dot{y}^f = 0,$$

where ϕ_{ef} is the degenerate tensor-field in (12).

Proof. Straightforward, after multiplying relation (13) with $\dfrac{1}{\mu}a^{cd}g_{ds}$, using (21) and the relation in proof of preceding Lemma.

7 Conclusions

The general Randers-Kropina Finsler metric is presented, and the coefficients of the Cartan connection are determined. It is shown that the general character of the metric permits to obtain, as particular cases, fundamental metrics with relevant applications to physics. The classical equations of Finslerian horizontal and vertical geodesics and of their deviations are inferred, and shown to be particular cases of equations of stationary curves. Also, the C-reducible case is characterized and the associated equations of vertical geodesics are provided.

References

[1] Asanov G.S., *Motion of the Rest Frame of the Electric Charge Defined by the Finslerian Structure of the Electromagnetic Field*, Rep. Math. Phys., 11, 2 (1977), 221-226.

[2] Asanov G.S., Stavrinos P.C., *Finslerian Deviations of Geodesics over Tangent Bundle*, Rep. Math. Phys., 30, (1991), 63-69.

[3] Balan V., *On Gedesics and Deviations of Geodesics in Fibered Finslerian Approach*, Mathematical Reports, Rom.Academy Eds., 4.6, 4(1994), 415-422.

[4] Balan V., Stavrinos P.C., *Deviations of Geodesics in the Fibered Finslerian Approach*, in vol. of Kluwer Acad. Publishers, FTPH no.76, 1996, 65-74.

[5] Balan V., Stavrinos P.C., *The Study of Geodesics and of Their Deviations in Higher-order Geometries*, Proceedings of the 3-rd Panhellenic Conference of Differential Geometry, May 29-30, 1997, Athens, Greece, Athens University Editors, 1998, 78-84.

[6] Balan V., Trencevski K., *Deviations of Stationary Curves in the Extended Fibered Finslerian Approach. Characterization and Analytical Solutions for E.D.G.*, Rep.Math.Phys., Torun, 1995, 53-61.

[7] Beil R.G., *Finsler and a Unified Field Theory*, Contemporary Mathematics, 196(1996), 265-271.

[8] Ikeda S., *On the Theory of Gravitational Field in Finsler Spaces*, Tensor N.S., 50(1991), 256-262.

[9] Matsumoto M., *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaisheisha Press, Kyoto, 1986.

[10] Miron R., General Randers Spaces, in Kluwer Acad. Publishers, FTPH, Lagrange and Finsler Geometry, 1996, 123-140.

[11] Miron R., *The Geometry of Higher Order Lagrange Spaces. Applications to Mechanics and Physics*, Kluwer Acad. Publishers, FTPH, 1996.

[12] Miron R., Anastasiei M., *The Geometry of Vector Bundles. Theory and Applications*, Kluwer Acad. Publishers, FTPH, no.59, 1994.

[13] Miron R., Balan V., Stavrinos P.C., Tsagas Gr., *Deviations of Stationary Curves in the Bundle* $Osc^{(2)}(M)$, Proceedings of the I-st Conference of the Balkan Society of Geometers 1996, Bucharest, Romania, Journal of the Balkan Society of Geometers, Bucharest, v.1, n.2, 1997.

[14] Rund H., The Differential Geometry of Finsler Spaces, Springer-Verlag, 1959.

[15] Shibata C., *On Finsler Spaces with Kropina Metric*, Rep.Math.Phys., 13, 1(1978), 117-128.

[16] Stavrinos P.C., *Generalized Finslerian Equations of Geodesic Deviation*, Rep.Math.Phys., 32, 3 (1993), 339-342.

[17] Stavrinos P.C., *Tidal Forces in Vertical Spaces of Finslerian Space-Time*, Rep.Math.Phys., 31, 1 (1992), 1-4.

[18] Stavrinos P.C., Kawaguchi H., *Deviation of Geodesics in the Gravitational Field of Finslerian Space-Time*, Memoirs of Shonan Inst. of Technology, 27, 1 (1993), 35-40.

[19] Stavrinos P.C., Nikolopoulos D., Prezas N., *Finslerian Metric Based on the Gravitational and Electromagnetic Fields*, Proc. of the Nat. Conf. of Gauge Theory, Bistriţa, Romania, 1995.

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