THE GEOMETRY OF RELATIVISTIC RHEONOMIC LAGRANGE SPACES

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Abstract

In this paper we shall present a geometrization of time-dependent Lagrangians. The reader is invited to compare this geometrization with that contained in the book of Miron and Anastasiei [11]. In this direction, Section 1 describes the main geometrical aspects of the 1-jet space $J^1(R, M)$, in the sense of d-tensors, d-connections, d-torsions and d-curvatures. Section 2 introduces the notion of *Relativistic Rheonomic Lagrange Space*, which naturally generalizes that of *Classical Rheonomic Lagrange Space* [11], and constructs its canonical nonlinear connection Γ as well as its Cartan canonical Γ-linear connection. From a physical point of view, our time-dependent Lagrange geometry gives a relativistic model for both gravitational and electromagnetic field. The relativistic rheonomic Lagrangian electromagnetism together with its Maxwell equations are presented in Section 4. Section 5 describes the Einstein's gravitational field equations of a relativistic rheonomic Lagrange space.

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1 The geometry of $J^1(R, M)$

1.1 Some physical aspects

Let us consider the usual time axis represented by the set of real numbers R and a real, smooth and n-dimensional manifold M that we regard like a *"spatial"* manifold [16]. We suppose that the temporal manifold R is coordinated by t while the spatial manifold M is coordinated by $(x^i)_{i=\overline{1,n}}$. Note that, throughout this paper, the latin letters $i, j, k \ldots$ run from 1 to n.

Let $J^1(R, M) \equiv R \times TM$ be the usual 1-jet vector bundle over the product manifold base $R \times M$. The local coordinates on $J^1(R, M)$ are denoted (t, x^i, y^i). From a physical point of view, the fibre bundle

$$J^1(R, M) \to R \times M, \quad (t, x^i, y^i) \to (t, x^i), \qquad (1.1.1)$$

is regarded like a *bundle of configurations*, in mechanics terms. The gauge group of this bundle of configurations is

$$
\begin{cases}
\tilde{t} = \tilde{t}(t) \\
\tilde{x}^i = \tilde{x}^i(x^j) \\
\tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j} \dfrac{dt}{d\tilde{t}} y^j.
\end{cases}
\qquad (1.1.2)
$$

We remark that the form of the gauge group 1.1.2 stands out by the *relativistic* character of the time coordinate t. The above reasons determine us to consider the jet vector bundle of order one $J^1(R, M)$ like a natural house for a subsequent development of a *relativistic rheonomic Lagrangian mechanics*.

Comparatively, the configuration bundle used in the *classical rheonomic Lagrangian mechanics* [11], is the vector bundle

$$\pi : R \times TM \to M, \ (t, x^i, y^i) \to (x^i), \qquad (1.1.3)$$

whose geometrical invariance group is

$$
\begin{cases}
\tilde{t} = t \\
\tilde{x}^i = \tilde{x}^i(x^j) \\
\tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j} y^j.
\end{cases}
\qquad (1.1.4)
$$

The structure of the gauge group 1.1.4 emphasizes the *absolute* character of the time coordinate t from the classical rheonomic Lagrangian mechanics. At the same time, we point out that the gauge group 1.1.4 is a subgroup of 1.1.2.

Like a conclusion, we underline that the gauge group of the jet bundle of order one (i. e., the configuration bundle of the relativistic rheonomic Lagrangian mechanics) is more general than that of the configuration bundle of the classical rheonomic Lagrangian mechanics, which ignores the temporal reparametrizations. Consequently, this will be an essential difference between our subsequent time-dependent Lagrangian geometrization and that constructed by Miron and Anastasiei.

Finally, we emphasize that the main physical aspects of Miron-Anastasiei's time-dependent Lagrangian geometry are presented by Ikeda in [6] and [12]. From our point of view, these physical interpretations can be easily and naturally extended to the relativistic approach of the rheonomic Lagrangian geometry that we develope in this paper.

1.2 Time-dependent sprays. Harmonic curves

Let us consider that the temporal manifold R is endowed with a semi-Riemannian metric $h = (h_{11}(t))$. In order to develope the geometrical background of the relativistic rheonomic Lagrangian mechanics on the 1-jet fibre bundle $E = J^1(R, M)$, we will

introduce a collection of important geometrical concepts. An important geometrical concept on $J^1(R, M)$ is that of time-dependent spray, which naturally generalizes the notion of time-dependent spray on $R \times M$, used in [11] and [22]. To introduce this concept, let us consider the following notions:

Definition 1.2.1 A global tensor H (resp. G) on E, locally expressed by

$$H = dt \otimes \frac{\partial}{\partial t} - 2H_{(1)1}^{(j)} dt \otimes \frac{\partial}{\partial y^j}, \qquad (1.2.1)$$

respectively

$$G = y^j dt \otimes \frac{\partial}{\partial x^j} - 2G_{(1)1}^{(j)} dt \otimes \frac{\partial}{\partial y^j} \qquad (1.2.2)$$

is called a *temporal* (resp. *spatial*) *spray* on E.

Because the sprays H and G are global tensors, using the coordinate transformations 1.1.2 of the 1-jet space E, it is easy to deduce the following [16]

Theorem 1.2.1 *To give a temporal (spatial) spray on E is equivalent to give a set of local functions $H = (H_{(1)1}^{(j)})$ (resp. $G = (G_{(1)1}^{(j)})$) which transform by the rules*

$$2\tilde{H}_{(1)1}^{(k)} = 2H_{(1)1}^{(j)} \left(\frac{dt}{d\tilde{t}}\right)^2 \frac{\partial \tilde{x}^k}{\partial x^j} - \frac{dt}{d\tilde{t}} \frac{\partial \tilde{y}^k}{\partial t}, \qquad (1.2.3)$$

respectively

$$2\tilde{G}_{(1)1}^{(k)} = 2G_{(1)1}^{(j)} \left(\frac{dt}{d\tilde{t}}\right)^2 \frac{\partial \tilde{x}^k}{\partial x^j} - \frac{\partial x^i}{\partial \tilde{x}^j} \frac{\partial \tilde{y}^k}{\partial x^i} \tilde{y}^j. \qquad (1.2.4)$$

The previous theorem allows us to offer the following important examples of temporal and spatial sprays. The importance of these sprays comes from their using in the description of the local equations of harmonic maps between two semi-Riemannian manifolds [4].

Example 1.2.1 Let $h = (h_{11})$ (resp. $\varphi = (\varphi_{ij})$) be a semi-Riemannian metric on R (resp. M) and H_{11}^1 (resp. γ_{jk}^i) its Christoffel symbols. In this context, taking into account the transformation rules of the Christoffel symbols H_{11}^1 and γ_{jk}^i, we deduce that the components $2H_{(1)1}^{(j)} = -H_{11}^1 y^j$ (resp. $2G_{(1)1}^{(j)} = \gamma_{kl}^j y^k y^l$) represent a temporal (resp. spatial) spray which is called the *canonical temporal* (resp. *spatial*) *spray associated to the metric h* (resp. φ).

Definition 1.2.2 A pair (H, G), which consists of a temporal spray and a spatial one, is called a *time-dependent spray on $J^1(R, M)$*.

Following the geometrical development of the classical rheonomic Lagrange mechanics, we introduce a natural generalization of the notion of path of a time-dependent spray, used in [11].

Definition 1.2.3 A curve $c \in C^{\infty}(R, M)$ is called a *harmonic curve of the time-dependent spray* (H, G) on $J^1(R, M)$, *with respect to the semi-Riemannian temporal metric* $h = (h_{11}(t))$ *on* R, if c is a solution of the DEs system of order two

$$h^{11} \left\{ \frac{d^2 x^i}{dt^2} + 2G_{(1)1}^{(i)} + 2H_{(1)1}^{(i)} \right\} = 0, \tag{1.2.5}$$

where $h^{11} h_{11} = 1$ and the curve c is locally expressed by $R \ni t \to (x^i(t))_{i=\overline{1,n}} \in M$.

Remarks 1.2.1 i) Under the coordinate transformations of $J^1(R, M)$, the left term of the equations 1.2.5 modifies like a d-tensor, that is,

$$\left[h^{11} \left\{ \frac{d^2 x^i}{dt^2} + 2G_{(1)1}^{(i)} + 2H_{(1)1}^{(i)} \right\} \right] = \frac{\partial x^i}{\partial \tilde{x}^j} \left[\tilde{h}^{11} \left\{ \frac{d^2 \tilde{x}^j}{d\tilde{t}^2} + 2\tilde{G}_{(1)1}^{(j)} + 2\tilde{H}_{(1)1}^{(j)} \right\} \right]. \tag{1.2.6}$$

Consequently, the equations 1.2.5 are global on $J^1(R, M) \equiv R \times TM$ (i. e. their geometrical invariance group is 1.1.2).

ii) Comparatively, the equations of a path on $R \times TM$ (see [11]), that we generalized by 1.2.5, are invariant only under the gauge group 1.1.4.

Example 1.2.2 Let us consider the canonical sprays asociated to the metrics h and φ, which are locally expressed by

$$H_{(1)1}^{(i)} = -\frac{1}{2} H_{11}^1 y^i, \quad G_{(1)1}^{(i)} = \frac{1}{2} \gamma_{jk}^i y^j y^k. \tag{1.2.7}$$

The equations of the harmonic curves attached to these sprays, with respect to the semi-Riemannian temporal metric h, reduce to

$$h^{11} \left\{ \frac{d^2 x^i}{dt^2} - H_{11}^1 \frac{dx^i}{dt} + \gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} \right\} = 0, \tag{1.2.8}$$

that is, exactly the equations whose solutions are the well known classical harmonic maps between the semi-Riemannian manifolds (R, h) and (M, φ) [4]. Particularly, if we regard the temporal manifold R endowed with the euclidian metric $h = \delta$, we recover the classical equations of geodesics of the semi-Riemannian manifold M. These facts emphasize the naturalness of our previous definition.

1.3 Nonlinear connections. Adapted bases.

It is well known the importance of the nonlinear connections in the study of the geometry of a fibre bundle E. A nonlinear connection (i. e. a supplementary horizontal distribution of the vertical distribution of E) offers the possibility of construction of the *vector* or *covector adapted bases*. These allow a more simple description of the geometrical objects or properties of the total space E. In this sense, considering our particular case $E = J^1(R, M)$, we proved in [16]

Theorem 1.3.1 *A nonlinear connection* Γ *on the jet fibre bundle of order one* E *is determined by a pair of local function sets* $M^{(i)}_{(1)1}$ *and* $N^{(i)}_{(1)j}$ *which modify by the transformation laws*

$$\tilde{M}^{(j)}_{(1)1} \frac{d\tilde{t}}{dt} = M^{(k)}_{(1)1} \frac{dt}{d\tilde{t}} \frac{\partial \tilde{x}^j}{\partial x^k} - \frac{\partial \tilde{y}^j}{\partial t}, \tag{1.3.1}$$

$$\tilde{N}^{(j)}_{(1)k} \frac{\partial \tilde{x}^k}{\partial x^i} = N^{(k)}_{(1)i} \frac{dt}{d\tilde{t}} \frac{\partial \tilde{x}^j}{\partial x^k} - \frac{\partial \tilde{y}^j}{\partial x^i}. \tag{1.3.2}$$

Definition 1.3.1 A set of local functions $M^{(i)}_{(1)1}$ (resp. $N^{(i)}_{(1)j}$) on $J^1(R, M)$, which transform by the rules 1.3.1 (resp. 1.3.2) is called a *temporal nonlinear connection* (resp. *spatial nonlinear connection*) on $E = J^1(R, M)$.

Example 1.3.1 Studying the transformation rules of the local components

$$M^{(i)}_{(1)1} = -H^1_{11} y^i, \quad N^{(i)}_{(1)j} = \gamma^i_{jk} y^k, \tag{1.3.3}$$

where H^1_{11} (resp. γ^i_{jk}) are the Christoffel symbols of a temporal (resp. spatial) semi-Riemannian metric h (resp. φ), we conclude that $\Gamma_0 = (M^{(i)}_{(1)1}, N^{(i)}_{(1)j})$ represents a nonlinear connection on E, which is called the *canonical nonlinear connection attached to the metric pair* (h, φ).

Taking into account the transformation laws 1.2.3, 1.2.4 and 1.3.1, 1.3.2, we deduce without difficulties that the notion of temporal (resp. spatial) spray is intimately connected to the notion of temporal (resp. spatial) nonlinear connection.

Theorem 1.3.2 *i) If* $M^{(i)}_{(1)1}$ *are the components of a temporal nonlinear connection, then the components*

$$H^{(i)}_{(1)1} = \frac{1}{2} M^{(i)}_{(1)1} \tag{1.3.4}$$

represent a temporal spray.

ii) Conversely, if $H^{(i)}_{(1)1}$ *are the components of a temporal spray, then*

$$M^{(i)}_{(1)1} = 2H^{(i)}_{(1)1} \tag{1.3.5}$$

are the components of a temporal nonlinear connection.

Theorem 1.3.3 *i) If* $G^{(i)}_{(1)1}$ *are the components of a spatial spray, then the components*

$$N^{(i)}_{(1)j} = \frac{\partial G^i_{(1)1}}{\partial y^j} \tag{1.3.6}$$

represent a spatial nonlinear connection.

ii) Conversely, the spatial nonlinear connection $N^{(i)}_{(1)j}$ *induces the spatial spray*

$$2G^{(i)}_{(1)1} = N^{(i)}_{(1)j} y^j. \tag{1.3.7}$$

Remark 1.3.1 The previous theorems allow us to conclude that a time-dependent spray (H, G) induces naturally a nonlinear connection Γ on E, which is called the *canonical nonlinear connection associated to the time-dependent spray* (H, G). We point out that the canonical nonlinear connection Γ attached to the time-dependent spray (H, G) is a natural generalization of the canonical nonlinear connection N induced by a time-dependent spray G from the classical rheonomic Lagrangian geometry contained in [11].

Let $\Gamma = (M_{(1)1}^{(i)}, N_{(1)j}^{(i)})$ be a nonlinear connection on the 1-jet fibre bundle E. At the same time, let us consider the geometrical objects,

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} - M_{(1)1}^{(j)} \frac{\partial}{\partial y^j},$$

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_{(1)i}^{(j)} \frac{\partial}{\partial y^j}, \qquad (1.3.8)$$

$$\delta y^i = dy^i + M_{(1)1}^{(i)} dt + N_{(1)j}^{(i)} dx^j.$$

It is easy to deduce that the set of vector fields $\left\{ \dfrac{\delta}{\delta t}, \dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i} \right\} \subset \mathcal{X}(E)$ and of covector fields $\{dt, dx^i, \delta y^i\} \subset \mathcal{X}^*(E)$ are dual bases on E.

Definition 1.3.2 The basis $\left\{ \dfrac{\delta}{\delta t}, \dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i} \right\} \subset \mathcal{X}(E)$ and its dual basis $\{dt, dx^i, \delta y^i\} \subset \mathcal{X}^*(E)$ are called the *adapted bases* on E, determined by the nonlinear connection Γ.

The big advantage of the adapted bases is that the transformation laws of its elements are simple and natural.

Proposition 1.3.4 *The transformation laws of the elements of the adapted bases attached to the nonlinear connection* Γ *are*

$$\begin{cases} \dfrac{\delta}{\delta t} = \dfrac{d\tilde{t}}{dt} \dfrac{\delta}{\delta \tilde{t}} \\[2mm] \dfrac{\delta}{\delta x^i} = \dfrac{\partial \tilde{x}^j}{\partial x^i} \dfrac{\delta}{\delta \tilde{x}^j} \\[2mm] \dfrac{\partial}{\partial y^i} = \dfrac{\partial \tilde{x}^j}{\partial x^i} \dfrac{dt}{d\tilde{t}} \dfrac{\delta}{\delta \tilde{y}^j}, \end{cases} \qquad (1.3.9)$$

$$\begin{cases} dt = \dfrac{dt}{d\tilde{t}} d\tilde{t} \\[2mm] dx^i = \dfrac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j \\[2mm] \delta y^i = \dfrac{\partial x^i}{\partial \tilde{x}^j} \dfrac{d\tilde{t}}{dt} \delta \tilde{y}^j. \end{cases} \qquad (1.3.10)$$

Remark 1.3.2 The simple transformation rules 1.3.9 and 1.3.10 determine us to describe the objects with geometrical and physical meaning from our subsequent theory, in adapted components.

1.4   Γ-linear connections

In order to develope the theory of Γ-linear connections on the 1-jet space E, we need the following

Proposition 1.4.1 *i) The Lie algebra $\mathcal{X}(E)$ of vector fields decomposes as*

$$\mathcal{X}(E) = \mathcal{X}(\mathcal{H}_T) \oplus \mathcal{X}(\mathcal{H}_M) \oplus \mathcal{X}(\mathcal{V}),$$

where

$$\mathcal{X}(\mathcal{H}_T) = Span\left\{\frac{\delta}{\delta t}\right\}, \quad \mathcal{X}(\mathcal{H}_M) = Span\left\{\frac{\delta}{\delta x^i}\right\}, \quad \mathcal{X}(\mathcal{V}) = Span\left\{\frac{\partial}{\partial y^i}\right\}.$$

ii) The Lie algebra $\mathcal{X}^*(E)$ of covector fields decomposes as

$$\mathcal{X}^*(E) = \mathcal{X}^*(\mathcal{H}_T) \oplus \mathcal{X}^*(\mathcal{H}_M) \oplus \mathcal{X}^*(\mathcal{V}),$$

where

$$\mathcal{X}^*(\mathcal{H}_T) = Span\{dt^\alpha\}, \quad \mathcal{X}^*(\mathcal{H}_M) = Span\{dx^i\}, \quad \mathcal{X}^*(\mathcal{V}) = Span\{\delta y^i\}.$$

Let us consider h_T, h_M (horizontal) and v (vertical) as the canonical projections of the above decompositions.

Definition 1.4.1 A linear connection $\nabla : \mathcal{X}(E) \times \mathcal{X}(E) \to \mathcal{X}(E)$ is called *a Γ-linear connection on E* if $\nabla h_T = 0$, $\nabla h_M = 0$ and $\nabla v = 0$.

In order to describe in local terms a Γ-linear connection ∇ on E, we need nine unique local components,

$$\nabla\Gamma = (\bar{G}^1_{11}, G^k_{i1}, G^{(k)(1)}_{(1)(i)1}, \bar{L}^1_{1j}, L^k_{ij}, L^{(k)(1)}_{(1)(i)j}, \bar{C}^{1(1)}_{1(j)}, C^{k(1)}_{i(j)}, C^{(k)(1)(1)}_{(1)(i)(j)}), \qquad (1.4.1)$$

which are locally defined by the relations

$$\nabla_{\frac{\delta}{\delta t}}\frac{\delta}{\delta t} = \bar{G}^1_{11}\frac{\delta}{\delta t}, \qquad \nabla_{\frac{\delta}{\delta t}}\frac{\delta}{\delta x^i} = G^k_{i1}\frac{\delta}{\delta x^k}, \qquad \nabla_{\frac{\delta}{\delta t}}\frac{\partial}{\partial y^i} = G^{(k)(1)}_{(1)(i)1}\frac{\partial}{\partial y^k},$$

$$\nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta t} = \bar{L}^1_{1j}\frac{\delta}{\delta t}, \qquad \nabla_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i} = L^k_{ij}\frac{\delta}{\delta x^k}, \qquad \nabla_{\frac{\delta}{\delta x^j}}\frac{\partial}{\partial y^i} = L^{(k)(1)}_{(1)(i)j}\frac{\partial}{\partial y^k},$$

$$\nabla_{\frac{\partial}{\partial y^j}}\frac{\delta}{\delta t} = \bar{C}^{1(1)}_{1(j)}\frac{\delta}{\delta t}, \qquad \nabla_{\frac{\partial}{\partial y^j}}\frac{\delta}{\delta x^i} = C^{k(1)}_{i(j)}\frac{\delta}{\delta x^k}, \qquad \nabla_{\frac{\partial}{\partial y^j}}\frac{\partial}{\partial y^i} = C^{(k)(1)(1)}_{(1)(i)(j)}\frac{\partial}{\partial y^k}.$$

Now, using the transformation laws 1.3.9 of the elements $\left\{\frac{\delta}{\delta t}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}$ together with the properties of the Γ-linear connection ∇, we obtain by computations

Theorem 1.4.2 *i) The coefficients of the Γ-linear connection ∇ modify by the rules.*

$$(h_T) \quad \begin{cases} \bar{G}_{11}^1 \dfrac{d\tilde{t}}{dt} = \tilde{\bar{G}}_{11}^1 \left(\dfrac{d\tilde{t}}{dt}\right)^2 + \dfrac{d^2\tilde{t}}{dt^2} \\[3mm] G_{i1}^k = \bar{G}_{j1}^m \dfrac{\partial x^k}{\partial \tilde{x}^m} \dfrac{\partial \tilde{x}^j}{\partial x^i} \dfrac{d\tilde{t}}{dt} \\[3mm] G_{(1)(i)1}^{(k)(1)} = \bar{G}_{(1)(j)1}^{(m)(1)} \dfrac{\partial x^k}{\partial \tilde{x}^m} \dfrac{\partial \tilde{x}^j}{\partial x^i} \dfrac{d\tilde{t}}{dt} + \delta_i^k \left(\dfrac{d\tilde{t}}{dt}\right)^2 \dfrac{d^2 t}{d\tilde{t}^2}, \end{cases}$$

$$(h_M) \quad \begin{cases} \bar{L}_{1j}^1 \dfrac{\partial x^j}{\partial \tilde{x}^l} = \tilde{\bar{L}}_{1l}^1 \\[3mm] L_{ij}^m \dfrac{\partial \tilde{x}^r}{\partial x^m} = \tilde{L}_{pq}^r \dfrac{\partial \tilde{x}^p}{\partial x^i} \dfrac{\partial \tilde{x}^q}{\partial x^j} + \dfrac{\partial^2 \tilde{x}^r}{\partial x^i \partial x^j} \\[3mm] L_{(1)(i)j}^{(m)(1)} \dfrac{\partial \tilde{x}^r}{\partial x^m} = \tilde{L}_{(1)(p)q}^{(r)(1)} \dfrac{\partial x^p}{\partial \tilde{x}^i} \dfrac{\partial \tilde{x}^q}{\partial x^j} + \dfrac{\partial^2 \tilde{x}^r}{\partial x^i \partial x^j}, \end{cases}$$

$$(v) \quad \begin{cases} \bar{C}_{1(i)}^{1(1)} = \tilde{\bar{C}}_{1(j)}^{1(1)} \dfrac{\partial \tilde{x}^j}{\partial x^i} \dfrac{dt}{d\tilde{t}} \\[3mm] C_{i(j)}^{k(1)} = \bar{C}_{p(r)}^{s(1)} \dfrac{\partial x^k}{\partial \tilde{x}^s} \dfrac{\partial \tilde{x}^p}{\partial x^i} \dfrac{\partial \tilde{x}^r}{\partial x^j} \dfrac{dt}{d\tilde{t}} \\[3mm] C_{(1)(i)(j)}^{(k)(1)(1)} = \tilde{C}_{(1)(p)(q)}^{(r)(1)(1)} \dfrac{\partial x^k}{\partial \tilde{x}^r} \dfrac{\partial \tilde{x}^p}{\partial x^i} \dfrac{\partial \tilde{x}^q}{\partial x^j} \dfrac{dt}{d\tilde{t}}. \end{cases}$$

ii) Conversely, to give a Γ-linear connection ∇ on the 1-jet space E is equivalent to give a set of nine local coefficients 1.4.1 whose local transformations laws are described in i.

The previous theorem allows us to offer an important example of Γ-linear connection on $J^1(R, M)$.

Example 1.4.1 Let h_{11} (resp. φ_{ij}) be a semi-Riemannian metric on the temporal (resp. spatial) manifold R (resp. M) and H_{11}^1 (resp. γ_{ij}^k) its Christoffel symbols. Let us consider $\Gamma_0 = (M_{(1)1}^{(i)}, N_{(1)j}^{(i)})$, where $M_{(1)1}^{(i)} = -H_{11}^1 y^i$, $N_{(1)j}^{(i)} = \gamma_{jk}^i y^k$, the canonical nonlinear connection on E attached to the metric pair (h_{11}, φ_{ij}). Using the transformation rules of the Christoffel symbols, we deduce that the following set of local coefficients [15]

$$B\Gamma_0 = (\bar{G}_{11}^1, 0, G_{(1)(i)1}^{(k)(1)}, 0, L_{ij}^k, L_{(1)(i)j}^{(k)(1)}, 0, 0, 0), \tag{1.4.2}$$

where $\bar{G}_{11}^1 = H_{11}^1$, $G_{(1)(i)1}^{(k)(1)} = -\delta_i^k H_{11}^1$, $L_{ij}^k = \gamma_{ij}^k$ and $L_{(1)(i)j}^{(k)(1)} = \delta_1^k \gamma_{ij}^k$, is a Γ_0-linear connection. This is called *the Berwald Γ_0-linear connection of the metric pair* (h_{11}, φ_{ij}).

Note that a Γ-linear connection ∇ on E, defined by the local coefficients 1.4.1, induces a natural linear connection on the d-tensors set of the jet fibre bundle $J^1(R, M)$,

Definition 1.5.1 A Γ-linear connection ∇ on E, defined by the local coefficients

$$\nabla\Gamma = (\bar{G}^1_{11}, G^k_{i1}, G^{(k)(1)}_{(1)(i)1}, \bar{L}^1_{1j}, L^k_{ij}, L^{(k)(1)}_{(1)(i)j}, \bar{C}^{1(1)}_{1(j)}, C^{k(1)}_{i(j)}, C^{(k)(1)(1)}_{(1)(i)(j)}),$$

that verifies the relations $\bar{G}^1_{11} = H^1_{11}$, $\bar{L}^1_{1j} = 0$, $\bar{C}^{1(1)}_{1(j)} = 0$ and $\nabla J = 0$, is called a *h-normal Γ-linear connection*.

Remark 1.5.1 Taking into account the local covariant R-horizontal $"_{/1}"$, M-horizontal $"_{|k}"$ and vertical $"|^{(1)}_{(k)}"$ covariant derivatives induced by ∇, the condition $\nabla J = 0$ is equivalent to

$$J^{(i)}_{(1)1j/1} = 0, \quad J^{(i)}_{(1)1j|k} = 0, \quad J^{(i)}_{(1)1j}|^{(1)}_{(k)} = 0. \tag{1.5.1}$$

In this context, we can prove the following

Theorem 1.5.1 *The coefficients of a h-normal Γ-linear connection ∇ verify the identities*

$$\bar{G}^1_{11} = H^1_{11}, \qquad \bar{L}^1_{1j} = 0, \qquad \bar{C}^{1(1)}_{1(j)} = 0,$$
$$G^{(k)(1)}_{(1)(i)1} = G^k_{i1} - \delta^k_i H^1_{11}, \quad L^{(k)(1)}_{(1)(i)j} = L^k_{ij}, \quad C^{(k)(1)(1)}_{(1)(i)(j)} = C^{k(1)}_{i(j)}. \tag{1.5.2}$$

Proof. The first three relations come from the definition of a *h*-normal Γ-linear connection.

The condition $\nabla J = 0$ implies locally that

$$h_{11}G^{(i)(1)}_{(1)(j)1} = h_{11}G^i_{j1} + \delta^i_j\left[-\frac{\partial h_{11}}{\partial t} + H_{111}\right],$$
$$h_{11}L^{(i)(1)}_{(1)(j)} = h_{11}L^i_{jk}, \tag{1.5.3}$$
$$h_{11}C^{(i)(1)(1)}_{(1)(j)(k)} = h_{11}C^{i(1)}_{j(k)},$$

where $H_{111} = H^1_{11}h_{11}$ represent the Christoffel symbols of the first kind attached to the semi-Riemannian metric h_{11}. Contracting the above relations by h^{11}, one obtains the last three identities of the theorem. ∎

Remarks 1.5.2 i) The preceding theorem implies that a *h*-normal Γ-linear on E is determined just by four effective coefficients

$$\nabla\Gamma = (H^1_{11}, G^k_{i1}, L^k_{ij}, C^{k(1)}_{i(j)}).$$

ii) Considering the particular case of the temporal metric $h = \delta$, we remark that a δ-normal Γ-linear connection on $J^1(R, M)$ is a natural generalization of the notion of N-linear connection used in the [11].

Example 1.5.1 Using the previous theorem, we deduce that the canonical Berwald Γ_0-linear connection associated to the metric pair (h_{11}, φ_{ij}) is a *h*-normal Γ_0-linear connection, defined by the local coefficients $B\Gamma_0 = (H^1_{11}, 0, \gamma^k_{ij}, 0)$.

1.6   d-Torsions and d-Curvatures

The study of the torsion $\mathbf{T}$ and curvature $\mathbf{R}$ d-tensors of a h-normal Γ-linear connection ∇ was made in [15]. We proved there that the adapted components $\bar{T}^1_{11}$, $\bar{T}^1_{1j}$, $\bar{P}^{1(1)}_{1(j)}$ and $R^{(m)}_{(1)11}$ of the torsion d-tensor $\mathbf{T}$ of ∇ vanish. Consequently, we obtain the following [15]

Theorem 1.6.1 *The torsion d-tensor* $\boldsymbol{T}$ *of the h-normal Γ-linear connection* ∇ *is determined by eight local d-tensors*

.	h_T	h_M	v
$h_T h_T$	0	0	0
$h_M h_T$	0	T^m_{1j}	$R^{(m)}_{(1)1j}$
$h_M h_M$	0	T^m_{ij}	$R^{(m)}_{(1)ij}$
$v h_T$	0	0	$P^{(m)\,(1)}_{(1)1(j)}$
$v h_M$	0	$P^{m(1)}_{i(j)}$	$P^{(m)\,(1)}_{(1)i(j)}$
$v v$	0	0	$S^{(m)(1)(1)}_{(1)(i)(j)}$

$$(1.6.1)$$

where
$$P^{(m)\,(1)}_{(1)1(j)} = \frac{\partial M^{(m)}_{(1)1}}{\partial y^j} - G^m_{j1} + \delta^m_j H^1_{11}, \qquad P^{(m)\,(1)}_{(1)i(j)} = \frac{\partial N^{(m)}_{(1)i}}{\partial y^j} - L^m_{ji},$$

$$R^{(m)}_{(1)1j} = \frac{\delta M^{(m)}_{(1)1}}{\delta x^j} - \frac{\delta N^{(m)}_{(1)j}}{\delta t}, \quad R^{(m)}_{(1)ij} = \frac{\delta N^{(m)}_{(1)i}}{\delta x^j} - \frac{\delta N^{(m)}_{(1)j}}{\delta x^i}, \quad S^{(m)(1)(1)}_{(1)(i)(j)} = C^{m(1)}_{i(j)} - C^{m(1)}_{j(i)},$$

$$T^m_{1j} = -G^m_{j1}, \quad T^m_{ij} = L^m_{ij} - L^m_{ji}, \quad P^{m(1)}_{i(j)} = C^{m(1)}_{i(j)}.$$

Remark 1.6.1 For the Berwald Γ_0-linear connection associated to the metrics h_{11} and φ_{ij}, all torsion d-tensors vanish, except $R^{(m)}_{(\mu)ij} = r^m_{ijl}x^l_\mu$, where (resp. r^m_{ijl}) are the curvature tensors of the metric φ_{ij}.

In the same context, following the paper [15], we deduce that the number of the effective adapted components of the curvature d-tensor $\mathbf{R}$ of a h-normal Γ-linear connection ∇ is five.

Theorem 1.6.2 *The curvature d-tensor* $\boldsymbol{R}$ *of* ∇ *is determined by the following effective local d-curvatures*

	h_T	h_M	v
$h_T h_T$	0	0	0
$h_M h_T$	0	R^l_{i1k}	$R^{(l)(1)}_{(1)(i)1k} = R^l_{i1k}$
$h_M h_M$	0	R^l_{ijk}	$R^{(l)(1)}_{(1)(i)jk} = R^l_{ijk}$
$v h_T$	0	$P^{l\,(1)}_{i1(k)}$	$P^{(l)(1)\,(1)}_{(1)(i)1(k)} = P^{l\,(1)}_{i1(k)}$
$v h_M$	0	$P^{l\,(1)}_{ij(k)}$	$P^{(l)(1)\,(1)}_{(1)(i)j(k)} = P^{l\,(1)}_{ij(k)}$
$v v$	0	$S^{l(1)(1)}_{i(j)(k)}$	$S^{(l)(1)(1)(1)}_{(1)(i)(j)(k)} = S^{l(1)(1)}_{i(j)(k)}$

$$(1.6.2)$$

where

$$R^l_{i1k} = \frac{\delta G^l_{i1}}{\delta x^k} - \frac{\delta L^l_{ik}}{\delta t} + G^m_{i1} L^l_{mk} - L^m_{ik} G^l_{m1} + C^{l(1)}_{i(m)} R^{(m)}_{(1)1k},$$

$$R^l_{ijk} = \frac{\delta L^l_{ij}}{\delta x^k} - \frac{\delta L^l_{ik}}{\delta x^j} + L^m_{ij} L^l_{mk} - L^m_{ik} L^l_{mj} + C^{l(1)}_{i(m)} R^{(m)}_{(1)jk},$$

$$P^{l\ (1)}_{i1(k)} = \frac{\partial G^l_{i1}}{\partial y^k} - C^{l(1)}_{i(k)/1} + C^{l(1)}_{i(m)} P^{(m)\ (1)}_{(1)1(k)},$$

$$P^{l\ (1)}_{ij(k)} = \frac{\partial L^l_{ij}}{\partial y^k} - C^{l(1)}_{i(k)|j} + C^{l(1)}_{i(m)} P^{(m)\ (1)}_{(1)j(k)},$$

$$S^{l(1)(1)}_{i(j)(k)} = \frac{\partial C^{l(1)}_{i(j)}}{\partial y^k} - \frac{\partial C^{l(1)}_{i(k)}}{\partial y^j} + C^{m(1)}_{i(j)} C^{l(1)}_{m(k)} - C^{m(1)}_{i(k)} C^{l(1)}_{m(j)}.$$

Remark 1.6.2 In the case of the Berwald Γ_0-linear connection associated to the metric pair (h_{11}, φ_{ij}), all curvature d-tensors vanish, except $R^l_{ijk} = r^l_{ijk}$, where r^l_{ijk} are the curvature tensors of the metric φ_{ij}.

2   Relativistic rheonomic Lagrange geometry

2.1   Some aspects of classical rheonomic Lagrange geometry

A lot of geometrical models in Mechanics, Physics or Biology are based on the notion of ordinary Lagrangian. Thus, the concept of Lagrange space which generalizes that of Finsler space was introduced. In order to geometrize the fundamental concept in mechanics, that of Lagrangian, we recall that a Lagrange space $L^n = (M, L(x,y))$ is defined as a pair which consists of a real, smooth, n-dimensional manifold M and a regular Lagrangian $L : TM \to R$, not necessarily homogenous with respect to the direction $(y^i)_{i=\overline{1,n}}$. The differential geometry of Lagrange spaces is now considerably developped and used in various fields to study the natural processes where the dependence on position, velocity or momentum is involved [11]. It is very important that the geometry of Lagrange spaces gives a model for both the gravitational and electromagnetic field, in a very natural blending of the geometrical structure of the space with the characteristic properties of these physical fields.

At the same time, there are many problems in Physics and Variational calculus in which time-dependent Lagrangians (i. e., smooth real functions on $R \times TM$) are involved. Consequently, a geometrization of the time-dependent Lagrangians was required. In this direction, a geometry derived from a time-dependent Lagrangian was sketched in [11] by Miron and Anastasiei. This is called the *"Rheonomic Lagrange Geometry"*. From our point of view, this time-dependent Lagrangian geometrization has an important inconvenience that we will describe.

In the context exposed in the book [11], the energy action functional $\mathcal{E}$, attached to a given time-dependent Lagrangian,

$$L : R \times TM \to R, \quad (t, x^i, y^i) \to L(t, x^i, y^i),$$

not necessarily homogenous with respect to the direction $(y^i)_{i=\overline{1,n}}$, is of the form

$$\mathcal{E}(c) = \int_a^b L(t, x^i(t), \dot{x}^i(t))\, dt, \qquad (2.1.1)$$

where $[a, b] \subset R$ and $c : [a, b] \to M$ is a smooth curve, locally expressed by $t \to (x^i(t))$, and having the velocity $\dot{x} = (\dot{x}^i(t))$. It is obvious that the non-homogeneity of the Lagrangian L, regarded as a smooth function on the product manifold $R \times TM$, implies that the energy $\mathcal{E}(c)$ is *dependent of the parametrizations of the curve c*. In order to remove this difficulty, the authors regard the space $R \times TM$ like a fibre bundle over M. In this context, the geometrical invariance group of $R \times TM$ is given by 1.1.4. In other words, to remove the parametrization dependence of $\mathcal{E}$, they ignore the temporal repametrizations on $R \times TM$. Naturally, in these conditions, their energy functional becomes a well defined one, but their approach stands out by the *"absolute"* character of the time coordinate t, that is, an other inconvenience.

In our geometrical approach, we try to remove the deficiency of the time-dependent Lagrangian geometrization constructed by Miron and Anastasiei. For that reason we regard the space $R \times TM \equiv J^1(R, M)$ like a fibre bundle over $R \times M$. The gauge group of this bundle of configurations is given by 1.1.2. Consequently, our gauge group does not ignore the temporal reparametrizations, hence, it stands out by the *relativistic* character of the time coordinate t. In this approach, using a given semi-Riemannian metric $h_{11}(t)$ on R, we construct a more general and natural energy action functional attached to the time-dependent Lagrangian L, setting

$$\mathcal{E}(c) = \int_a^b L(t, x^i(t), \dot{x}^i(t))\sqrt{|h_{11}|}\, dt. \qquad (2.1.2)$$

Obviously, $\mathcal{E}$ is well defined and is *independent of the curve parametrizations*.

In conclusion, we consider that the deficiency of the classical rheonomic geometry comes from a puzzling utilization of the notion of Lagrangian. From this point of view, we point out that, in our geometrical development, we use the distinct notions:

i) *time-dependent Lagrangian function* $-$ A smooth real function on $J^1(R, M) \equiv R \times TM$;

ii) *time-dependent Lagrangian* (Olver's terminology) $-$ A local function $\mathcal{L}$ on $J^1(R, M)$, which transforms by the rule $\bar{\mathcal{L}} = \mathcal{L}|dt/d\bar{t}|$. If L is a Lagrangian function on 1-jet fibre bundle, then $\mathcal{L} = L\sqrt{|h_{11}|}$ represents a Lagrangian on $J^1(R, M)$.

Finally, we point out that the geometrization attached to a time-dependent Lagrangian function that we will construct, can be called *"Relativistic Rheonomic Lagrange Geometry"*. The reasons which motivates us to use a such terminology was given at the begin of this paper. .

2.2   Relativistic rheonomic Lagrange spaces

In order to develope our time-dependent Lagrange geometry, we start the study considering $L : E \to R$ a smooth Lagrangian function on $E = J^1(R, M)$, which

is locally expressed by $E \ni (t, x^i, y^i) \rightarrow L(t, x^i, y^i) \in R$. The *vertical fundamental metrical d-tensor* of L is defined by

$$G_{(i)(j)}^{(1)(1)} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}. \tag{2.2.1}$$

Let $h = (h_{11})$ be a semi-Riemannian metric on the temporal manifold R.

Definition 2.2.1 A Lagrangian function $L : E \rightarrow R$ whose vertical fundamental metrical d-tensor is of the form

$$G_{(i)(j)}^{(1)(1)}(t, x^k, y^k) = h^{11}(t) g_{ij}(t, x^k, y^k), \tag{2.2.2}$$

where $g_{ij}(t, x^k, y^k)$ is a d-tensor on E, symmetric, of rank n and having a constant signature on E, is called a *Kronecker h-regular Lagrangian function, with respect to the temporal semi-Riemannian metric $h = (h_{11})$*.

In this context, we can introduce the following

Definition 2.2.2 A pair $RL^n = (J^1(R, M), L)$, where $n = \dim M$, which consists of the 1-jet fibre bundle and a Kronecker h-regular Lagrangian function $L : J^1(T, M) \rightarrow R$ is called a *relativistic rheonomic Lagrange space*.

Remark 2.2.1 In the geometrization of the time-dependent Lagrangian function L that we will construct, all entities with geometrical or physical meaning will be directly arised from the vertical fundamental metrical d-tensor $G_{(i)(j)}^{(1)(1)}$. This fact emphasizes the *metrical character* (see [5]) and the naturalness of the subsequent relativistic rheonomic Lagrangian geometry.

Examples 2.2.1 i) Suppose that the spatial manifold M is also endowed with a semi-Riemannian metric $g = (g_{ij}(x))$. Then, the time dependent Lagrangian function $L_1 : J^1(R, M) \rightarrow R$ defined by

$$L_1 = h^{11}(t) g_{ij}(x) y^i y^j \tag{2.2.3}$$

is a Kronecker h-regular time dependent Lagrangian function. Consequently, the pair $RL^n = (J^1(R, M), L_1)$ is a relativistic rheonomic Lagrange space. We underline that the Lagrangian $\mathcal{L}_1 = L_1 \sqrt{|h_{11}|}$ is exactly the energy Lagrangian whose extremals are the harmonic maps between the semi-Riemannian manifolds (R, h) and (M, g). At the same time, this Lagrangian is a basic object in the physical theory of bosonic strings.

ii) In above notations, taking $U_{(i)}^{(1)}(t, x)$ as a d-tensor field on E and $F : R \times M \rightarrow R$ a smooth map, the more general Lagrangian function $L_2 : E \rightarrow R$ defined by

$$L_2 = h^{11}(t) g_{ij}(x) y^i y^j + U_{(i)}^{(1)}(t, x) y^i + F(t, x) \tag{2.2.4}$$

is also a Kronecker h-regular Lagrangian. The relativistic rheonomic Lagrange space $RL^n = (J^1(R, M), L_2)$ is called the *autonomous relativistic rheonomic Lagrange space*

of electrodynamics because, in the particular case $h_{11} = 1$, we recover the classical Lagrangian space of electrodynamics which governs the movement law of a particle placed concomitantly into a gravitational field and an electromagnetic one [11]. From a physical point of view, the semi-Riemannian metric $h_{11}(t)$ (resp. $g_{ij}(x)$) represents the *gravitational potentials* of the space R (resp. M), the d-tensor $U_{(i)}^{(1)}(t, x)$ stands for the *electromagnetic potentials* and F is a function which is called *potential function*. The non-dynamic character of the spatial gravitational potentials $g_{ij}(x)$ motivates us to use the term of *"autonomous"*.

iii) More general, if we consider $g_{ij}(t, x)$ a d-tensor field on E, symmetric, of rank n and having a constant signature on E, we can define the Kronecker h-regular Lagrangian function $L_3 : E \to R$, setting

$$L_3 = h^{11}(t)g_{ij}(t, x)y^i y^j + U_{(i)}^{(1)}(t, x)y^i + F(t, x). \tag{2.2.5}$$

The pair $RL^n = (J^1(R, M), L_3)$ is a relativistic rheonomic Lagrange space which is called the *non-autonomous relativistic rheonomic Lagrange space of electrodynamics*. Physically, we remark that the gravitational potentials $g_{ij}(t, x)$ of the spatial manifold M are dependent of the temporal coordinate t, emphasizing their dynamic character.

2.3   Canonical nonlinear connection of RL^n

Let us consider $h = (h_{11})$ a fixed semi-Riemannian metric on R and a rheonomic Lagrange space $RL^n = (J^1(R, M), L)$, where L is a Kronecker h-regular Lagrangian function. Let $[a, b] \subset R$ be a compact interval in the temporal manifold R. In this context, we can define the *energy action functional* of RL^n, setting

$$\mathcal{E} : C^\infty(R, M) \to R, \quad \mathcal{E}(c) = \int_a^b L(t, x^i, y^i)\sqrt{|h|}\,dt,$$

where the smooth curve c is locally expressed by $(t) \to (x^i(t))$ and $y^i = \dfrac{dx^i}{dt}$.

The extremals of the energy functional $\mathcal{E}$ verifies the Euler-Lagrange equations

$$2G_{(i)(j)}^{(1)(1)}\frac{d^2x^j}{dt^2} + \frac{\partial^2 L}{\partial x^j \partial y^i}\frac{dx^j}{dt} - \frac{\partial L}{\partial x^i} + \frac{\partial^2 L}{\partial t \partial y^i} + \frac{\partial L}{\partial y^i}H_{11}^1 = 0, \quad \forall\, i = \overline{1, n}, \tag{2.3.1}$$

where H_{11}^1 are the Christoffel symbols of the semi-Riemannian metric h_{11}.

Taking into account the Kronecker h-regularity of the Lagrangian function L, it is possible to rearrange the Euler-Lagrange equations 2.3.1 of the Lagrangian $\mathcal{L} = L\sqrt{|h|}$, in the Poisson form [16],

$$\Delta_h x^k + 2\mathcal{G}^k(t, x^m, y^m) = 0, \quad \forall\, k = \overline{1, n}, \tag{2.3.2}$$

where

$$\Delta_h x^k = h^{11}\left\{\frac{d^2x^k}{dt^2} - H_{11}^1\frac{dx^k}{dt}\right\}, \; y^m = \frac{dx^m}{dt},$$

$$2\mathcal{G}^k = \frac{g^{ki}}{2}\left\{\frac{\partial^2 L}{\partial x^j \partial y^i}y^j - \frac{\partial L}{\partial x^i} + \frac{\partial^2 L}{\partial t \partial y^i} + \frac{\partial L}{\partial y^i}H_{11}^1 + 2g_{ij}h^{11}H_{11}^1 y^j\right\}. \tag{2.3.3}$$

Theorem 2.3.1 *Denoting* $G_{(1)1}^{(r)} = h_{11}\mathcal{G}^r$, *the geometrical object* $G = (G_{(1)1}^{(r)})$ *is a spatial spray on the 1-jet space* E.

Proof. By a direct calculation, we deduce that the local geometrical entities of the 1-jet space $J^1(R, M)$,

$$2\mathcal{S}^k = \frac{g^{ki}}{2}\left\{ \frac{\partial^2 L}{\partial x^j \partial y^i}y^j - \frac{\partial L}{\partial x^i} \right\},$$

$$2\mathcal{H}^k = \frac{g^{ki}}{2}\left\{ \frac{\partial^2 L}{\partial t \partial y^i} + \frac{\partial L}{\partial y^i}H_{11}^1 \right\}, \tag{2.3.4}$$

$$2\mathcal{J}^k = h^{11}H_{11}^1 y^j,$$

verify the following transformation rules:

$$2\mathcal{S}^p = 2\tilde{\mathcal{S}}^r\frac{\partial x^p}{\partial \tilde{x}^r} + h^{11}\frac{\partial x^p}{\partial \tilde{x}^l}\frac{d\tilde{t}}{dt}\frac{\partial \tilde{x}_\gamma^l}{\partial x^j}y^j,$$

$$2\mathcal{H}^p = 2\tilde{\mathcal{H}}^r\frac{\partial x^p}{\partial \tilde{x}^r} + h^{11}\frac{\partial x^p}{\partial \tilde{x}^l}\frac{d\tilde{t}}{dt}\frac{\partial \tilde{y}^l}{\partial t}, \tag{2.3.5}$$

$$2\mathcal{J}^p = 2\tilde{\mathcal{J}}^r\frac{\partial x^p}{\partial \tilde{x}^r} - h^{11}\frac{\partial x^p}{\partial \tilde{x}^l}\frac{d\tilde{t}}{dt}\frac{\partial \tilde{y}^l}{\partial t}.$$

Consequently, the local entities $2\mathcal{G}^p = 2\mathcal{S}^p + 2\mathcal{H}^p + 2\mathcal{J}^p$ modify by the transformation laws

$$2\tilde{\mathcal{G}}^r = 2\mathcal{G}^p\frac{\partial \tilde{x}^r}{\partial x^p} - h^{11}\frac{\partial x^p}{\partial \tilde{x}^j}\frac{\partial \tilde{x}_\mu^r}{\partial x^p}\tilde{y}^j. \tag{2.3.6}$$

Hence, multiplying the relation 2.3.6 by h_{11} and regarding the equations 1.2.4, we obtain what we were looking for. ∎

Taking into account the harmonic curve equations 1.2.5 of a time-dependent spray on E, we can give the following natural geometrical interpretation of the Euler-Lagrange equations 2.3.2 attached to the Lagrangian $\mathcal{L}$.

Theorem 2.3.2 *The extremals of the energy functional attached to a Kronecker h-regular Lagrangian function L on $J^1(R, M)$ are harmonic curves of the time-dependent spray (H, G), with respect to the semi-Riemannian metric h, defined by the temporal components*

$$H_{(1)1}^{(i)} = -\frac{1}{2}H_{11}^1(t)y^i \tag{2.3.7}$$

and the local spatial components

$$G_{(1)1}^{(i)} = \frac{h_{11}g^{ik}}{4}\left[\frac{\partial^2 L}{\partial x^j \partial y^k}y^j - \frac{\partial L}{\partial x^k} + \frac{\partial^2 L}{\partial t \partial y^k} + \frac{\partial L}{\partial x^k}H_{11}^1 + 2h^{11}H_{11}^1 g_{kl}y^l \right]. \tag{2.3.8}$$

Definition 2.3.1 The time-dependent spray (H, G) constructed in the preceding theorem is called the *canonical time-dependent spray attached to the relativistic rheonomic Lagrange space* RL^n.

Remark 2.3.1 In the particular case of an autonomous electrodynamics relativistic rheonomic Lagrange space (i. e., $g_{ij}(t, x^k, y^k) = g_{ij}(x^k)$), the canonical spatial spray G is given by the components

$$G_{(1)1}^{(i)} = \frac{1}{2}\gamma_{jk}^i y^j y^k + \frac{h_{11}g^{li}}{4}\left[U_{(l)j}^{(1)}y^j + \frac{\partial U_{(l)}^{(1)}}{\partial t} + U_{(l)}^{(1)}H_{11}^1 - \frac{\partial F}{\partial x^l}\right], \qquad (2.3.9)$$

where $U_{(i)j}^{(1)} = \dfrac{\partial U_{(i)}^{(1)}}{\partial x^j} - \dfrac{\partial U_{(j)}^{(1)}}{\partial x^i}$.

In the sequel, using the theorems 1.3.2 and 1.3.3, we obtain the following

Theorem 2.3.3 *The pair of local functions* $\Gamma = (M_{(1)1}^{(i)}, N_{(1)j}^{(i)})$, *which consists of the temporal components*

$$M_{(1)1}^{(i)} = 2H_{(1)1}^{(i)} = -H_{11}^1 y^i, \qquad (2.3.10)$$

and the spatial components

$$N_{(1)j}^{(i)} = \frac{\partial G_{(1)1}^i}{\partial y^j}, \qquad (2.3.11)$$

where $H_{(1)1}^{(i)}$ *and* $G_{(1)1}^{(i)}$ *are the components of the canonical time-dependent spray of* RL^n, *represents a nonlinear connection on* $J^1(R, M)$.

Definition 2.3.2 The nonlinear connection $\Gamma = (M_{(1)1}^{(i)}, N_{(1)j}^{(i)})$ from the previous theorem is called the *canonical nonlinear connection of the relativistic rheonomic Lagrange space* RL^n.

Remark 2.3.2 i) In the case of an autonomous electrodynamics relativistic rheonomic Lagrange space (i. e., $g_{ij}(t, x^k, y^k) = g_{ij}(x^k)$), the canonical nonlinear connection becomes $\Gamma = (M_{(1)1}^{(i)}, N_{(1)j}^{(i)})$, where

$$M_{(1)1}^{(i)} = -H_{11}^1 y^i, \quad N_{(1)j}^{(i)} = \gamma_{jk}^i y^k + \frac{h_{11}g^{ik}}{4}U_{(k)j}^{(1)}. \qquad (2.3.12)$$

2.4 Cartan canonical metrical connection of RL^n

The main theorem of this paper is the theorem of existence of the *Cartan canonical h-normal linear connection* $C\Gamma$ which allows the subsequent development of the *relativistic rheonomic Lagrangian geometry of physical fields* which will be exposed in the next Sections.

Theorem 2.4.1 *(of existence and uniqueness of Cartan canonical connection)*
On the relativistic rheonomic Lagrange space $RL^n = (J^1(R, M), L)$, *endowed with its canonical nonlinear connection* Γ, *there is a unique h-normal* Γ-*linear connection*

$$C\Gamma = (H_{11}^1, G_{j1}^k, L_{jk}^i, C_{j(k)}^{i(1)}),$$

having the metrical properties

i) $g_{ij|k} = 0$, $\quad g_{ij}|_{(k)}^{(1)} = 0$,

ii) $G_{j1}^k = \dfrac{g^{ki}}{2}\dfrac{\delta g_{ij}}{\delta t}$, $\quad L_{ij}^k = L_{ji}^k$, $\quad C_{j(k)}^{i(1)} = C_{k(j)}^{i(1)}$.

Proof. Let $C\Gamma = (\bar{G}_{11}^1, G_{j1}^k, L_{jk}^i, C_{j(k)}^{i(1)})$ be a h-normal Γ-linear connection whose coefficients are defined by $\bar{G}_{11}^1 = H_{11}^1$, $G_{j1}^k = \dfrac{g^{ki}}{2}\dfrac{\delta g_{ij}}{\delta t}$ and

$$L_{jk}^i = \frac{g^{im}}{2}\left(\frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{km}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m}\right),$$

$$C_{j(k)}^{i(1)} = \frac{g^{im}}{2}\left(\frac{\partial g_{jm}}{\partial y^k} + \frac{\partial g_{km}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^m}\right). \tag{2.4.1}$$

By computations, one easily verifies that $C\Gamma$ satisfies the conditions i and ii.

Conversely, let us consider $\tilde{C}\Gamma = (\tilde{\bar{G}}_{11}^1, \tilde{G}_{j1}^k, \tilde{L}_{jk}^i, \tilde{C}_{j(k)}^{i(1)})$ a h-normal Γ-linear connection which satisfies i and ii. It follows directly that

$$\tilde{\bar{G}}_{11}^1 = H_{11}^1, \text{ and } \tilde{G}_{j1}^k = \frac{g^{ki}}{2}\frac{\delta g_{ij}}{\delta t}.$$

The condition $g_{ij|k} = 0$ is equivalent with

$$\frac{\delta g_{ij}}{\delta x^k} = g_{mj}\tilde{L}_{ik}^m + g_{im}\tilde{L}_{jk}^m.$$

Applying a Christoffel process to the indices $\{i, j, k\}$, we find

$$\tilde{L}_{jk}^i = \frac{g^{im}}{2}\left(\frac{\delta g_{jm}}{\delta x^k} + \frac{\delta g_{km}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^m}\right).$$

By analogy, using the relations $C_{j(k)}^{i(1)} = C_{k(j)}^{i(1)}$ and $g_{ij}|_{(k)}^{(1)} = 0$ and following a Christoffel process applied to the indices $\{i, j, k\}$, we obtain

$$\tilde{C}_{j(k)}^{i(1)} = \frac{g^{im}}{2}\left(\frac{\partial g_{jm}}{\partial y^k} + \frac{\partial g_{km}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^m}\right).$$

In conclusion, the uniqueness of the Cartan canonical connection $C\Gamma$ is clear. ∎

Remarks 2.4.1 i) Replacing the canonical nonlinear connection Γ by a general one, the previous theorem holds good.

ii) As a rule, the Cartan canonical connection of a relativistic rheonomic Lagrange space RL^n verifies also the metrical properties

$$h_{11/1} = h_{11|k} = h_{11}|_{(k)}^{(1)} = 0 \text{ and } g_{ij/1} = 0. \tag{2.4.2}$$

iii) Particularly, the coefficients of the Cartan connection of an autonomous relativistic rheonomic Lagrange space of electrodynamics (i. e., $g_{ij}(t, x^k, y^k) = g_{ij}(x^k)$) are the same with those of the Berwald connection, namely, $C\Gamma = (H_{11}^1, 0, \gamma_{jk}^i, 0)$. Note that the Cartan connection is a Γ-linear connection, where Γ is the canonical nonlinear connection of the relativistic rheonomic Lagrangian space while the Berwald connection is a Γ_0-linear connection, Γ_0 being the canonical nonlinear connection associated to the metric pair (h_{11}, g_{ij}). Consequently, the Cartan and Berwald connections are distinct.

iv) The torsion d-tensor **T** of the Cartan canonical connection of a relativistic rheonomic Lagrange space is determined by only six local components, because the properties of the Cartan canonical connection imply the relations $T_{ij}^m = 0$ and $S_{(1)(j)(k)}^{(i)(1)(1)} = 0$. At the same time, we point out that the number of the curvature local d-tensors of the Cartan canonical connection not reduces. In conclusion, the curvature d-tensor **R** of the Cartan canonical connection is determined by five effective local d-tensors. Their expressions was described in Section 1.

Definition 2.4.1 The torsion and curvature d-tensors of the Cartan canonical connection of an RL^n are called the torsion and curvature of RL^n.

By a direct calculation, we obtain

Theorem 2.4.2 *i) All torsion d-tensors of an autonomous relativistic rheonomic Lagrange space of electrodynamics vanish, except*

$$\begin{cases} R_{(1)1j}^{(m)} = -\dfrac{h_{11}g^{mk}}{4}\left[H_{11}^1 U_{(k)j}^{(1)} + \dfrac{\partial U_{(k)j}^{(1)}}{\partial t}\right], \\[4mm] R_{(1)ij}^{(m)} = r_{ijk}^m y^k + \dfrac{h_{11}g^{mk}}{4}\left[U_{(k)i|j}^{(1)} + U_{(k)j|i}^{(1)}\right], \end{cases} \tag{2.4.3}$$

where r_{ijk}^m are the curvature tensors of the semi-Riemannian metric g_{ij}.

ii) All curvature d-tensors of an autonomous relativistic rheonomic Lagrange space of electrodynamics vanish, except $R_{ijk}^l = r_{ijk}^l$.

3 Relativistic rheonomic Lagrangian electromagnetism

3.1 Electromagnetic field

Let us consider $RL^n = (J^1(R, M), L)$ a relativistic rheonomic Lagrange space and $\Gamma = (M_{(1)1}^{(i)}, N_{(1)j}^{(i)})$ its canonical nonlinear connection. At the same time, we denote $C\Gamma = (H_{11}^1, G_{i1}^k, L_{ij}^k, C_{i(j)}^{k(1)})$ the Cartan canonical connection of RL^n.

Using the *canonical Liouville d-tensor field* $C = y^i \dfrac{\partial}{\partial y^i}$, we can introduce the *deflection d-tensors*

$$\bar{D}^{(i)}_{(1)1} = y^i_{/1}, \quad D^{(i)}_{(1)j} = y^i_{|j}, \quad d^{(i)(1)}_{(1)(j)} = y^i|^{(1)}_{(j)}, \tag{3.1.1}$$

where $"_{/1}"$, $"_{|j}"$ and $"|^{(1)}_{(j)}"$ are the local covariant derivatives induced by $C\Gamma$.

By a direct calculation, we find

Proposition 3.1.1 *The deflection d-tensors of the rheonomic Lagrange space RL^n have the expressions*

$$
\begin{aligned}
\bar{D}^{(i)}_{(1)1} &= \frac{g^{ik}}{2} \frac{\delta g_{km}}{\delta t} y^m, \\
D^{(i)}_{(1)j} &= -N^{(i)}_{(1)j} + L^i_{jm} y^m, \\
d^{(i)(1)}_{(1)(j)} &= \delta^i_j + C^{i(1)}_{m(j)} y^m.
\end{aligned} \tag{3.1.2}
$$

Remark 3.1.1 For an autonomous relativistic rheonomic Lagrange space of electrodynamics (i. e., $g_{ij} = g_{ij}(x^k)$), the deflection d-tensors reduce to

$$\bar{D}^{(i)}_{(1)1} = 0, \quad D^{(i)}_{(1)j} = -\frac{1}{4} g^{ik} h_{11} U^{(1)}_{(k)j}, \quad d^{(i)(1)}_{(1)(j)} = \delta^i_j. \tag{3.1.3}$$

In the sequel, using the *vertical fundamental metrical d-tensor* $G^{(1)(1)}_{(i)(k)} = h^{11} g_{ij}$ of the relativistic rheonomic Lagrange space RL^n we construct the *metrical deflection d-tensors*,

$$
\begin{aligned}
\bar{D}^{(1)}_{(i)1} &= G^{(1)(1)}_{(i)(k)} \bar{D}^{(k)}_{(1)1} = y_{i/1} \\
D^{(1)}_{(i)j} &= G^{(1)(1)}_{(i)(k)} D^{(k)}_{(1)j} = y_{i|j} \\
d^{(1)(1)}_{(i)(j)} &= G^{(1)(1)}_{(i)(k)} d^{(k)(1)}_{(1)(j)} = y_i|^{(1)}_{(j)},
\end{aligned} \tag{3.1.4}
$$

where $y_i = G^{(1)(1)}_{(i)(k)} y^k = h^{11} g_{ik} y^k$. Using the expressions 3.1.2 of the deflection distinguished tensors, it follows

Proposition 3.1.2 *The metrical deflection d-tensors of the relativistic rheonomic Lagrange space RL^n are given by the formulas*

$$
\begin{aligned}
\bar{D}^{(1)}_{(i)1} &= \frac{h^{11}}{2} \frac{\delta g_{im}}{\delta t} y^m, \\
D^{(1)}_{(i)j} &= h^{11} g_{ik} \left[-N^{(k)}_{(1)j} + L^k_{jm} y^m \right], \\
d^{(1)(1)}_{(i)(j)} &= h^{11} \left[g_{ij} + g_{ik} C^{k(1)}_{m(j)} y^m \right].
\end{aligned} \tag{3.1.5}
$$

Remark 3.1.2 In the particular case of an autonomous relativistic rheonomic Lagrange space of electrodynamics (i. e., $g_{ij} = g_{ij}(x^k)$), we have

$$\bar{D}^{(1)}_{(i)1} = 0, \quad D^{(1)}_{(i)j} = -\frac{1}{4}U^{(1)}_{(i)j}, \quad d^{(1)(1)}_{(i)(j)} = h^{11}g_{ij}. \tag{3.1.6}$$

In order to construct the relativistic rheonomic Lagrangian theory of electromagnetism, we introduce the following

Definition 3.1.1 The distinguished 2-form on $E = J^1(R, M)$,

$$F = F^{(1)}_{(i)j}\delta y^i \wedge dx^i + f^{(1)(1)}_{(i)(j)}\delta y^i \wedge \delta y^j, \tag{3.1.7}$$

where

$$F^{(1)}_{(i)j} = \frac{1}{2}\left[D^{(1)}_{(i)j} - D^{(1)}_{(j)i}\right], \quad f^{(1)(1)}_{(i)(j)} = \frac{1}{2}\left[d^{(1)(1)}_{(i)(j)} - d^{(1)(1)}_{(j)(i)}\right], \tag{3.1.8}$$

is called the *electromagnetic d-form* of the relativistic rheonomic Lagrange space RL^n.

Using the above definition, by a direct calculation, we obtain

Proposition 3.1.3 *The following expressions of the electromagnetic components,*

$$F^{(1)}_{(i)j} = \frac{h^{11}}{2}\left[g_{jm}N^{(m)}_{(1)i} - g_{im}N^{(m)}_{(1)j} + (g_{ik}L^k_{jm} - g_{jk}L^k_{im})y^m\right]. \quad f^{(1)(1)}_{(i)(j)} = 0, \tag{3.1.9}$$

hold good.

Remark 3.1.3 We emphasize that, in the particular case of an autonomous relativistic rheonomic Lagrange space (i. e. $g_{ij} = g_{ij}(x^k)$), the electromagnetic local components get the form

$$F^{(1)}_{(i)j} = \frac{1}{8}\left[U^{(1)}_{(j)i} - U^{(1)}_{(i)j}\right], \quad f^{(1)(1)}_{(i)(j)} = 0. \tag{3.1.10}$$

3.2 Maxwell equations

The main result of the electromagnetic relativistic rheonomic Lagrangian geometry is given by

Theorem 3.2.1 *The electromagnetic local components $F^{(1)}_{(i)j}$ of a relativistic rheonomic Lagrange space $RL^n = (J^1(R, M), L)$ are governed by the Maxwell equations*

$$F^{(1)}_{(i)k/1} = \frac{1}{2}\mathcal{A}_{\{i,k\}}\left\{\bar{D}^{(1)}_{(i)1|k} + D^{(1)}_{(i)m}T^m_{1k} + d^{(1)(1)}_{(i)(m)}R^{(m)}_{(1)1k} - \left[T^p_{1i|k} + C^{p(1)}_{k(m)}R^{(m)}_{(1)1i}\right]y_p\right\},$$

$$\sum_{\{i,j,k\}} F^{(1)}_{(i)j|k} = -\frac{1}{2}\sum_{\{i,j,k\}} C^{(1)(1)(1)}_{(i)(l)(m)}R^{(m)}_{(1)jk}y^l, \qquad \sum_{\{i,j,k\}} F^{(1)}_{(i)j}|^{(1)}_{(k)} = 0.$$

where $y_p = G^{(1)(1)}_{(p)(q)}y^q$ and $C^{(1)(1)(1)}_{(i)(l)(m)} = G^{(1)(1)}_{(l)(q)}C^{q(1)}_{i(m)} = \frac{h^{11}}{2}\frac{\partial^3 L}{\partial y^i \partial y^l \partial y^m}.$

Proof. Firstly, we point out that the Ricci identities [18], applied to the spatial metrical d-tensor g_{ij}, imply that the following curvature d-tensor identities

$$R_{mi1k} + R_{im1k} = 0, \quad R_{mijk} + R_{imjk} = 0, \quad P_{mij(k)}^{(1)} + P_{imj(k)}^{(1)} = 0,$$

where $R_{mi1k} = g_{ip}R^p_{m1k}$, $R_{mijk} = g_{ip}R^p_{mjk}$ and $P_{mij(k)}^{(1)} = g_{ip}P^p_{mj(k)}{}^{(1)}$, are true.

Now, let us consider the following general deflection d-tensor identities [18]:

d_1) $\bar{D}_{(1)1|k}^{(p)} - D_{(1)k/1}^{(p)} = y^m R^p_{m1k} - D_{(1)m}^{(p)} T^m_{1k} - d_{(1)(m)}^{(p)(1)} R_{(1)1k}^{(m)},$

d_2) $D_{(1)j|k}^{(p)} - D_{(1)k|j}^{(p)} = y^m R^p_{mjk} - d_{(1)(m)}^{(p)(1)} R_{(1)jk}^{(m)},$

d_3) $D_{(1)j}^{(p)}|_{(k)}^{(1)} - d_{(1)(k)|j}^{(p)(1)} = y^m P^p_{mj(k)}{}^{(1)} - D_{(1)m}^{(p)} C_{j(k)}^{m(1)} - d_{(1)(m)}^{(p)(1)} P_{(1)j(k)}^{(m)(1)}.$

Contracting these deflection d-tensor identities by $G_{(i)(p)}^{(1)(1)}$ and using the above curvature d-tensor equalities, we obtain the following metrical deflection d-tensors identities:

d'_1) $\bar{D}_{(i)1|k}^{(1)} - D_{(i)k/1}^{(1)} = -y_m R^m_{i1k} - D_{(i)m}^{(1)} T^m_{1k} - d_{(i)(m)}^{(1)(\mu)} R_{(1)1k}^{(m)},$

d'_2) $D_{(i)j|k}^{(1)} - D_{(i)k|j}^{(1)} = -y_m R^m_{ijk} - d_{(i)(m)}^{(1)(\mu)} R_{(1)jk}^{(m)},$

d'_3) $D_{(i)j}^{(1)}|_{(k)}^{(1)} - d_{(i)(k)|j}^{(1)(1)} = -y_m P^m_{ij(k)}{}^{(1)} - D_{(i)m}^{(1)} C_{j(k)}^{m(1)} - d_{(i)(m)}^{(1)(1)} P_{(1)j(k)}^{(m)(1)}.$

At the same time, we recall that the following Bianchi identities [15]

b_1) $\mathcal{A}_{\{j,k\}} \left\{ R^l_{j1k} + T^l_{1j|k} + C_{k(m)}^{l(1)} R_{(1)1j}^{(m)} \right\} = 0,$

b_2) $\sum_{\{i,j,k\}} \left\{ R^l_{ijk} - C_{k(m)}^{l(1)} R_{(1)ij}^{(m)} \right\} = 0,$

b_3) $\mathcal{A}_{\{j,k\}} \left\{ P^l_{jk(p)}{}^{(1)} + C_{j(p)|k}^{l(1)} + C_{k(m)}^{l(1)} P_{(1)j(p)}^{(m)(1)} \right\} = 0,$

where $\mathcal{A}_{\{j,k\}}$ means alternate sum and $\sum_{\{i,j,k\}}$ means cyclic sum, hold good.

In order to obtain the first Maxwell identity, we permute i and k in d'_1 and we subtract the new identity from the initial one. Finally, using the Bianchi identity b_1, we obtain what we were looking for.

Doing a cyclic sum by the indices $\{i,j,k\}$ in d'_2 and using the Bianchi identity b_2, it follows the second Maxwell equation.

Applying a Christoffel process to the indices $\{i,j,k\}$ in d'_3 and combining with the Bianchi identity b_3 and the relation $P_{(1)j(p)}^{(m)(1)} = P_{(1)p(j)}^{(m)(1)}$, we get a new identity. The cyclic sum by the indices $\{i,j,k\}$ applied to this last identity implies the third Maxwell equation. ∎

Remark 3.2.1 In the case of an autonomous relativistic rheonomic Lagrange space of electrodynamics (i. e., $g_{ij} = g_{ij}(x^k)$), the Maxwell equations take the simple form

$$F_{(i)k/1}^{(1)} = \frac{1}{2}\mathcal{A}_{\{i,k\}}h^{11}g_{im}R_{(1)1k}^{(m)}, \quad \sum_{\{i,j,k\}} F_{(i)j|k}^{(1)} = 0, \quad \sum_{\{i,j,k\}} F_{(i)j}^{(1)}|_{(k)}^{(1)} = 0. \quad (3.2.1)$$

4 Relativistic rheonomic Lagrangian gravitational theory

4.1 Gravitational field

Let $h = (h_{11})$ be a fixed semi-Riemannian metric on the temporal manifold R and $\Gamma = (M^{(i)}_{(1)1}, N^{(i)}_{(1)j})$ a fixed nonlinear connection on the 1-jet space $J^1(R, M)$. In order to develope a relativistic rheonomic Lagrange theory of gravitational field on $J^1(R, M)$, we introduce the following

Definition 4.1.1 From physical point of view, an adapted metrical d-tensor G on $J^1(R, M)$, expressed locally by

$$G = h_{11}dt \otimes dt + g_{ij}dx^i \otimes dx^j + h^{11}g_{ij}\delta y^i \otimes \delta y^j,$$

where $g_{ij} = g_{ij}(t, x^k, y^k)$ is a d-tensor on $J^1(R, M)$, symmetric, of rank $n = \dim M$ and having a constant signature on E, is called a *gravitational h-potential* on E.

Now, taking $RL^n = (J^1(R, M), L)$ a relativistic rheonomic Lagrange space, via its vertical fundamental metrical d-tensor

$$G^{(1)(1)}_{(i)(j)} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} = h^{11}(t)g_{ij}(t, x^k, y^k),$$

and its canonical nonlinear connection $\Gamma = (M^{(i)}_{(1)1}, N^{(i)}_{(1)j})$, one induces a natural gravitational h-potential on $J^1(R, M)$, setting

$$G = h_{11}dt \otimes dt + g_{ij}dx^i \otimes dx^j + h^{11}g_{ij}\delta y^i \otimes \delta y^j. \qquad (4.1.1)$$

4.2 Einstein equations and conservation laws

Let us consider $C\Gamma = (H^1_{11}, G^k_{j1}, L^i_{jk}, C^{i(1)}_{j(k)})$ the Cartan canonical connection of the relativistic rheonomic Lagrange space RL^n.

We postulate that the Einstein equations, which govern the gravitational h-potential G of the relativistic rheonomic Lagrange space RL^n, are the Einstein equations attached to the Cartan canonical connection of RL^n and the adapted metric G on $J^1(R, M)$, that is,

$$Ric(C\Gamma) - \frac{Sc(C\Gamma)}{2}G = \mathcal{K}\mathcal{T}, \qquad (4.2.1)$$

where $Ric(C\Gamma)$ represents the Ricci d-tensor of the Cartan connection, $Sc(C\Gamma)$ is its scalar curvature, $\mathcal{K}$ is the Einstein constant and $\mathcal{T}$ is an intrinsec tensor of matter which is called the *stress-energy* d-tensor.

In the adapted basis $(X_A) = \left(\dfrac{\delta}{\delta t}, \dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i}\right)$, the curvature d-tensor $\mathbf{R}$ of the Cartan connection is expressed locally by $\mathbf{R}(X_C, X_B)X_A = R^D_{ABC}X_D$. Hence, it

follows that we have $R_{AB} = Ric(C\Gamma)(X_A, X_B) = R^D_{ABD}$ and $Sc(C\Gamma) = G^{AB}R_{AB}$, where

$$G^{AB} = \begin{cases} h_{11}, & \text{for } A = 1, \ B = 1 \\ g^{ij}, & \text{for } A = i, \ B = j \\ h_{11}g^{ij}, & \text{for } A = \binom{(i)}{(1)}, \ B = \binom{(j)}{(1)} \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.2}$$

Taking into account the expressions of the local curvature d-tensors of the Cartan connection and the form of the vertical fundamental metrical d-tensor G^{AB}, we deduce

Proposition 4.2.1 *The Ricci d-tensor of the Cartan canonical connection of RL^n is determined by the following six effective local Ricci d-tensors,*

$$R_{11} \overset{not}{=} H_{11} = H^1_{111} = 0, \quad R_{i1} = R^m_{i1m}, \quad R_{ij} = R^m_{ijm}, \quad R^{(1)}_{i(j)} \overset{not}{=} P^{(1)}_{i(j)} = -P^m_{im(j)}^{(1)},$$

$$R^{(1)}_{(i)1} \overset{not}{=} P^{(1)}_{(i)1} = P^m_{i1(m)}^{(1)}, \quad R^{(1)}_{(i)j} \overset{not}{=} P^{(1)}_{(i)j} = P^m_{ij(m)}^{(1)}, \quad R^{(1)(1)}_{(i)(j)} \overset{not}{=} S^{(1)(1)}_{(i)(j)} = S^{m(1)(1)}_{i(j)(m)}.$$

Consequently, denoting $R = g^{ij}R_{ij}$ and $S = h_{11}g^{ij}S^{(1)(1)}_{(i)(j)}$, we obtain

Proposition 4.2.2 *The scalar curvature of the Cartan canonical connection of RL^n has the expression*

$$Sc(C\Gamma) = R + S. \tag{4.2.3}$$

Remark 4.2.1 In the particular case of an autonomous relativistic rheonomic Lagrange space of electrodynamics (i. e., $g_{ij} = g_{ij}(x^k)$), all Ricci d-tensors vanish, except $R_{ij} = r_{ij}$, where r_{ij} are the Ricci tensors associated to the semi-Riemannian metric g_{ij}. It follows that the scalar curvature of a such space is $Sc(C\Gamma) = r$, where r is the scalar curvature of the semi-Riemannian metric g_{ij}.

Using the above results, we can establish the following

Theorem 4.2.3 *The Einstein equations, which govern the gravitational h-potential G induced by the Kronecker h-regular Lagrangian function of a relativistic rheonomic Lagrange space RL^n, have the form*

$$(E_1) \quad \begin{cases} -\dfrac{R+S}{2}h_{11} = \mathcal{K}T_{11} \\[2mm] R_{ij} - \dfrac{R+S}{2}g_{ij} = \mathcal{K}T_{ij} \\[2mm] S^{(1)(1)}_{(i)(j)} - \dfrac{R+S}{2}h^{11}g_{ij} = \mathcal{K}T^{(1)(1)}_{(i)(j)}, \end{cases}$$

$$(E_2) \quad \begin{cases} 0 = T_{1i}, \quad R_{i1} = \mathcal{K}T_{i1}, \quad P^{(1)}_{(i)1} = \mathcal{K}T^{(1)}_{(i)1} \\[2mm] 0 = T^{(1)}_{1(i)}, \quad P^{(1)}_{i(j)} = \mathcal{K}T^{(1)}_{i(j)}, \quad P^{(1)}_{(i)j} = \mathcal{K}T^{(1)}_{(i)j}, \end{cases}$$

where T_{AB}, $A, B \in \{1, i, \binom{1}{(i)}\}$ *are the adapted local components of the stress-energy*
d-tensor T.

Remark 4.2.2 i) Note that, in order to have the compatibility of the Einstein equations, it is necessary that the certain adapted local components of the stress-energy d-tensor vanish "*a priori*".

ii) In the particular case of an **autonomous relativistic rheonomic Lagrange** space of electrodynamics (i. e., $g_{ij} = g_{ij}(x^k)$), the following Einstein equations of gravitational field,

$$(E_1) \qquad \begin{cases} r_{ij} - \dfrac{r}{2}g_{ij} = \mathcal{K}T_{ij} \\[2mm] -\dfrac{r}{2}h^{11}g_{ij} = \mathcal{K}T^{(1)(1)}_{(i)(j)}, \end{cases}$$

$$(E_2) \qquad \begin{cases} 0 = T_{1i}, \quad 0 = T_{i1}, \quad 0 = T^{(1)}_{(i)1} \\[2mm] 0 = T^{(1)}_{1(i)}, \quad 0 = T^{(1)}_{i(j)}, \quad 0 = T^{(1)}_{(i)j}, \end{cases}$$

hold good.

It is well known that, from physical point of view, the stress-energy d-tensor T must verifies the local *conservation laws* $T^B_{A|B} = 0$, $\forall\ A \in \{1, i, \binom{1}{(i)}\}$, where $T^B_A = G^{BD}T_{DA}$.

Denoting $T^1_1 = h^{11}T_{11}$, $T^m_j = g^{mr}T_{rj}$ and $T^{(m)(1)}_{(1)(j)} = h_{11}g^{mr}T^{(1)(1)}_{(r)(j)}$, by computations, we find

Theorem 4.2.4 *In the relativistic rheonomic Lagrangian gravitational theory, the conservation laws of the stress-energy d-tensor* T *have the local form:*

$$\begin{cases} T^1_{1/1} = R^m_{1|m} - P^{(m)}_{(1)1}|^{(1)}_{(m)} \\[2mm] T^m_{j|m} = -P^{(m)}_{(1)j}|^{(1)}_{(m)} \\[2mm] T^{(m)(1)}_{(1)(j)}\Big|^{(1)}_{(m)} = -P^{m(1)}_{(j)|m}, \end{cases} \qquad (4.2.4)$$

where $R^i_1 = g^{im}R_{m1}$, $P^{(i)}_{(1)1} = h_{11}g^{im}P^{(1)}_{(m)1}$, $P^{(i)}_{(1)j} = h_{11}g^{im}P^{(1)}_{(m)j}$, $P^{i(1)}_{(j)} = g^{im}P^{(1)}_{m(j)}$.

References

[1] G. S. Asanov, *Finsler Geometry, Relativity and Gauge Theories*, D. Reidel Publishing Company, 1985.

[2] G. S. Asanov, *Gauge-Covariant Stationary Curves on Finslerian and Jet Fibrations and Gauge Extension of Lorentz Force*, Tensor N. S. , Vol 50 (1991), 122-137.

[3] L. A. Cordero, C. T. J. Dodson, M. de Léon, *Differential Geometry of Frame Bundles*, Kluwer Academic Publishers, 1989.

[4] J. Eells, L. Lemaire. *A report on harmonic maps*, Bull. London Math. Soc. 10 (1978), 1-68.

[5] M. J. Gotay, J. Isenberg, J. E. Marsden, *Momentum Maps and the Hamiltonian Structure of Classical Relativistic Fields*, http://xxx.lanl.gov/hep/9801019.

[6] S. Ikeda, *Some Remarks on the Lagrangian Theory of Electromagnetism*, Tensor, N. S. , 49, (1990), 204-208.

[7] R. Hermann, *Differential Geometry and the Calculus of Variation*, Academic Press, New York, 1968.

[8] N. Kamron et P. J. Olver, *Le Probléme d'equivalence à une divergence prés dans le calcul des variations des intégrales multiples*, C. R. Acad. Sci. Paris, t. 308, Série I, p. 249-252. 1989.

[9] O. Krupkova, *The Geometry of Ordinary Variational Equations*, Springer-Verlag, 1997.

[10] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaisheisha Press, Kyoto, 1982.

[11] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, 1994.

[12] R. Miron, M. Anastasiei, *Fibrate rectoriale. Spaţii Lagrange. Aplicaţii în teoria relativităţii*, Ed. Acad. R. S. R. , Bucureşti, 1987, in Romanian.

[13] M. Morse, *Global Variational Analysis. Weierstrass Integrals on a Riemannian manifold*, Princeton University Press and University of Tokyo Press, Princeton, New Jersey, 1976.

[14] M. Neagu, *Harmonic Maps between Generalized Lagrange Spaces*, Southeast Asian Bulletin of Mathematics. Springer-Verlag, 2000-2001, in press.

[15] M. Neagu, *Upon h-normal Γ-linear connection on $J^1(T, M)$*, 2000, http://xxx.lanl.gov/math.DG/0009070.

[16] M. Neagu, C. Udrişte. *Multi-Time Dependent Sprays and Harmonic Maps on $J^1(T, M)$*. Third Conference of Balkan Society of Geometers, Politehnica University of Bucharest. Romania, July 31-August 3. 2000; http://xxx.lanl.gov/math.DG/0009049. 2000.

[17] M. Neagu, C. Udrişte, *The Geometry of Metrical Multi-Time Lagrange Spaces*, http://xxx.lanl.gov/math.DG/0009071, 2000.

[18] M. Neagu, C. Udrişte, *Torsions and Curvatures on Jet Fibre Bundle $J^1(T, M)$*, http://xxx.lanl.gov/math.DG/0009069, 2000.

[19] J. T. Oden, J. N. Reddy, *Variational Methods in Theoretical Mecanics*, Springer-Verlag, 1976.

[20] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer-Verlag, 1986.

[21] D. Saunders, *The Geometry of Jet Bundles*, Cambridge University Press, New York, London, 1989.

[22] Z. Shen, *Geometric Methods for Second Order Ordinary Differential Equations*, preprint, 2000.

[23] C. Udrişte, *Geometric Dynamics*, Second Conference of Balkan Society of Geometers, Aristotle University of Thessaloniki, Greece, June 23-26, 1998; Southeast Asian Bulletin of Mathematics, Springer-Verlag, 24 (2000), 313-322; Kluwer Academic Publishers, 2000.

[24] C. Udrişte, *Solutions of DEs and PDEs as potential maps using first order Lagrangians*, Centennial Vrânceanu, Romanian Academy, University of Bucharest, June 30-July 4, 2000; http://xxx.lanl.gov/math.DS/0007061, 2000.

[25] C. Udrişte, M. Neagu, *Geometrical Interpretation of Solutions of Certain PDEs*, Balkan Journal of Geometry and Its Applications, 4,1 (1999), 145-152.

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