# SYGNOMIAL TYPE LAGRANGIANS

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**Abstract**

This paper studies the sygnomial type Lagrangians and their extremals. The main results are:

- the only minimal surfaces of sygnomial form are the planes and the regions of the plane.

- there exist sygnomial type Lagrangians whose extremals are described by the classical Laplace equation.

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**Key words**: sygnomial Lagrangians, Euler-Lagrange PDEs, Monge-Ampére equation, Laplace equation

## 1 Euler-Lagrange PDE of Sygnomial Lagrangians

**Definition**. An expression of the form

$$\sum_{i=1}^{m} c_i \prod_{j=1}^{n} (x^j)^{a_{ij}},$$

where $(x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$, $c_i \in \mathbb{R}$, $a_{ij} \in \mathbb{R}$, $\forall i = \overline{1,m}$, $j = \overline{1,n}$ is called *sygnom*. If the coefficients $c_i$ are positive numbers, then the expression above is called *posinom*.

If $a_{ij} \in \mathbb{N}$ we obtain polynomials.

We search the extremals of the sygnomial type Lagrangians

$$L(x, y, f, f_x, f_y) = \sum_{i=1}^{q} (c_i x^{a_i} y^{b_i} f^{m_i} f_x^{n_i} f_y^{p_i})^{\alpha}$$

on a compact domain $D \subset \mathbb{R}^2$.

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We write the Euler-Lagrange PDEs

$$
\begin{aligned}
&f_{xx}c_i c_i f^{m_i+m_j} f_x^{n_i+n_j-1} f_y^{p_i+p_j}(2(\alpha-1)n_i n_j + n_i(n_i-1)+\\
&+n_j(n_j-1)) + 2f_{xy}c_i c_j f^{m_i+m_j} f_x^{n_i+n_j-1} f_y^{p_i+p_j-1}\cdot\\
&\cdot((\alpha-1)p_i n_j + (\alpha-1)p_j n_i + n_i p_i + n_j p_j) + f_{yy}c_i c_j f^{m_i+m_j}\cdot\\
&\cdot f_x^{n_i+n_j} f_y^{p_i+p_j-2}(2(\alpha-1)p_i p_j + p_i(p_i-1) + p_j(p_j-1))+\\
&+c_i c_j f^{m_i+m_j-1} f_x^{n_i+n_j} f_y^{p_i+p_j}((\alpha-1)m_i n_j + (\alpha-1)m_j n_i+\\
&+m_i n_i + m_j n_j + (\alpha-1)m_i p_j + (\alpha-1)m_j p_i + m_i p_i + m_j p_j) = 0,
\end{aligned}
\tag{1}
$$

for all $i, j$ from 1 to $q$; satisfying

$$
\begin{cases}
(\alpha-1)a_i n_j + (\alpha-1)a_j n_i + a_i n_i + a_j n_j = 0\\
(\alpha-1)b_i p_j + (\alpha-1)b_j p_i + b_i p_i + b_j p_j = 0.
\end{cases}
$$

## 2 Illustrative Examples

**Example.** 1. The Euler-Lagrange equation of the Lagrangian is $L = \left(c_i f_x^{n_i} f_y^{p_i}\right)^\alpha$

$$
\begin{aligned}
&f_{xx} f_x^{n_i+n_j-1} f_y^{p_i+p_j} c_i c_j [2(\alpha-1)n_i n_j + n_i(n_i-1) + n_j(n_j-1)]+\\
&+2f_{xy} f_x^{n_i+n_j-1} f_y^{p_i+p_j-1} c_i c_j [(\alpha-1)n_i p_j + (\alpha-1)n_j p_i + n_i p_i + n_j p_j]+\\
&+f_{yy} f_x^{n_i+n_j} f_y^{p_i+p_j-2} c_i c_j [2(\alpha-1)p_i p_j + p_i(p_i-1) + p_j(p_j-1)] = 0.
\end{aligned}
$$

This is a generalization of the minimal surfaces equation. Therefore, we can search surface which are solutions of sygnomial type. Particularly, we have

**Proposition.** *The planes and the regions of the planes are the only sygnomial minimal surfaces.*

**Proof.** We look for, $z = f(x, y) = \sum_{i=1}^{q} c_i x^{\alpha_i} y^{\beta_i}$, $q \geq 1$, like solution of the minimal surfaces PDE. Then the coefficients $\alpha_i$, $\beta_i$ satisfy

$$
\begin{cases}
\alpha_i^2\beta_j^2 - \alpha_i^2\beta_j + \alpha_j^2\beta_i^2 - \alpha_j\beta_i^2 = 0\\
\alpha_j^2\beta_i\beta_k - \alpha_j\beta_i\beta_k - \alpha_j\alpha_k\beta_i\beta_k + \alpha_j\alpha_k\beta_j^2 - \alpha_j\alpha_k\beta_j = 0\\
\alpha_j(\alpha_j - 1) = 0\\
\beta_j(\beta_j - 1) = 0,
\end{cases}
$$

whole considering $c_i \neq 0\ \forall i = \overline{1, q}$. From the two last relations we see that $\alpha_j \in \{0, 1\}$ and $\beta_j \in \{0, 1\}$, so we have as solutions the regions of the planes. For $c_i > 0$ we get semiplanes.

**Remark.** It follows that there are no minimal surfaces of the form $z = f(x, y) = \ln(\alpha x + \beta y)$, where $\alpha x + \beta y > 0$, and the only solutions of minimal surfaces described by $z = f(x, y) = \alpha e^{\beta x + \alpha y}$ are the semiplanes as well.

**Example** 2. Let us consider a Riemannian manifold $(\mathbf{R}^2, g)$ and $(g_{ij})_{i,j=\overline{1,2}}$ the matrix of the components of the metric $g$. We want to find the metric for which the Euler-Lagrange equation derived of a sygnomial Lagrangian is a generalized Laplace equation, $g^{ij} f_{ij} = 0$.

$L(x, y, f, f_x, f_y) = x^{a_1} y^{b_1} f_y^2 + x^{a_2} y^{b_2} f_x^2$ leeds to $a_i n_i + a_j n_j = 0$, i.e. $a_2 = 0$ and similary $b_1 = 0$, and $f_{xx}(2f_x^2 + f_y^2) + f_{yy}(f_x^2 + 2f_y^2) = 0$.

We choose $g^{11} = 2f_x^2 + f_y^2$, $g^{12} = g^{21} = 0$, $g^{22} = f_x^2 + 2f_y^2$ and $g = \det(g_{ij})_{i,j=\overline{1,n}}$. Then, excepting the case of the constant function $f$, we obtain the metric

$$\begin{pmatrix} \dfrac{1}{2f_x^2 + f_y^2} & 0 \\ 0 & \dfrac{1}{f_x^2 + 2f_y^2} \end{pmatrix}$$

which is nondegenerate, symmetric and positive definite

**Remark**. We cannot obtain a Monge-Ampére equation from the Euler-Lagrange equation of a sygnomial type Lagrangian, since there are no terms containing $f_{xx} f_{yy}$ or $f_{xy}^2$.

Many physical processes depend on time and happen in a space at least two dimensions. That is why, we consider a real function $f : D \to \mathbf{R}$, $D \subseteq \mathbf{R}^3$ and the sygnomial Lagrangins

$$L(x, y, t, f, f_x, f_y, f_t) = \sum_{i=1}^{s} r_i x^{a_i} y^{b_i} t^{c_i} f_x^{n_i} f_y^{p_i} f_t^{q_i}, \; s \in \mathbf{N}^*.$$

The associated Euler-Lagrange equation is

$$\begin{aligned} & f_{xx} r_i n_i (n_i - 1) f_x^{n_i - 2} f_y^{p_i} f_t^{q_i} + f_{yy} r_i p_i (p_i - 1) f_x^{n_i} f_y^{p_i - 2} f_t^{q_i} + \\ & + f_{tt} r_i q_i (q_i - 1) f_x^{n_i} f_y^{p_i} f_t^{q_i - 2} + 2 f_{xy} r_i n_i p_i f_x^{n_i - 1} f_y^{p_i - 1} f_t^{q_i} + \\ & + 2 f_{xt} r_i n_i q_i f_x^{n_i - 1} f_y^{p_i} f_t^{q_i - 1} + 2 f_{ty} r_i q_i p_i f_x^{n_i} f_y^{p_i - 1} f_t^{q_i - 1} = 0. \end{aligned} \tag{2}$$

From the conditions

$$\begin{cases} r_i n_i a_i = 0 \\ r_i p_i b_i = 0 \\ r_i q_i c_i = 0, \end{cases}$$

satisfied for all $i = \overline{1,s}$, we note that we cannot have, in the same term of the sum, a certain variable and the partial derivative of the function $f$ with respect to the same variable. Without loss of generality, we can put $r_i = 1$, $\forall i = \overline{1,s}$.

**Example**. 3. We use the canonic Riemannian metric $g_{ij} = \delta_{ij}$, $\forall i, j = \overline{1,3}$, and the Lagrangian $L = y^{b_1} t^{c_1} f_x^2 + x^{a_2} t^{c_2} f_y^2 + x^{a_3} y^{b_3} f_t^2$. Then, relation (2) implies

$$f_{xx} + f_{yy} + f_{tt} = 0,$$

which is ordinary Laplace equation.

**Example** 4. Now we build a generalized Laplace equation too, using a semi-Riemannian metric. If we consider $L = f_x^2 + f_t f_y$, then the relation (2) becomes $f_{xx} + f_{ty} = 0$. Thie PDE can be written $g^{ij} f_{ij} = 0$, where $(g^{ij})_{i,j=\overline{1,3}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ comes from a semi-Riemannian metric $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

**Example**. 5. We consider the Lagrangian

$$L(x, y, t, f, f_x, f_y, f_t) = t^{c_1} f_x^2 f_y + y^{b_2} f_x f_t^2 + x^{a_3} f_y^2 f_t$$

and the associated Euler-Lagrange equation

$$f_{xx} f_y + f_{yy} f_t + f_{tt} f_x + 2f_{xy} f_x + 2f_{tx} f_t + 2f_{ty} f_y = 0$$

We regard the coefficients $f_x, f_y, f_t, 2f_x, 2f_y, 2f_t$ as the entries of the matrix

$$(g^{ij})_{i,j=\overline{1,3}} = \begin{pmatrix} f_y & f_x & f_t \\ f_x & f_t & f_y \\ f_t & f_y & f_x \end{pmatrix}$$

of determinant $g = 3f_x f_y f_t - f_x^3 - f_y^3 - f_t^3$. This metric is non-degenerate if and only if $f_x, f_y, f_t$ are different from each other and $f_x + f_y + f_t \neq 0$. In this case,

$$(g_{ij})_{i,j=\overline{1,3}} = \frac{1}{g} \begin{pmatrix} f_x f_t - f_y^2 & f_t f_y - f_x^2 & f_x f_y - f_t^2 \\ f_t f_y - f_x^2 & f_x f_y - f_t^2 & f_x f_t - f_y^2 \\ f_x f_y - f_t^2 & f_x f_t - f_y^2 & f_t f_y - f_x^2 \end{pmatrix}.$$

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