

ON THE STOCHASTIC BEHAVIOUR OF THE DIGITS IN THE MODIFIED ENGEL-TYPE ALTERNATING SERIES REPRESENTATIONS FOR REAL NUMBERS

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Abstract

Let $I = [0, 1]$ and \mathcal{B}_I the σ -algebra of all Borel subsets of I . Then any number $x \in I$ can be uniquely expressed as a modified Engel-type alternating representation by a general alternating series algorithm which produce a sequence of digits.

In the present paper we investigate the stochastic behaviour of the digits in the modified Engel-type alternating expansion. Particularly, we obtain that the digits are stochastically independent and identically distributed random variables under the normal probability space $(I, \mathcal{B}_I, \lambda)$. Consequently all the classic limit theorems may be used in the study of the metrical problems occurring in the context of alternating series representations.

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1 Introduction

Firstly we define a general series algorithm by A. and J. Knopfmacher (see [3], [4]) analogous to a positive one of Oppenheim (see [5]), that leads to a general alternating series representations for real numbers in terms of rationals. In particular, this algorithm is used to prove the existence of an alternating series expansion which is analogous to a modified Engel expansion.

1.1 A general alternating series algorithm of A. and J. Knopfmacher

Given any real number A , let

$$\alpha_0 = [A] \text{ (the integral part of } A) \text{ and}$$

$$A_1 = A - \alpha_0 = \{A\} \text{ (the fractional part of } A).$$

Then we define recursively the digits α_n , for any $n \geq 1$, by

$$\alpha_n = \left\lceil \frac{1}{A_n} \right\rceil \geq 1, \text{ for } A_n > 0, \quad (1.1.1)$$

where

$$A_{n+1} = \left(\frac{1}{\alpha_n} - A_n \right) \cdot (c_n/b_n), \text{ for } \alpha_n > 0.$$

In the above formula

$$b_i = b_i(\alpha_1, \alpha_2, \dots, \alpha_i), \quad c_i = c_i(\alpha_1, \alpha_2, \dots, \alpha_i)$$

are positive numbers (usually integers), functions of the first i digits $\alpha_1, \alpha_2, \dots, \alpha_i$, chosen so that $A_n \leq 1$, for $n \geq 1$. Note that $A_{n+1} \geq 0$, since $\alpha_n \leq \frac{1}{A_n}$, for $A_n > 0$.

1.2 Modified Engel-type alternating series expansion

Using the above mentioned algorithm we prove the following result.

Theorem 1 *Any real number A has a unique alternating series representation of the form*

$$\begin{aligned} A &= \alpha_0 + \frac{1}{\alpha_1} - \frac{1}{\alpha_1 + 1} \cdot \frac{1}{\alpha_2} + \frac{1}{(\alpha_1 + 1)(\alpha_2 + 1)} \cdot \frac{1}{\alpha_3} - \dots \\ &\equiv (\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n, \dots), \end{aligned} \quad (1.2.1)$$

where $\alpha_1 \geq 1$, $\alpha_{n+1} \geq \alpha_n$, for $n \geq 1$.

Proof. Repeated application of the general alternating series algorithm yields

$$\begin{aligned} A &= \alpha_0 + A_1 = \alpha_0 + \frac{1}{\alpha_1} - \frac{b_1}{c_1} \cdot A_2 = \\ &= \alpha_0 + \frac{1}{\alpha_1} - \frac{b_1}{c_1} \cdot \frac{1}{\alpha_2} + \frac{b_1 \cdot b_2}{c_1 \cdot c_2} \cdot \frac{1}{\alpha_3} - \dots + (-1)^{n-1} \frac{b_1 \cdot b_2 \dots b_{n-1}}{c_1 \cdot c_2 \dots c_{n-1}} \cdot A_n. \end{aligned}$$

Now, since $\alpha_n = \left\lceil \frac{1}{A_n} \right\rceil$ implies that $\frac{1}{\alpha_1 + 1} < A_n \leq \frac{1}{\alpha_n}$, for $0 < A_n \leq 1$, we obtain that:

$$A_{n+1} = \left(\frac{1}{\alpha_n} - A_n \right) \cdot \frac{c_n}{b_n} < \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_{n+1}} \right) \cdot \frac{c_n}{b_n} = \frac{1}{\alpha_n(\alpha_n + 1)} \cdot \frac{c_n}{b_n}, \text{ if } 0 < A_n \leq 1.$$

By setting $b_n = 1$ and $c_n = \alpha_n + 1$, for all n , we obtain that:

$$\alpha_{n+1} = \left[\frac{1}{A_{n+1}} \right] \geq \alpha_n, \text{ provided } A_i > 0 \text{ for } i \leq n.$$

Furthermore we have:

$$\frac{A_{n+1}}{(\alpha_1 + 1)(\alpha_2 + 1)\dots(\alpha_n + 1)} \leq \frac{1}{2^n} \rightarrow 0, \text{ as } n \rightarrow \infty,$$

since $\alpha_1 \geq 1$.

Consequently, it follows that any real number A has the following alternating expansion

$$\begin{aligned} A = & \alpha_0 + \frac{1}{\alpha_1} - \frac{1}{\alpha_1 + 1} \cdot \frac{1}{\alpha_2} + \frac{1}{(\alpha_1 + 1)(\alpha_2 + 1)} \cdot \frac{1}{\alpha_3} - \dots \\ & + (-1)^{n-1} \frac{1}{(\alpha_1 + 1)(\alpha_2 + 1)\dots(\alpha_{n-1} + 1)} \cdot \frac{1}{\alpha_n} + \dots, \end{aligned}$$

where $\alpha_{n+1} \geq \alpha_n \geq 1, n \geq 1. \square$

Representation (1.2.1) is called "modified Engel-type" alternating expansion for real numbers, while the positive integers $\alpha_0, \alpha_1, \dots, \alpha_n, \dots$, are called digits of the above-mentioned expansion (see [1], [6], [7], [2]).

2 Stochastic behaviour of the sequence of digits

Let $I = [0, 1]$, $X = N^*$, where $N^* = \{1, 2, \dots\}$ and \mathcal{B}_I the σ -algebra of all Borel subsets of the unit interval I . Then the functions $\alpha_n(\cdot) : I \rightarrow X$, for any $n \in N^*$, defined by the algorithm (1.1.1), may be considered as random variables defined almost surely on I with respect to any probability measure on the σ -algebra \mathcal{B}_I (in particular, with respect to the Lebesgue measure λ).

We focus on the modified Engel-type expansion (1.2.1) for any real number $x \in I$. We are going to study the stochastic properties of the sequence of digits $(\alpha_n(x))_{n \in N^*}$, of the expansion (1.2.1) for any $x \in I$.

Particularly, we are going to prove that the digits $\alpha_n(\cdot), n \in N^*$ are stochastically independent and identically distributed random variables with respect to the normal probability space $(I, \mathcal{B}_I, \lambda)$.

2.1 Auxiliary results

Firstly we consider:

$$\begin{aligned} I_{k_1 k_2 \dots k_n} = & \{x = (\alpha_1(x), \alpha_2(x), \dots, \alpha_n(x), \dots) \in I : \alpha_1(x) = k_1, \dots, \alpha_n(x) = k_n, \\ & \alpha_{n+m}(x) \in N^*, m \geq 1\}, k_1 k_2 \dots k_n \in N^*, \end{aligned}$$

the set of all $x \in I$ which have a unique expansion of the form (1.2.1) such that its digits $\alpha_1(x), \alpha_2(x), \dots, \alpha_n(x)$ have concrete values k_1, k_2, \dots, k_n respectively, while all the others are positive integers.

Then we take the following:

Proposition 2 *The set $I_{k_1 k_2 \dots k_n}$ is bounded and its corresponding bounds are given by the relations*

$$\begin{aligned} M_n &= \sup I_{k_1 k_2 \dots k_n} = \frac{1}{k_1} - \frac{1}{(k_1 + 1)} \cdot \frac{1}{k_2} + \dots \\ &\quad + (-1)^{n-1} \frac{1}{(k_1 + 1)(k_2 + 1) \dots (k_{n-1} + 1)} \cdot \frac{1}{k_n}, \\ m_n &= \inf I_{k_1 k_2 \dots k_n} = \frac{1}{k_1} - \frac{1}{(k_1 + 1)} \cdot \frac{1}{k_2} + \dots \\ &\quad + (-1)^{n-1} \frac{1}{(k_1 + 1)(k_2 + 1) \dots (k_{n-1} + 1)} \cdot \frac{1}{k_n} + \\ &\quad (-1)^n \frac{1}{(k_1 + 1)(k_2 + 1) \dots (k_n + 1)}, \end{aligned}$$

where n is odd. If n is even then the above relations are inverted.

Proof. Using expansion (1.2.1) we have in general that

$$\begin{aligned} \text{if } \alpha_n(x) = k_n, \quad \text{then } r_{n-1}(x) &= \frac{1}{k_n} - \frac{1}{(k_n + 1)} \cdot r_n(x), \quad n \in N^*, \\ \text{where } r_n(x) &= \frac{1}{\alpha_{n+1}(x)} - \frac{1}{(\alpha_{n+1}(x) + 1)} \cdot \frac{1}{\alpha_{n+2}(x)} + \dots, \\ \text{with } \alpha_{n+m}(x) &\in N^*, \quad \text{for any } m \geq 1, \quad x \in I_{k_1 k_2 \dots k_n}. \end{aligned}$$

Therefore if $x \in I_{k_1 k_2 \dots k_n}$, then

$$\begin{aligned} x &= \frac{1}{k_1} - \frac{1}{(k_1 + 1)} \cdot \frac{1}{k_2} + \dots + (-1)^{n-1} \frac{1}{(k_1 + 1)(k_2 + 1) \dots (k_{n-1} + 1)} \cdot \frac{1}{k_n} + \\ &\quad + (-1)^n \frac{1}{(k_1 + 1)(k_2 + 1) \dots (k_n + 1)} \frac{1}{\alpha_{n+1}(x)} + \\ &\quad (-1)^{n+1} \frac{1}{(k_1 + 1) \dots (k_n + 1)(\alpha_{n+1}(x) + 1)} \frac{1}{\alpha_{n+2}(x)} + \dots \\ &= \frac{1}{k_1} - \frac{1}{(k_1 + 1)} \cdot \frac{1}{k_2} + \dots + (-1)^{n-1} \frac{1}{(k_1 + 1)(k_2 + 1) \dots (k_{n-1} + 1)} \cdot \frac{1}{k_n} + \\ &\quad + (-1)^n \frac{1}{(k_1 + 1)(k_2 + 1) \dots (k_n + 1)} \cdot \left(\frac{1}{\alpha_{n+1}(x)} - \frac{1}{(\alpha_{n+1}(x) + 1)} \cdot \frac{1}{\alpha_{n+2}(x)} + \dots \right) \end{aligned}$$

$$= \frac{1}{k_1} - \frac{1}{(k_1+1)} \cdot \frac{1}{k_2} + \dots + (-1)^{n-1} \frac{1}{(k_1+1)(k_2+1)\dots(k_{n-1}+1)} \cdot \frac{1}{k_n} + (-1)^n \frac{1}{(k_1+1)(k_2+1)\dots(k_n+1)} r_n(x),$$

where

$$r_n(x) = \frac{1}{\alpha_{n+1}(x)} - \frac{1}{(\alpha_{n+1}(x)+1)} \cdot r_{n+1}(x), \quad r_n(x) \in [0, 1], \quad n = 0, 1, 2, \dots$$

We have the following cases:

(i) $n = 2k + 1, k = 0, 1, 2, \dots$

If $r_n(x) = 0$, then

$$M_n = \sup I_{k_1 k_2 \dots k_n} = \frac{1}{k_1} - \frac{1}{(k_1+1)} \cdot \frac{1}{k_2} + \dots + (-1)^{n-1} \frac{1}{(k_1+1)(k_2+1)\dots(k_{n-1}+1)} \cdot \frac{1}{k_n}$$

while, if $r_n(x) = 1$, then

$$m_n = \inf I_{k_1 k_2 \dots k_n} = \frac{1}{k_1} - \frac{1}{(k_1+1)} \cdot \frac{1}{k_2} + \dots + \frac{1}{(k_1+1)(k_2+1)\dots(k_{n-1}+1)} \cdot \frac{1}{k_n} - \frac{1}{(k_1+1)(k_2+1)\dots(k_n+1)}.$$

(ii) $n = 2k, k = 0, 1, 2, \dots$

Using the same argument we get:

$$M_n = \sup I_{k_1 k_2 \dots k_n} = \frac{1}{k_1} - \frac{1}{(k_1+1)} \cdot \frac{1}{k_2} + \dots - \frac{1}{(k_1+1)(k_2+1)\dots(k_{n-1}+1)} \cdot \frac{1}{k_n} + \frac{1}{(k_1+1)(k_2+1)\dots(k_{n-1}+1)}$$

$$m_n = \inf I_{k_1 k_2 \dots k_n} = \frac{1}{k_1} - \frac{1}{(k_1+1)} \cdot \frac{1}{k_2} + \dots + \frac{1}{(k_1+1)(k_2+1)\dots(k_{n-1}+1)} \cdot \frac{1}{k_n}. \quad \square$$

2.2 Stochastic properties of the digits

Now, using the above results we are prepared to prove

Theorem 3 *The digits $\alpha_n, n = 1, 2, \dots$, of the expansion (1.2.1) are stochastically independent and identically distributed random variables with respect to Lebesgue measure λ , with*

$$\lambda(\alpha_n(x) = k) = \frac{1}{k+1}, \quad k = 1, 2, \dots, \text{ for any } n = 1, 2, \dots$$

Therefore we get

$$\lambda(\{x \in I : \alpha_1(x) = k_1, \dots, \alpha_n(x) = k_n\}) = \prod_{j=1}^n \lambda(\alpha_j(x) = k_j) \quad (2.2.1)$$

$$= \prod_{j=1}^n \frac{1}{k_j + 1}, \quad k_j = 1, 2, \dots, \text{ for any } j = 1, 2, \dots .$$

Proof. Firstly we prove:

$$\lambda(\alpha_j(x) = k_j) = \frac{1}{k_j + 1}, \quad k_j = 1, 2, \dots \text{ for any } j = 1, 2, \dots .$$

If $\alpha_j(x) = k_j$, then

$$r_{j-1}(x) = \frac{1}{k_j} - \frac{1}{k_j + 1} r_j(x), \quad r_j(x) \in [0, 1], \quad j \in N^* .$$

So if $r_j(x) = 0$, then $r_{j-1}(x) = \frac{1}{k_j}$. Equivalently we obtain $\sup I_{k_j} = \frac{1}{k_j}$.

If $r_j(x) = 1$, then

$$r_{j-1}(x) = \frac{1}{k_j(k_j + 1)} .$$

So $\inf I_{k_j} = \frac{1}{k_j(k_j + 1)}$.

Therefore we get

$$\lambda(\alpha_j(x) = k_j) = \lambda(I_{k_j}) = \frac{1}{k_j + 1}, \quad j = 1, 2, \dots .$$

In order to prove (2.2.1) we take the following cases:

(i) $n = 2k + 1$, $k = 0, 1, 2, \dots$.

Then

$$\lambda(\{x \in I : \alpha_1(x) = k_1, \dots, \alpha_n(x) = k_n\}) =$$

$$\lambda(I_{k_1 k_2 \dots k_n}) = M_n - m_n = \frac{1}{(k_1 + 1)(k_2 + 1) \dots (k_n + 1)} = \prod_{j=1}^n \frac{1}{k_j + 1} = \prod_{j=1}^n \lambda(\alpha_j(x) = k_j) .$$

We repeat the same argument if

(i) $n = 2k$, $k = 0, 1, 2, \dots$. □

As a consequence we will be able to apply all the classic limit theorems (Laws of Large numbers, Central limit theorems etc.) in the study of the metrical problems of the sequence of digits of expansion (1.2.1) such as the mean value of appearance of a digits, asymptotic problems Gauss-type, etc. (see [2]).

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