COMBINATORICS OF THE HYPERPLANE ARRANGEMENTS AND INTEGRABLE MODELS

V. A. Golubeva and V. P. Leksin

Abstract

The generalized differential Knizhnik-Zamolodchikov (KZ) operators associated to the root systems are considered. The Hamiltonians representable by the sum of squares of KZ operators are given in explicit form. The models having such Hamiltonians appears as integrable ones. The ground and excited states of such Hamiltonians are the solutions of the generalized KZ equations. A method of analyzing the asymptotical behavior of the ground states based on the associators of Drinfeld and on blowing-ups of the divisor singularities is given.

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The integrability of the models of non-relativistic quantum mechanics with potentials of the Calogero–Suthetland–Moser (CSM) type is usually proved following to one of two schemes presented in [2] and [3]: either by means of calculating the complete system of the integrals of the model as traces of the degrees of L-matrices in (L, A) Lax pair (see [1]), or by means of the construction of the commutative set of differential-difference Dunkl operators ∇_i, $i = 1, \ldots, n$. The restriction of the operators and their degrees to the space of symmetric or skew-symmetric functions with respect to the action of finite group for which the Dunkl operators are constructed turns the mentioned operators into the scalar differential operators.Then the proof of the integrability of the quantum problem with Hamiltonian

$$\sum_{i=1}^{n} \nabla_i^2 \tag{1}$$

is reduced to the consideration of the functionally independent differential operators that are obtained by the substitution of the operators ∇_i to the first n symmetrical polynomials on n variables.

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The scheme given in the paper, of constructing the integrable models associated to the root systems, differs from the above mentioned. We expose it briefly.

Consider the linear integrable Pfaffian system

$$d\Psi = \Omega\Psi, \tag{2}$$

where

$$\Omega = \sum_{i=1}^{n} \Omega_i(z) dz_i,$$

and the function $\Psi(z)$, $z \in \mathbf{C^n}$, which takes its values in the linear space V on which the coefficients $\Omega_i(z)$ of the 1-form Ω act as linear operators. It is supposed that the 1-form Ω satisfies the complete integrability condition $d\Omega = \Omega \wedge \Omega$. A various set of examples of this kind are known till now (see [3]–[7]). In [3]–[7] it is shown that the set of differential operators of the form

$$D_i = \partial - \Omega_i, i = 1, \dots, n,$$

is commutative one. Let us construct the Hamiltonian

$$H = \sum_{i=1}^{n} D_i^2. \tag{3}$$

The proof of the integrability of the corresponding quantum problem is performed using the same scheme as above: the operators D_i are substituted in the first n symmetrical polynomials of n variables, which give the Hamiltonian (3) and higher Hamiltonians. In our opinion it is an interesting problem to find the physical models corresponding to such Hamiltonians.

Unfortunately none of the known Hamiltonians, except (1), has been investigated starting from its representativity in the form (3). But it is known that any Hamiltonian CSM associated to the root system can be presented in the following form:

$$H = \sum_{i=1}^{n} D_i^* D_i,$$

where D_i^* is the formal adjoint operator that often can be really adjoint with respect to some scalar product (see [8], [9]). Using this representation, the operators D_i^* and D_i do not commute, but the operators $D_i^* D_i$ commute. It implies the existence of the integrals of even degree for the quantum many-body problem.

The paper consists of two parts. The first part contains the scheme of calculation of the Hamiltonians H in the form (3) for a sufficiently general class of KZ operators associated to the root systems. Some nontrivial examples are given. The second part is devoted to the local investigation of the eigenstates of the Hamiltonians considered above. This problem is equivalent to the analysis of the generalized KZ equations. An example of the root system A_3 is considered. In this case, using the changes of variables of blow-up type (used firstly by Cherednik [10]), it is proved that the KZ system is reduced to the systems of hypergeometric type.

1 r-Matrices, generalized Knizhnik–Zamolodchikov equations associated with root systems and corresponding Hamiltonians

Generalized KZ equations associated with different root systems was initially introduced by I. Cherednik [4] and then investigated by A. Matsuo ([3]), A. Liebman ([11]), V. Golubeva and V. Leksin ([6]), Concini and Procesi ([12, 13]). These equations are closely related with the classical r-matrices corresponding to the root systems. Initially, in the papers of Takhtadjan and Faddeev, the r-matrices were appeared as an unification tool for proving the complete integrability of different quantum models (see [14] and the references therein). Generalized KZ equation is the linear Pfaffian system (2), where the 1-form Ω has the form

$$\Omega = \sum_{\alpha \in R_+} r_\alpha(z)\, d\alpha(z),$$

R_+ is a positive root system and $r_\alpha(z)$ is defined by the following conditions (Cherednik):

1.

$$r_\alpha(z) = \frac{k_\alpha t_\alpha}{\alpha(z)} + b_\alpha(\alpha(z)),$$

where b_α is an holomorphic function, k_α is an arbitrary W-invariant set of constants and t_α is a W-invariant set of operators in the space of values of Ψ.

2. ${}^w r_{\alpha(z)} = w^{-1} r_\alpha(\alpha(w^{-1}z))w = r_{w\alpha}(z)$.

3. $r_\alpha + r_{-\alpha} = 0$.

4. The classical Yang–Baxter equations are satisfied:

$$\sum_{\alpha,\beta \in L\cap R_+} [r_\alpha, r_\beta] = 0,$$

where $L \subset \mathbf{R^n}$ is an arbitrary two-dimensional subspace in $\mathbf{R^n}$.

Let us construct a Hamiltonian for KZ-operators. Consider the KZ-operator (2) associated with a reduced irreducible root system R, as follows. The r-matrices have the form

$$r_\alpha(z) = \frac{k_\alpha}{\alpha(z)}\sigma_\alpha + e_\alpha(\lambda), \tag{4}$$

where σ_α is the multiplication operator in the group algebra $C[W(R)]$ using $s_\alpha \in W(R)$, the function $\Psi(z)$ takes values in $C[W(R)]$, and the operator $e_\alpha(\lambda)$ acts on the base element w in $C[W(R)]$ by the rule:

$$e_\alpha(\lambda)[w] = (w\lambda, \alpha)[w], \tag{5}$$

where $(,)$ is the natural W-invariant bilinear form in $\mathbf{C^n}$.

Similarly, we can consider the case of trigonometric r-matrices

$$r_\alpha(z) = k_\alpha(\coth(\alpha(z)) \cdot \sigma_\alpha + \sigma_\alpha \varepsilon_\alpha) + e(\lambda), \tag{6}$$

where

$$\varepsilon_\alpha(w) = \begin{cases} 1, & \text{if } w\alpha \in R_+, \\ -1, & \text{if } w\alpha \notin R_+, \end{cases}$$

and R_+ is the positive root system.

The following assertions hold.

Proposition 1 *In the rational case, the Hamiltonian*

$$H = \sum_{i=1}^{n} D_i^2$$

takes the form

$$H = \Delta + \sum_{\alpha\in R_+} \frac{(\alpha,\alpha)k_\alpha(k_\alpha - \sigma_\alpha)}{(\alpha,z)^2} - 2\sum_{\alpha\in R_+} \frac{k_\alpha}{(\alpha,z)}\sigma_\alpha \cdot \partial_\alpha + e(\partial_\lambda) + (\lambda,\lambda), \qquad (7)$$

where the action of $e(\partial_\lambda)$ is defined by the formula:

$$e(\partial_\lambda)\Psi(z) = e(\partial_\lambda)\Bigg(\sum_{w\in W(R)} \Psi_w(z)[w]\Bigg) = \sum_{w\in W(R)} \partial_{w\lambda}\Psi_w(z)\cdot[w].$$

Proof. We have:

$$H = \sum_{i=1}^{n} D_i^2 = \sum(\partial_i - \sum_{\alpha\in R_+} \frac{k_\alpha(\alpha,e_i)}{(\alpha,z)}\sigma_\alpha - e_i(\lambda))^2 =$$

$$= \sum_{i=1}^{n}(\partial_i^2 - \partial_i\circ\Omega_i - \Omega_i\circ\partial_i + \Omega_i^2)$$

$$= \Delta - \sum_{i=1}^{n}(\partial_i(\Omega_i) - \Omega_i^2) - 2\sum\Omega_i\circ\partial_i.$$

Further, we have:

$$\sum_{i=1}^{n}\Omega_i\circ\partial_i = \sum_{i=1}^{n}\frac{k_\alpha(\alpha,e_i)}{(\alpha,z)}\sigma_\alpha\partial_i + \sum_{i=1}^{n} e_i(\lambda)\partial_i,$$

$$= \sum_{\alpha\in R_+}\frac{k_\alpha^2\cdot\sigma_\alpha}{(\alpha,z)}\partial_\alpha,$$

$$\partial_i(\Omega_i) = -\sum_{i=1}^{n}\sum_{\alpha\in R_+}\frac{k_\alpha(\alpha,e_i)(\alpha,e_i)}{(\alpha,z)^2}\sigma_\alpha = -\sum\frac{k_\alpha(\alpha,\alpha)}{(\alpha,z)^2}\sigma_\alpha.$$

Since $\sum_{i=1}^n (\alpha, e_i)(e_i, \alpha) = (\alpha, \alpha)$, then

$$\sum_{i=1}^n \Omega_i^2 = \sum_{i=1}^n \sum_{\alpha \in R_+} k_a^2 \frac{(\alpha, e_i)(e_i, \alpha)}{(\alpha, z)^2} \sigma_\alpha^2 +$$

$$\sum_{i=1}^n \sum_{\alpha, \beta \in R_+} \frac{k_\alpha k_\beta (\alpha, e_i)(e_i, \beta)}{(\alpha, z)(\beta, z)} \sigma_\alpha \sigma_\beta + \sum_{i=1}^n e_i^2(\lambda) +$$

$$+ \Big(\sum_{i=1}^n e_i(\lambda) \sum_{\alpha \in R_+} \frac{k_\alpha (\alpha, e_i)}{(a, z)} + \sum_{i=1}^n \sum_{\alpha \in R_+} \frac{k_\alpha (\alpha, e_i)}{(\alpha, z)} \sigma_\alpha e_i(\lambda) \Big).$$

Using the Dunkl identity it follows that the second term is equal to zero ; the forth term is also equal to 0. The third term coincides with the constant scalar operator (λ, λ), i.e. $\sum_{i=1}^n e_i^2(\lambda) = (\lambda, \lambda)$. □

The following Proposition can be proved similarly by means of the trigonometric Dunkl identity.

Proposition 2 *In the trigonometric case, the Hamiltonian* $H = \sum_{i=1}^n D_i^2$ *takes the form*

$$H = \Delta - \sum_{\alpha \in R_+} \frac{(\alpha, \alpha) k_\alpha (k_\alpha - \sigma_\alpha)}{\sinh^2(\alpha, z)} - 2 \sum_{\alpha \in R_+} k_\alpha [\coth(\alpha, z) \sigma_\alpha + \sigma_\alpha \varepsilon_\alpha] \cdot \partial_\alpha + e(\partial_\lambda) + (\lambda, \lambda), \tag{8}$$

where k *are some constant functions on the orbits of the Weyl group.*

It is interesting to point out physical models which can be described by these Hamiltonians.

2 Asymptotical analysis of ground and excited states

We consider the generalized equations of the form (2) with the 1-form Ω given by

$$\Omega = \sum_{\alpha \in R_+} r_\alpha(z) d\alpha(z), \tag{9}$$

where the operators r_α have the form

$$r_\alpha(z) = \frac{k_\alpha \cdot t_\alpha}{\alpha(z)} + b_\alpha(\alpha(z)), \tag{10}$$

where $b_\alpha(u)$, $\alpha \in R_+$, are holomorphic functions of one variable u. Let us suppose that the W-invariant set of constants $\{k_\alpha\}_{\alpha \in \mathbf{R}_+}$, $k_{w\alpha} = k_\alpha$ is generic. It means that that the operators $k_\alpha t_\alpha$ and $\sum_{\alpha \in R_+} k_\alpha t_\alpha$ have all the eigenvalues as integers. The following principal theorem gives the structure of the fundamental solution of the generalized KZ system (2), (4) and (5) in a neighbourhood of the origin $z_1 = \ldots = z_n = 0$.

Theorem 1 *For any W-invariant generic set of complex numbers k_α, $\alpha \in R$, the fundamental solution of the generalized KZ system (2), (9) and (10) can be represented in some neighbourhood of the origin in the form*

$$\Psi = FG, \tag{11}$$

where G is the solution of a KZ equation of the form

$$dG = \left(\sum_{\alpha \in R_+} \frac{k_\alpha \cdot t_\alpha}{\alpha(z)} d(\alpha(z)) \right) G, \tag{12}$$

and F is an holomorphic function in this neighborhood.

The behavior of G in a neighbourhood of the origin is characterized by the following set of combinatorial data:

1) A polyhedron K_n, where each vertex A is marked by an oriented tree S_A such that its nodes are associated with some simple chosen roots from the set of simple roots $\Delta = \{a_1, \dots, \alpha_n\} \in R$. It is also required, that any connected part of the Dynkin diagram has a maximal element with respect to the partial order in the set $\Delta = \{a_1, \dots, \alpha_n\}$ defined by the tree S_A.

2) To every one-dimensional edge of the polyhedron K_n which connects the vertices (A, B) there is related the generalized associator of Drinfeld [15] ϕ_{AB}, i.e. a function of some $t'_\alpha s$, $\alpha \in R_+$ which do not depend of z. The associator $(A, B) \to \phi_{AB}$ defines a cocycle of a one-dimensional skeleton of K_n.

3) A vertex $A \in K_n$ is associated with a set of variables, as follows: $u_i^A = \dfrac{\alpha_i(z)}{\alpha_{i'}(z)}$. if α_i is not a maximal element with respect to the order defined by the tree S_A, where $\alpha_{i'}$ means that it proceed to the element α_i and $u_i^A = \alpha_i(z)$ if α_i is a maximal root in S_A.

We associate with every vertex A and with every variable u_i the set of operators t_A^i defined by the tree combinatorial sum of the elements t_{α_j}, $j = 1, \dots, n$.

For these combinatorial data and the variables u_i the solution G can be represented in the form

$$G = H(u_1, \dots, u_n) \cdot \prod_{i=1}^{n} u_i^{t_A^i}, \tag{13}$$

where $H(u_1, \dots, u_n)$ is an holomorphic and holomorphic invertible function in a neighbourhood of the origin.

Proof. The representation (11) follows from theorem of Takano using the generic condition for k_α (non resonance case) (see [16]). The polyhedron K_n is constructed from the scheme exposed in [12] and [13], using the blow-up processes for the hyperplane arrangements $(\alpha, z) = 0$, $\alpha \in R_+$. The method of marking the vertices was given by Cherednik and was generalized by Concini and Procesi. □

The representation of the solution of the equation (12) in the form (13) follows from the generity condition for k_α and from the general theorem of Bolibruch concerning the structure of the fundamental matrix of a Fuchsian system in the neighbourhood of the point of a normal intersection of the singular divisors ([17]; see also [18]).

Remark 1 Every neighbourhood of the point $u_i^A = 0, \quad i = 1, \ldots, n$, corresponds to a conic neighbourhood of the maximal root of the tree, associated to the vertex A. Counting the action of the Weyl group, these conic neighborhoods cover the whole neighborhood of the origin in the original space. The combinatorial analysis described above corresponds to the investigation of the solution in different directions in a neighbourhood of the origin.

Remark 2 The polyhedron has the property that there exists a map

$$\mu_{nm} : K_n \times K_m \to K_{n+m-1}.$$

On vertices this map is constructed easily by means of the engrafting operation for the trees.

The system of maps μ_{nm} is equivalent in some sense to the structure called *operad* (see [19]).

3 Examples

Consider the KZ equation associated with the root system A_3 and the reduction of the Fuchsian system with singular divisors, such that the divisors of the hypergeometric functions are those of Appell and Kampé de Feriet, F_2 and F_3. Notice that some examples of KZ equations corresponding to the two-dimensional root systems A_2, B_2 and G_2 were considered in the papers [12] and [13].

Since all roots A_3 have the same length (we have $k_{w\alpha} = k_\alpha = k$ for all $\alpha \in R$), the KZ system can be written using the Pfaffian form Ω in the following form:

$$\frac{1}{k}\Omega = t_{12}\frac{d(z_1 - z_2)}{z_1 - z_2} + t_{23}\frac{d(z_2 - z_3)}{z_2 - z_3}$$

$$+t_{34}\frac{d(z_3 - z_4)}{z_3 - z_4} + t_{13}\frac{d(z_1 - z_3)}{z_1 - z_3}$$

$$+t_{14}\frac{d(z_1 - z_4)}{z_1 - z_4} + t_{24}\frac{d(z_2 - z_4)}{z_2 - z_4}.$$

Different trees which satisfy the conditions of the theorem and correspond to the set of simple roots

$$\alpha_1 = z_1 - z_2, \alpha_2 = z_2 - z_3, \alpha_3 = z_3 - z_4$$

have one of the following types:

1. Ordering $\alpha_1, \alpha_2, \alpha_3$ such that the corresponding tree has the following form:

$$\alpha_{i_1} \to \alpha_{i_2} \to \alpha_{i_3},$$

where (i_1, i_2, i_3) is a permutation of the numbers (1,2,3). Let us denote it as Γ_{i_1,i_2,i_3}.

2. A tree

$$\begin{array}{ccc} & & \alpha_2 \\ & \nearrow & \\ \alpha_1 & & \\ & \searrow & \\ & & \alpha_3 \end{array}$$

is denoted by Γ.

For example, for Γ_{i_1,i_2,i_3} the change of variables has the form

$$u_1 = z_1 - z_2, u_2 = \frac{z_2 - z_3}{z_1 - z_2}, u_3 = \frac{z_3 - z_4}{z_2 - z_3},$$

that is

$$\alpha_1 = u_1, \quad \alpha_2 = u_1 u_2, \quad \alpha_3 = u_1 u_2 u_3.$$

The other roots are expressed in this case by the formulas

$$\alpha_{13} = \alpha_1 + \alpha_2 = u_1 + u_1 u_2 = u_1(1 + u_2),$$

$$\alpha_{14} = \alpha_1 + \alpha_2 + \alpha_3 = u_1 + u_1 u_2 + u_1 u_2 u_3 = u_1(1 + u_2 + u_2 u_3),$$

$$\alpha_{24} = \alpha_2 + \alpha_3 = u_1 u_2 + u_1 u_2 u_3 = u_1 u_2 (1 + u_3),$$

where $\alpha_{ij} = z_i - z_j$. The variables $(z_1, z_2, z_3) \to (u_1, u_2 u_3)$ gives the resolution of variables of the hyperplane arrangement

$$\bigcup_{i<j} (z_i - z_j) = 0.$$

Consider the transformation of the form Ω under such change of variables. We have:

$$\frac{1}{k}\Omega_{A_3} = (t_{12} + t_{13} + t_{14} + t_{23} + t_{24} + t_{34})\frac{du_1}{u_1}$$

$$+(t_{23} + t_{24} + t_{34})\frac{du_2}{u_2} + t_{34}\frac{d(1+u_2)}{1+u_2}$$

$$+t_{24}\frac{d(1+u_3)}{1+u_3} + t_{14}\frac{d(1+u_2+u_2u_3)}{1+u_2+u_2u_3}.$$

The singular divisor of this form coincides (up to the change of signs of variables) with the projective singular divisor of the Fuchsian system of the hypergeometric function F_3. In small neighborhood of the origin $u_1 = u_2 = u_3 = 0$ the form Ω has the singular divisor with normal intersections. Using the conditions of complete integrability of the form Ω it follows that the operator $\tau = \sum_{i<j} t_{ij}$ commute with all the operators t_{ij} and the gauge transformation of the form Ω using the function u_1^τ reduces it to the form

$$\frac{1}{k}\Omega_{A_3} = (t_{23} + t_{24} + t_{34})\frac{du_2}{u_2} + t_{34}\frac{d(1+u_2)}{1+u_2}$$

$$+t_{24}\frac{d(1+u_3)}{1+u_3}+t_{14}\frac{d(1+u_2+u_2u_3)}{1+u_2+u_2u_3}.$$

The singularities of this form (by means of the change of the sign of the variables) are reduced to the singular divisor of the hypergeometric function F_3. The system which corresponds to the tree Γ_{321} has the same form as the system for Γ_{123}. For the other types of singularities (for example, for Γ_{i_1,i_2,i_3}) the systems are reduced to some with the same type of singularities. The singular divisors are obtained simply by means of the change of signs of variables or by the permutation of the components of the divisor. It is easy to see that by a fractional-linear change this divisor is reduced to the divisor of the hypergeometric function F_3.

For the tree Γ the change

$$u_2=\alpha_2(z),\quad u_1=\frac{\alpha_1(z)}{\alpha_2(z)}=\frac{z_1-z_2}{z_2-z_3},\qquad u_3=\frac{\alpha_3(z)}{\alpha_2(z)}=\frac{z_3-z_4}{z_2-z_3}$$

reduces the form Ω_{A_3} to

$$\frac{1}{k}\Omega_{A_3}=\tau\frac{du_2}{u_2}+t_{12}\frac{du_1}{u_1}+t_{34}\frac{du_3}{u_3}$$

$$+t_{13}\frac{d(u_1+1)}{u_1+1}+t_{24}\frac{d(u_3+1)}{u_3+1}+t_{14}\frac{d(u_1+u_3+1)}{u_1+u_3+1},$$

where $\tau=\sum_{i<j}t_{ij}$. The gauge transformation by means of the function u_2^τ gives for Ω:

$$\frac{1}{k}\Omega_{A_3}=t_{12}\frac{du_1}{u_1}+t_{34}\frac{du_3}{u_3}$$

$$+t_{13}\frac{d(u_1+1)}{u_1+1}+t_{24}\frac{d(u_3+1)}{u_3+1}+t_{14}\frac{d(u_1+u_3+1)}{u_1+u_3+1}.$$

Up to change of signs of variables, the singular divisor of this form coincides with the singularity divisor of the hypergeometric function F_2. The local behavior in a neighborhood of the origin of the solutions of the reduced system of equations $d\Psi=\tilde{\Omega}\Psi$ is defined by the general theory of multidimensional Fuchsian systems (see [17], [18]). The other properties of these solutions can be derived from the theory of hypergeometric functions of many variables. The Drinfeld's associators connect the solutions corresponding to the different trees Γ_{i_1,i_2,i_3} using the permutation of the indices. A similar situation occurs in the case of the tree Γ.

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