UNINORM AGGREGATION OPERATORS FOR INTUITIONISTIC FUZZY SETS

Toader T. Buhaescu

Abstract

The paper introduces two types of uni-norms operators in intuitionistic fuzzy sets class, starting from the families of the uninorms R^* and R_* defined in [5].

We study their behavior and show that there are some invariable subclasses trough the action of these operators. Also, we prove some properties in connection with the identity elements.

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1 Introduction.

The set theoretical operators have had on important role since in the beginning of fuzzy set theory. Starting from Zadeh's operators min and max many other operators were introduced in the fuzzy set literature. All types of the particular operators were included in the general concepts of t-norm and t-conorm. The corresponding operators, defined on the unit interval, are distinguished each from other by their identity elements. These elements are one for t-norm and zero for t-conorm. The identity elements do not affect the aggregation value.

Commutativity, monotonicity and associativity are the common properties of a t-norm and t-conorm. R. Yager and A. Rybalov ([5] in 1996) have unified and generalized these operators by allowing the identity to be any number in the unit interval. They call this generalized class of operators uni-norms.

The notion of intuitionistic fuzzy set was originally been introduced by K. T. Atanassov ([1] in 1986), as a generalization of the concept of fuzzy set. Basically, an identification of intuitionistic fuzzy set can be done with an ordered pair of mappings $\{A^+, A^-\}$ defined on an universe X as membership and nonmembership functions so that $0 \leq A^+(x) + A^-(x) \leq 1$ for all $x \in X$.

Two types of uni-norms operators are introduced in the paper in the intuitionistic fuzzy sets class and their behavior will be studied.

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2 Preliminaries

The membership of a fuzzy set is in an universe X having its values in the unit interval $I = [0,1] \subseteq R$.

Definition 1 An *intuitionistic fuzzy set* in X (denoted by IFS) is an object having the following form:

$$A = \left\{ \left\langle A^+(x),\ A^-(x) \right\rangle \mid A^+(x) + A^-(x) \leq 1,\, x \in X \right\}, \qquad (1)$$

where $A^+ : X \to [0,1]$, $A^- : X \to [0,1]$ define the degree of membership and the degree of nonmembership of the element $x \in X$ to the IFS A.

The set operations in the fuzzy sets class introduced by L. A. Zadeh have the following corresponding operators, defined on the unit interval:

$$\wedge : [0,1] \times [0,1] \to [0,1],\ \vee : [0,1] \times [0,1] \to [0,1],\ ^{\neg} : [0,1] \to [0,1],$$

$$a \wedge b = \min(a,b),\ a \vee b = \max(a,b),\ ^{\neg}a = 1 - a. \qquad (2)$$

Definition 2 For the class of IFS in X the corresponding operators defined in

$$L = \{a = \left\langle a^+,\, a^- \right\rangle \mid a^+ a^- \in [0,1], a^+ + a^- \leq 1\}$$

have the following form:

$$a \wedge b = \left\langle a^+ \wedge b^+,\, a^- \vee b^- \right\rangle, \qquad (3)$$

$$a \vee b = \left\langle a^+ \vee b^+,\, a^- \wedge b^- \right\rangle, \qquad (4)$$

$$^{\neg}a = \left\langle a^-, a^+ \right\rangle = {}^{c}a, \qquad (5)$$

where $a^+ \wedge b^+ = \min(a^+,\, b^+)$, $a^- \vee b^- = \max(a^-, b^-)$. We consider the following order relation:

$a \leq b$ if $a^+ \leq b^+$ and $a^- \geq b^-$(see [1] and [2]).

Definition 3 A *t-norm* T is a mapping

$$T : [0,1] \times [0,1] \to [0,1] \qquad (6)$$

having the following properties:

(a) $T(a,b) = T(b,a)$	*(Commutativity)*
(b) $T(a,b) \geq T(c,d)$, *if* $a \geq c$ *and* $b \geq d$	*(Monotonicity)*
(c) $T(a, T(b,c)) = T(T(a,b),c)$	*(Associativity)*
(d) $T(a,1) = a$, $\forall a \in [0,1]$	*(Boundary)*

Definition 4 A *t-conorm* S is a mapping

$$S : [0,1] \times [0,1] \to [0,1] \tag{7}$$

having the same (a), (b), (c) properties of a t-norm and the fourth property having the form:

$(d^{'})$ $S(a,0) = a$. *(Boundary)*

Definition 5 A *uni-norm* R is a mapping

$$R : [0,1] \times [0,1] \to [0,1] \tag{8}$$

having the following properties:
(1) $R(a,b) = R(b,a)$ *(Commutativity),*
(2) $R(a,b) \geq R(c,d)$ *if* $a \geq c$ *and* $b \geq d$ *(Monotonicity),*
(3) $R(a,R(b,c)) = R(R(a,b),c)$ *(Associativity),*
(4) There exist some elements $e \in [0,1]$ called the *identity element* such that for all $a \in [0,1]$, $R(a,e) = a$.

We observe (see [5]) that a uni-norm has the first three properties common of a t-norm and a t-conorm, but allows more options relative to property four by assigning it an identity element e. We see that the t-norm is a special case of uni-norm with $e = 1$, while the t-conorm is a special case with $e = 0$.

Proposition 1 . *The operator R_* defined such that:*
(1) $R_*(a,b) = \min(a,b)$, *if* $\min(a,b) < e$,
(2) $R_*(a,b) = \max(a,b)$, *if* $\min(a,b) \geq e$,
is a uni-norm operator with identity e.

Proposition 2 *The operator R^* defined such that:*
(1) $R^*(a,b) = \min(a,b)$, *if* $\max(a,b) \leq e$,
(2) $R^*(a,b) = \max(a,b)$, *if* $\max(a,b) > e$,
is a uni-norm operator with identity e.

For the Definition 5 and Propositions 1 and 2 see [5].

3 On uni-norm aggregators

Let R_* and R^* be the operators defined in Proposition 1 and Proposition 2 respectively. We denote the identity element of R_* by e_* and of R^* by e^*, that is:

$$R_*(a,e_*) = a;\ R^*(a,e^*) = a,\ (\forall)a \in [0,1]. \tag{9}$$

Theorem 3 *The mapping $\varphi : L \times L \to [0,1] \times [0,1]$ defined by*

$$\varphi(a,b) = \left\langle R^*(a^+, b^+),\, R_*(a^-, b^-)\right\rangle \tag{10}$$

has the following properties:

(1) $\varphi(a,b) = \varphi(b,a)$, *(Commutativity)*

(2) $\varphi(a, \varphi(b,c)) = \varphi(\varphi(a,b), c)$, *(Associativity)*

(3) $\begin{array}{l} \varphi(a,c) \le \varphi(b,c) \text{ if } a \le b, \\ \varphi(a,b) \le \varphi(a,c) \text{ if } b \le c, \end{array}$ *(Monotonicity)*

(4) $\varphi(a,u) = a$, *where* $u = \langle e^*,\, e_*\rangle$ *has the role of the identity element iff* $e^* + e_* \le 1$.

Proof. (1) and (2) follow from Propositions 1 and 2 and from (10), since R_* and R^* have these properties.

From Definition 2 $a \le b$, if $a^+ \le b^+$ and $a^- \ge b^-$ and therefore $R^*(a^+, c^+) \le R^*(a^+, c^+)$ and $R_*(a^-, c^-) \ge R_*(b^-, c^-)$, thus $\varphi(a,c) \le \varphi(a,b)$.

Identity: $\varphi(a,u) = \langle R^*(a^+, e^*),\, R_*(a^-, e_*)\rangle = \langle a^+,\, a^-\rangle = a; u \in L,$ if $e^* + e_* \le 1$. □

Definition 6 An *intuitionistic fuzzy* set $A = \langle A^+(x),\ A^-(x)\rangle$, $x \in X$ is called weak (WIFS), if $A^+(x),\ A^-(x) \in [0,\, \frac{1}{2}]$. We denote the corresponding lattice by WL,

$$WL = \left\{ a = \left\langle a^+,\, a^-\right\rangle \middle| a^+,\, a^- \in \left[0, \frac{1}{2}\right].\right\} \tag{11}$$

Proposition 4 *For any pair $(e^*,\, e_*)$ with $e^* + e_* \le 1$ the mapping φ defined by (10) is an aggregation operator on WL.*

Proof. From $e^* + e_* \le 1$ it follows that $u = \langle e^*,\, e_*\rangle \in L$.

If a, b are arbitrary elements in WL, then

$$\begin{array}{rcl} R^*(a^+, b^+) & = & \min(a^+, b^+)\,,\ \ R_*(a^-, b^-) = \min(a^-, b^-), \\ R^*(a^+, b^+) & = & \max(a^+, b^+)\,,\ \ R_*(a^-, b^-) = \max(a^-, b^-), \end{array} \tag{12}$$

for all relative position of $e^*$and e_*. All possible values of $\varphi(a,b)$ are:

$$\varphi(a,b) = \left\langle \min(a^+,\, b^+),\, \min(a^-,\, b^-)\right\rangle,\ \varphi(a,b) = \left\langle \min(a^+,\, b^+),\, \max(a^-,\, b^-)\right\rangle,$$

$$\varphi(a,b) = \left\langle \max(a^+,\, b^+),\, \min(a^-,\, b^-)\right\rangle,\ \varphi(a,b) = \left\langle \max(a^+,\, b^+),\, \max(a^-,\, b^-)\right\rangle.$$

For all cases $\varphi(a,b) \in WL$. □

Definition 7 The IFS $A = \langle A^+(x),\ A^-(x)\rangle$, $x \in X$ is called *strong* (SIFS) if:

$$A^-(x) \le \frac{1}{2} \le A^+(x).$$

The corresponding lattice will be noted by SL, that is

$$SL = \left\{ a = \left\langle a^+,\, a^- \right\rangle \middle| 0 \le a^- \le \frac{1}{2} \le a^+ \le 1,\, a^+ + a^- \le 1 \right\}. \tag{13}$$

Proposition 5 *If* $e^* < \dfrac{1}{2} < e_*,\ e^* + e_* \le 1$, *then the mapping* φ *defined by (10) is an aggregation operator on* SL.

Proof. For any $a, b \in SL$, $a^-, b^- \in [0, e_*^-)$ and $a^+, b^+ \in (e^*, 1]$, therefore $R^*(a^+, b^+) = \max(a^+, b^+)$ and $R_*(a^-, b^-) = \min(a^-, b^-)$. On the other hand $R_*(a^-, b^-) \le \dfrac{1}{2}$, $R^*(a^+, b^+) \ge \dfrac{1}{2}$ and $\varphi(a, b) = \langle \max(a^+, b^+),\ \min(a^-, b^-)\rangle$. Obviously $\varphi(a, b) \in SL$.

In this case we observe that the operator φ is a union operator.

Proposition 6 . *If* $e_* < \dfrac{1}{2} < e^*,\ e^* + e_* \le 1$, *then the mapping* φ *defined in (10) is an aggregation operator for all* $a, b \in SL$, *excepting the case:* $\max(a^+, b^+) + \max(a^-, b^-) > 1$.

Proof. According to the propositions 1 and 2, we have:

$$R^*(a^+, b^+) = \max(a^+, b^+),\ R^*(a^+, b^+) = \min(a^+, b^+) \tag{14}$$

$$R_*(a^-, b^-) = \min(a^-, b^-),\ R_*(a^-, b^-) = \max(a^-, b^-). \tag{15}$$

Hence, the mapping φ can takes the following values:

$$\varphi(a, b) = \left\langle \min(a^+,\, b^+),\, \min(a^-,\, b^-) \right\rangle,\ \varphi(a, b) = \left\langle \min(a^+,\, b^+),\, \max(a^-,\, b^-) \right\rangle,$$

$$\varphi(a, b) = \left\langle \max(a^+,\, b^+),\, \min(a^-,\, b^-) \right\rangle,\ \varphi(a, b) = \left\langle \max(a^+,\, b^+),\, \max(a^-,\, b^-) \right\rangle.$$

In the second case $\varphi(a, b) \in SL$ if $\max(a^+, b^+) + \max(a^-, b^-) \le 1$. In the first case φ is a union operator and in the forth case φ is an intersection operator.

We obtain new types of aggregation operators in the second case and in the third case.

Remark 1 If $a^-, b^-, a^+, b^+ \in [\,e^*, e_*]$ then φ is an intersection operator in SIFS.

Theorem 7 . *The mapping* $\psi : L \times L \to [0, 1] \times [0, 1]$ *defined by*

$$\psi(a, b) = \left\langle R_*(a^+, b^+),\, R^*(a^-, b^-) \right\rangle \tag{16}$$

has the following properties: commutativity, associativity, monotonicity and also $\psi(a, b) = a$, *where* $v = \langle e^*,\, e_*\rangle$ *with* $e^* + e_* \le 1$ *is the identity element.*

The proof is similar to Theorem 3.

Proposition 8 *. The mapping ψ defined by (16) is an aggregation operator on WL.*

The proof is the same of Proposition 4.

Proposition 9 *. If $e^* < \frac{1}{2} < e_*$, $e^* + e_* \leq 1$, then the mapping ψ defined by (16) is an aggregation operator on SL.*

The proof is similar to Proposition 5.

Proposition 10 *. If $e_* < \frac{1}{2} < e^*$, $e^* + e_* \leq 1$, then the mapping ψ defined in (16) is an aggregation operator for all $a, b \in SL$, excepting the case:* $\max(a^+, b^+) + \max(a^-, b^-) > 1$.

The proof is similar to Proposition 6.

Remark 2 If $a^-, b^- \in [0, e_*]$ and $a^+, b^+ \in [e^*, 1]$ then ψ is an union operator in SIFS.

Proposition 11 *. The connection between φ and ψ is given by the following relations:*

$$\varphi(a, b) = {}^c\psi({}^ca, {}^cb), \tag{17}$$

$$\psi(a, b) = {}^c\varphi({}^ca, {}^cb). \tag{18}$$

Proof. If a is $a = \langle a^+, a^- \rangle$, then ${}^ca = \langle a^-, a^+ \rangle$ (see Definition 2).

Hence $\varphi({}^ca, {}^cb) = \langle R^*(a^-, b^-), R_*(a^+, b^+) \rangle$ and
${}^c\varphi({}^ca, {}^cb) = \langle R_*(a^+, b^+), R^*(a^-, b^-) \rangle = \psi(a, b)$ that is (18) is true. Analogously we prove (17).

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Author's address:

Toader T. Buhaescu
Department of mathematics
University "Dunarea de Jos"
Galati - Romania