TIME-INVARIANT AND DIAGONAL OPERATORS ASSOCIATED TO A SYSTEM OF IMPRIMITIVITY

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Abstract

Time-invariant and diagonal operators associated to a system of imprimitivity are represented as multiplicators on the frequency domain and (under appropriate hypothesis) as convolution operators on the time-domain.

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1 Convolution Operators

In this section we study a class of convolution operators defined by an operator-valued measure on an abelian locally compact group.

1.Definition Throughout this paper $(G,+)$ is a locally compact abelian group and $(\Gamma,+)$ the dual group of G. $(L^1(G), \|\cdot\|_1)$ and $(L^2(G), \|\cdot\|_2)$ are the usual spaces of integrable (respectively square-integrable) complex valued functions with respect to the Haar measure on G. The Banach algebra (with convolution and total variation) of all bounded, Borelian, regular, complex- valued measures on G will be denoted by $(M(G), \|\cdot\|)$. Let $(K, <\cdot,\cdot>)$ be a Hilbert space and let $\mathcal{L}(K)$ be the Banach algebra of all bounded linear operators on K. We denote by $M(G, \mathcal{L}(K))$ the (noncommutative) Banach algebra of all bounded, Borelian, regular, $\mathcal{L}(K)$-valued measures on G. If μ and ν are in $M(G, \mathcal{L}(K))$, we denote by $\| \mu \|$ the total variation of μ and by $\mu * \nu$ the convolution of μ and ν. For every $\mu \in M(G, \mathcal{L}(K)$, we define the *Fourier transform* $\widehat{\mu} : \Gamma \to \mathcal{L}(K)$, by : $\widehat{\mu}(\sigma)h = \int_G h\sigma(-t)d\mu(t)$, for every $h \in K$ and $\sigma \in \Gamma$. Obviously, the integral is convergent and we have:

2. Proposition *(a) For every* $\mu \in M(G, \mathcal{L}(K)), \widehat{\mu}$ *is a uniformly continuous bounded*

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function on Γ.
(b) For every $\mu, \nu \in M(G, \mathcal{L}(K))$, *we have:* $\widehat{\mu * \nu} = \widehat{\mu} \cdot \widehat{\nu}$.
(c) If δ_t *is the Dirac measure concentrated at* $t \in G$, *then* $\widehat{\delta}_t(\sigma) = \sigma(-t) \cdot I$, *where* I *is the identity operator.*
(d) For every $\mu \in M(G, \mathcal{L}(K))$ *and* $h, k \in K$, *let* $\mu_{h,k}(A) = < \mu(A)h\,,\, k >$, *for every Borelian set* A *in* G. *Then* $\mu_{h,k} \in M(G)$ *and* $< \widehat{\mu}(\sigma)h\,,\, k > = \int_G \sigma(-t) d\mu_{h,k}(t)$.

Proof. (a) For every $\mu \in M(G, \mathcal{L}(K))$, $h \in K$ and $\sigma \in \Gamma$, we have:

$$\| \widehat{\mu}(\sigma)h \| \le \int_G \| h\sigma(-t) \| \, d \| \mu \| \, (t) \le \| h \| \cdot \| \mu \|,$$

hence $\| \widehat{\mu}(\sigma) \| \le \| \mu \|$. Let $\sigma, \tau \in \Gamma$, then we have:

$$\| \widehat{\mu}(\sigma) - \widehat{\mu}(\tau) \| = \sup\{ \| [\widehat{\mu}(\sigma) - \widehat{\mu}(\tau)](h) \| \, ; h \in K, \| h \| = 1\} \le$$

$$\le \sup\{ \| \int_G h[\sigma(-t) - \tau(-t)] d\mu(t) \| \, ; h \in K, \| h \| = 1\} \le \int_G | \, 1 - (\sigma - \tau)(-t) \, | \, d \| \mu \| \, (t).$$

The measure $\| \mu \|$ is regular, hence for every $\varepsilon > 0$ there is a compact $Q \subset G$ such that $\| \mu \| \, (G - Q) < \varepsilon$. Let $W(Q, \varepsilon) = \{\sigma \in \Gamma; \; | \, \sigma(t) - 1 \, | < \varepsilon, \forall t \in Q\}$; if $\sigma - \tau \in W(Q, \varepsilon)$, we have:

$$\| \widehat{\mu}(\sigma) - \widehat{\mu}(\tau) \| \le \int_G | \, 1 - (\sigma - \tau)(-t) \, | \; d \| \mu \| \, (t) + \int_{G-Q} | \, 1 - (\sigma - \tau)(-t) \, | \; d \| \mu \| \, (t) \le$$

$$\le \varepsilon \, \| \mu \| \, (Q) + 2 \, \| \mu \| \, (G - Q) \to 0 if \varepsilon \to 0.$$

(b) Let $\mu \, , \, \nu \in M(G, \mathcal{L}(K))$, $\gamma \in \Gamma$ and $h \in K$, we have:

$$(\widehat{\mu * \nu})(\gamma)h = \int_G h \, \gamma(-t) d(\mu * \nu)(t) = \int_G (\int_G h\gamma(-t - s) d\nu(s)) d\mu(t) =$$

$$= \int_G (\gamma(-t) \int_G h \, \gamma(-s) d\nu(s)) d\mu(t) = \widehat{\mu}(\gamma) \cdot \widehat{\nu}(\gamma)h.$$

The assertions (c) and (d) are obvious. □

3. Observation The Hilbert spaces of K-valued square integrable functions with respect to the Haar measure on G and on Γ respectively are denoted by $(L^2(G, K), \| \cdot \|_2)$ and $(L^2(\Gamma, K), \| \cdot \|_2)$ respectively. Let us observe that for every $x \in L^2(G, K)$ and $h \in K$, the function $x_h : G \to C$ defined by $x_h(t) = < x(t)\,,\, h >$ is in $L^2(G)$. Let $\widehat{x_h}$ be its Fourier transform and let $\widehat{x} : \Gamma \to K$ be defined by $< \widehat{x}(\sigma)\,,\, h > = \widehat{x_h}(\sigma)$, for every $\sigma \in \Gamma$. Moreover, the Fourier transform $\mathcal{F} : L^2(G, K) \to L^2(\Gamma, K)$, $\mathcal{F}(x) = \widehat{x}$ is a Hilbert space isomorphism ([2],[12]).

4. Lemma *Let* $\mu \in M(G, \mathcal{L}(K))$ *and* $x \in L^2(G, K)$.
(a) The function $(\mu * x)(t) = \int_G x(t - s) d\mu(s)$ *is in* $L^2(G, K)$ *and* $\| \mu * x \|_2 \le \| \mu \| \, \| x \|_2$.

(b) For every $\sigma \in \Gamma$, *we have* $\widehat{(\mu * x)}(\sigma) = \widehat{\mu}(\sigma)\widehat{x}(\sigma)$.
Proof. (a) If dt is the Haar measure on G, we have:

$$\| \mu * x \|_2^2 = \int_G \| (\mu * x)(t) \|^2 dt = \int_G \| \int_G x(t-s)d\mu(s) \|^2 dt \leq$$

$$\leq \int_G \int_G \| x(t-s) \|^2 dtd \| \mu \|^2 (s) = \int_G \| x \|_2^2 d \| \mu \|^2 (s) = \| x \|_2^2 \| \mu \|^2 .$$

(b) Let $L^1(G,K)$ be the space of K-valued integrable functions with respect to the Haar measure on G and let $x \in L^1(G,K) \bigcap L^2(G,K)$. Then, $\widehat{x}(\sigma) = \int\limits_G \sigma(-t)x(t)dt$ and for every $\mu \in M(G, \mathcal{L}(K)), \mu * x \in L^1(G,K) \bigcap L^2(G,K)$. For every $h \in K$, we have:

$$< \widehat{(\mu * x)}(\sigma), h > = < \int_G (\mu * x)(t)\sigma(-t)dt, h > = < \int_G [\sigma(-t) \int_G x(t-s)d\mu(s)]dt, h > =$$

$$= < \int_G [\int_G \sigma(-t)x(t-s)dt]d\mu(s), h > = < \int_G [\int_G x(u)\sigma(-u)\sigma(-s)du]d\mu(s), h > =$$

$$= < \int_G \sigma(-s)\widehat{x}(\sigma)d\mu(s), h > = < \widehat{\mu}(\sigma)\widehat{x}(\sigma), h > .$$

Hence for every $x \in L^1(G,K) \bigcap L^2(G,K)$, we have $\widehat{(\mu * x)}(\sigma) = \widehat{\mu}(\sigma)\widehat{x}(\sigma)$. Let $x \in L^2(G,K)$ and let $x_n \in L^1(G,K) \bigcap L^2(G,K)$, such that $x_n \to x$ in $L^2(G,K)$. It results that $\widehat{x}_n \to \widehat{x}$ in $L^2(\Gamma, K)$. Let $x_{k_n} \subseteq x_n$, such that $\widehat{x}_{k_n}(\sigma) \to \widehat{x}(\sigma)$ in H, for every $\sigma \in \Gamma$. We have $\widehat{(\mu * x_{k_n})}(\sigma) = \widehat{\mu}(\sigma)\widehat{x}_{k_n}(\sigma)$ and for $n \to \infty$ we get $\widehat{(\mu * x)}(\sigma) = \widehat{\mu}(\sigma)\widehat{x}(\sigma)$. □

5. Definition For every $\mu \in M(G, \mathcal{L}(K))$ we define the convolution operator on $L^2(G,K)$ by
$U_\mu x = \mu * x$. The restriction of the map $M(G, \mathcal{L}(K)) \ni \mu \to U_\mu \in \mathcal{L}(L^2(G,K))$ to the group G is the regular representation:$(U_{\delta_s}x)(t) = x(t-s)$, for every $x \in L^2(G,K)$ and $t, s \in G$. We shall denote $U_{\delta_s} = U^s$.

6. Definition Let m be the Haar measure on the group Γ, let Bor(Γ) be the family of Borelian sets of Γ and let $\phi : \Gamma \to \mathcal{L}(K)$,be a measurable function. Let : $\| \phi \|_\infty = \inf \{\sup \{\| \phi(\gamma) \|; \gamma \in \Gamma - A\}; A \in \text{Bor}(\Gamma), m(A) = 0\}$be the essential supremum of ϕ. We denote by $L^\infty(\Gamma, \mathcal{L}(K))$ the (noncommutative) Banach algebra of essential bounded functions. The unit element of $L^\infty(\Gamma, \mathcal{L}(K))$ is the constant function $1(\gamma) = I$, for every $\gamma \in \Gamma$, (here I is the identity operator on K). A function $\phi \in L^\infty(\Gamma, \mathcal{L}(K))$ is *invertible* (in this algebra) if there is $\psi \in L^\infty(\Gamma, \mathcal{L}(K))$ such that $\phi\psi = \psi\phi = 1$, (all the equalities and inequalities are true a.e. on Γ). For every $\varepsilon > 0$ and $T \in \mathcal{L}(K)$, let $B(T, \varepsilon) = \{S \in \mathcal{L}(K); \| S - T \| < \varepsilon\}$. The *essential range* of a function $\phi \in L^\infty(\Gamma, \mathcal{L}(K))$ is defined by : $\text{essran}(\phi) = \{T \in \mathcal{L}(K); m(\phi^{-1}(B(T, \varepsilon))) > 0, \forall \varepsilon > 0\}$. The set essran$(\phi)$ is a closed and bounded set in $\mathcal{L}(K)$ and $\sup\{\| T \|; T \in \text{essran}(\phi)\} \leq \| \phi \|_\infty$.

7. Proposition *(a) Let* $K = C^n$ *and let* $\phi \in L^\infty(\Gamma, \mathcal{L}(K))$. *Then* essran$(\phi)$ *is a compact set (in* $\mathcal{L}(K)$*) and* $\| \phi \|_\infty = \sup\{\| T \|, T \in \text{essran}(\phi)\}$.

(b) If K is an arbitrary Hilbert space, then for every $\mu \in M(G, \mathcal{L}(K))$, we have $\mathrm{essran}(\widehat{\mu}) = \mathrm{clos}\,(\widehat{\mu}(\Gamma))$ *and* $\| \widehat{\mu} \|_\infty = \sup\{\| \widehat{\mu}(\gamma) \|; \gamma \in \Gamma\} = \sup\{\| T \|; T \in \mathrm{essran}(\widehat{\mu})\}$. *Here clos denotes the closure.*
Proof. (a) Let $\phi \in L^\infty(\Gamma, \mathcal{L}(K))$ and let us suppose that there is no $T \in \mathrm{essran}(\phi)$, such that $\| T \| = \| \phi \|_\infty$; it results that there is $\varepsilon_T > 0$, such that $m(\{\gamma \in \Gamma; \| \phi(\gamma) - T \| < \varepsilon_T\}) = 0$.
Let $\mathcal{A} = \{T \in \mathcal{L}(C^n); \| T \| = \| \phi \|_\infty\}$. $\mathcal{A}$ is a compact set, hence there are $T_1, T_2, \ldots, T_n$ in $\mathcal{L}(C^n)$ and positive numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, such that

$$\bigcup_{i=1}^{n} B(T_i, \varepsilon_i) \supseteq \mathcal{A} \text{ and } m(\bigcup_{i=1}^{n} \phi^{-1}(B(T_i, \varepsilon_i))) = 0.$$

It results that there is $\varepsilon > 0$, such that $m(\{\gamma \in \Gamma; \| \phi(\gamma) \| \geq \| \phi \|_\infty - \varepsilon\} = 0$, contradiction.
(b) The assertion (b) is evident because $\widehat{\mu}$ is a continuous function. □
8. Example If the Hilbert space K has not finite dimension, then the essential range need not to be a compact set. Let $\Gamma = Z$ (hence G is the unit circle, denoted by $\mathcal{S}^1$), let $K = l^2(Z)$, let $W : l^2(Z) \to l^2(Z), (Wx)(n) = x(n-1)$ and let $\phi : Z \to \mathcal{L}(l^2(Z)), \phi(n) = W^n$. It results that $\| \phi \|_\infty = 1$ and $\mathrm{essran}(\phi)$ is not a compact set.
9. Proposition *(a) For every $\phi \in L^\infty(\Gamma, \mathcal{L}(K))$ and $\widehat{x} \in L^2(\Gamma, K)$, the function :*

$$\Gamma \ni \gamma \to \phi(\gamma)\widehat{x}(\gamma) \in K$$

is in $L^2(\Gamma, K)$ and the multiplication operator $M_\phi : L^2(\Gamma, K) \to L^2(\Gamma, K)$ defined by $(M_\phi \widehat{x})(\gamma) = \phi(\gamma)\widehat{x}(\gamma)$ is a continuous linear operator; moreover, $\| M_\phi \| = \| \phi \|_\infty$.
(b) Let $\phi : \Gamma \to \mathcal{L}(K)$ be a measurable function and let $(M_\phi \widehat{x})(\gamma) = \phi(\gamma)\widehat{x}(\gamma)$; if $M_\phi \widehat{x} \in L^2(\Gamma, K)$, for every $\widehat{x} \in L^2(\Gamma, K)$, then $\phi \in L^\infty(\Gamma, \mathcal{L}(K))$ and therefore M_ϕ is a continuous operator.
Proof. **(a)** Let $\phi \in L^\infty(\Gamma, \mathcal{L}(K))$ and $\widehat{x} \in L^2(\Gamma, K)$, we have (a.e. on Γ) :

$$\| \phi(\gamma)\widehat{x}(\gamma) \| \leq \| \phi(\gamma) \| \cdot \| \widehat{x}(\gamma) \| \leq \| \phi \|_\infty \cdot \| \widehat{x}(\gamma) \| .$$

It results : $\int\limits_\Gamma \| \phi(\gamma)\widehat{x}(\gamma) \|^2 \, d\gamma \leq \| \phi \|_\infty^2 \| \widehat{x} \|_2^2$, hence $\| M_\phi \| \leq \| \phi \|_\infty$.
(b) Let us first suppose that $M_\phi \widehat{x} \in L^2(\Gamma, K)$, for every $\widehat{x} \in L^2(\Gamma, K)$ and that M_ϕ is a continuous operator (we shall prove later that the continuity is not necessary). We now prove that $\| \phi(\gamma) \| \leq \| M_\phi \|$(a.e. on Γ). Let $A \in \mathrm{Bor}(\Gamma)$, such that $m(A) < \infty$ and $\| \phi(\gamma) \| > \| M_\phi \|$, for every $\gamma \in A$. If $m(A) = 0$, then $\| \phi \|_\infty \leq \| M_\phi \|$, hence $\phi \in L^\infty(\Gamma, \mathcal{L}(K))$ and the proof is over. Let us now suppose that $m(A) \neq 0$. For every $\gamma \in A$, let $h_\gamma \in K$, such that $\| h_\gamma \| = 1$ and $\| \phi(\gamma)h_\gamma \| > \| M_\phi \|$. Let $f : \Gamma \to K$, defined by $f(\gamma) = 0$, if $\gamma \notin A$ and $f(\gamma) = h_\gamma$, if $\gamma \in A$. The properties of f are : $f \in L^2(\Gamma, K)$and $\| f \|_2^2 = m(A)$, hence we have:

$$\| M_\phi f \|_2^2 = \int_\Gamma \| \phi(\gamma)f(\gamma) \|^2 \, d\gamma =$$

$$= \int_A \| \phi(\gamma)h_\gamma \|^2 \, d\gamma \; > \| M_\phi \|^2 \; m(A) = \| M_\phi \|^2 \| f \|_2^2 ,$$

which represents a contradiction.
We now renounce at the hypothesis of continuity of M_ϕ and we prove that it is a closed operator (hence, by the closed graph theorem it is continuous). Let $(\widehat{x}_n , \widehat{y}_n) \in$ graph(M_ϕ), hence $\widehat{y}_n(\gamma) = \phi(\gamma)\widehat{x}_n(\gamma)$, $(a.e)$. Let us suppose that $\widehat{x}_n \to \widehat{x}$ and $\widehat{y}_n \to \widehat{y}$ in $L^2(\Gamma, K)$. We can choose $\hat{x}_n$ and $\hat{y}_n$, such that $\widehat{x}_n(\gamma) \to \widehat{x}(\gamma)$ and $\widehat{y}_n(\gamma) \to \widehat{y}(\gamma)$for almost all $\gamma \in \Gamma$. It results that $\phi(\gamma)\widehat{x}_n(\gamma) \to \phi(\gamma)\widehat{x}(\gamma)$, $(a.e.)$, hence $\widehat{y}(\gamma) = \phi(\gamma)\widehat{x}(\gamma)$, $(a.e.)$, which completes the proof. □

10. Proposition *Let $\phi \in L^\infty(\Gamma, \mathcal{L}(K))$; then M_ϕ is an invertible operator if and only if ϕ is an invertible function.*

Proof. If ϕ is an invertible function, then there is $\psi \in L^\infty(\Gamma, \mathcal{L}(K))$, such that $\phi\psi = \psi\phi = 1$, hence M_ψ is the inverse of M_ϕ.
Conversely, let us suppose that M_ϕ is an invertible operator and let M_ϕ^{-1} be its inverse. We first prove that $\phi(\gamma)$ is an invertible operator for almost all $\gamma \in \Gamma$. Let us suppose that there is $B \in \mathrm{Bor}(\Gamma)$, such that $m(B) < \infty, m(B) > 0$ and $\phi(\gamma)$ is not an injective operator (for all $\gamma \in B$); it results that for every $\gamma \in B$, there is $h_\gamma \in K$, such that $\| h_\gamma \| = 1$ and $\phi(\gamma)h_\gamma = 0$. Let $f : \Gamma \to K, f(\gamma) = 0$, if $\gamma \not\in B$ and $f(\gamma) = h_\gamma$, if $\gamma \in B$. The properties of f are : $f \in L^2(\Gamma, K), \| f \|_2 \neq 0$ and $(M_\phi f)(\gamma) = \phi(\gamma)f(\gamma) = 0$, which is a contradiction with the injectivity of M_ϕ. Let us now suppose that there is $B \in \mathrm{Bor}(\Gamma)$, such that $m(B) > 0, m(B) < \infty$ and $\phi(\gamma)$ is not a surjective operator (for all $\gamma \in \Gamma$). It results that for every $\gamma \in B$, there is $h_\gamma \in K$, such that $\| h_\gamma \| = 1$ and $h_\gamma \notin \mathrm{ran}(\phi(\gamma))$. Let $g : \Gamma \to K$ defined by $g(\gamma) = 0$, if $\gamma \notin B$ and $g(\gamma) = h_\gamma$, if $\gamma \in B$; it results that $g \in L^2(\Gamma, K)$. We now prove that $g \notin \mathrm{ran}(M_\phi)$. If we suppose that there is $k \in L^2(\Gamma, K)$, such that $M_\phi k = g$, then we get:

$$\phi(\gamma)k(\gamma) = \begin{cases} h_\gamma, & \text{if } \gamma \in B \\ 0, & \text{if } \gamma \notin B, \end{cases}$$

hence $h_\gamma \in \mathrm{ran}(\phi(\gamma))$, which is a contradiction.
Let $\psi(\gamma) = (\phi(\gamma))^{-1}$; we have $(M_\phi^{-1}\widehat{x})(\gamma) = \psi(\gamma)\widehat{x}(\gamma)$, hence, by Proposition 9(b) it results that $\psi \in L^\infty(\Gamma, \mathcal{L}(K))$. The set $\{T \in \mathcal{L}(K) \; ; \; T$ is not invertible$\}$ is a closed set in $\mathcal{L}(K)$, hence $\phi^{-1}(\{T \in \mathcal{L}(K); T$ is not invertible$\})$ is a measurable set, which concludes the proof. □

11. Theorem *For every $\mu \in M(G, \mathcal{L}(K))$, we have:*
(a) $\mathcal{F} U_\mu \mathcal{F}^{-1} = M_{\widehat{\mu}}$.
(b) $\| U_\mu \| = \| \widehat{\mu} \|_\infty = \sup\{\| T \|; T \in \mathrm{essran}(\widehat{\mu})\} = \sup\{\| \widehat{\mu}(\gamma) \|; \gamma \in \Gamma\} = \| M_{\widehat{\mu}} \|$.
(c) spec$(U_\mu) = \mathrm{spec}(M_{\widehat{\mu}}) = \mathrm{spec}(\widehat{\mu})$ (here spec denotes the spectrum).
(d) If $\widehat{\mu}(\Gamma)$ is a compact subset in $\mathcal{L}(K)$, (for example, if $K = C^n$ or $\widehat{\mu}$ has compact support), then spec$(U_\mu) = \mathrm{spec}(M_{\widehat{\mu}}) = \bigcup_{T \in \mathrm{clos}(\widehat{\mu}(\Gamma))} \mathrm{spec}(T)$.

Proof. The assertions (a), (b) and (c) were proved in Lemma 4 and Propositions 7,9,10.
(d) Let us first suppose that every operator in $\mathrm{clos}(\widehat{\mu}(\Gamma))$ is an invertible operator. Let $\psi(\gamma) = (\widehat{\mu}(\gamma))^{-1}$, for every $\gamma \in \Gamma$. Let $f : \widehat{\mu}(\Gamma) \to \psi(\Gamma), f(T) = T^{-1}$, then $\psi(\Gamma) \subseteq f(\mathrm{clos}(\widehat{\mu}(\Gamma))$; it results that $\mathrm{clos}(\psi(\Gamma))$ is a compact set, hence $\psi \in L^\infty(\Gamma, \mathcal{L}(K))$and therefore $M_{\widehat{\mu}}$ is an invertible operator.
Conversely, if $M_{\widehat{\mu}}$ is an invertible operator, then $\widehat{\mu}$is an invertible function in the algebra $L^\infty(\Gamma, \mathcal{L}(K))$, hence every operator in $\widehat{\mu}(\Gamma)$ is invertible. Let $T \in \mathrm{clos}(\widehat{\mu}(\Gamma))$ and let $S_n \in \widehat{\mu}(\Gamma)$ be a sequence of operators, such that $S_n \to T$. The sequence S_n is bounded (because $\widehat{\mu}$ is invertible), hence there is a convergent subsequence $S_{k_n} \subseteq S_n$ and let $S \in \mathcal{L}(K)$be its limit. From the equalities $T_{k_n} S_{k_n} = S_{k_n} T_{k_n} = I$ it results that $S = T^{-1}$, hence T is an invertible operator and the proof is over.
□

2 The representation of time-invariant and diagonal operators

In this section we obtain a characterization of time-invariant operators (as convolution operators) and of diagonal operators associated to a system of imprimitivity.
12. Definition Let H be a Hilbert space and let $\mathcal{L}(H)$ be the Banach algebra of all bounded linear operators on H. Let $P : \mathrm{Bor(G)} \to \mathcal{L}(H)$ be a spectral measure and let $V : G \to \mathcal{L}(H)$ be a continuous unitary representation. The 4-uple (H, G, P, V) is *termed system of imprimitivity* if $V_t P(A) = P(A + t)V_t$, for every $t \in G$ and $A \in \mathrm{Bor}(G)$. For every given system of imprimitivity, Stone's Theorem allows one to construct a second spectral measure $\widehat{P} : \mathrm{Bor}(\Gamma) \to \mathcal{L}(H)$ and a continuous unitary representation $\widehat{V} : \Gamma \to \mathcal{L}(H)$, such that

$$V_t = \int_\Gamma \gamma(-t) d\widehat{P}(\gamma) \text{ and } \widehat{V}_\gamma = \int_G \gamma(-t) dP(t),$$

for every $t \in G$ and $\gamma \in \Gamma$. We term $(H, \Gamma, \widehat{P}, \widehat{V})$ the dual system of (H, G, P, V). Moreover, we have ([5],[6]): $V_t \widehat{V}_\gamma = \gamma(t) \widehat{V}_\gamma V_t$ and $\widehat{V}_\gamma \widehat{P}(B) = \widehat{P}(B - \gamma)\widehat{V}_\gamma$, for every $t \in G, \gamma \in \Gamma$ and $B \in \mathrm{Bor}(\Gamma)$, ([6]).

We give the usual example of system of imprimitivity. Let $(K, <\,,\,>)$ be a Hilbert space and let $L^2(G, K)$ and $L^2(\Gamma, K)$ be the usual Hilbert spaces of K-valued square integrable functions on G and Γ,respectively. For every $A \in \mathrm{Bor(G)}$, let χ_A be the characteristic function of A and let E be the usual spectral measure on $L^2(G, K)$, i.e. $E(A)x = \chi_A x$, for every $x \in L^2(G, K)$. Analogously, if $B \in \mathrm{Bor}(\Gamma)$, let $F(B)\hat{x} = \chi_B \hat{x}$, for every $\hat{x} \in L^2(\Gamma, K)$. We denote by U and W the left regular representations on G and Γ, respectively: $(U_t x)(s) = x(s - t)$, for all $t, s \in G$, $x \in L^2(G, K)$ and $(W_\gamma \hat{x})(\sigma) = \hat{x}(\sigma - \gamma)$, for all $\gamma, \sigma \in \Gamma$, $\hat{x} \in L^2(\Gamma, K)$. It results that $(L^2(G, K), G, E, U)$ and $(L^2(\Gamma, K), \Gamma, F, W)$ are systems of imprimitivity. The dual system of $(L^2(G, K), G, E, U)$ is $(L^2(G, K), \Gamma, \widehat{E}, \widehat{U})$, where $U_t = \int_\Gamma \gamma(-t) d\widehat{E}(\gamma)$ and $\widehat{U}_\gamma = \int_G \gamma(-t) dE(t)$, for all $t \in G$ and $\gamma \in \Gamma$. The dual system of $(L^2(\Gamma, K), \Gamma, F, W)$

is $(L^2(\Gamma, K), G, \widehat{F}, \widehat{W})$, where $W_\gamma = \int_G \gamma(-t)d\widehat{F}(t)$ and $\widehat{W_t} = \int_\Gamma \gamma(-t)dF(\gamma)$, for every $t \in G$ and $\gamma \in \Gamma$. The relations between $E, F, U, W, \widehat{E}, \widehat{F}, \widehat{U}, \widehat{W}$ and $\mathcal{F}$ are given in the following lemma ([9]).

13. Lemma *With the above notations, we have:*
(a) $(\mathcal{F}U_t\mathcal{F}^{-1}\hat{x})(\gamma) = \gamma(-t)\hat{x}(\gamma)$, *for every* $t \in G$, $\gamma \in \Gamma$ *and* $\hat{x} \in L^2(\Gamma, K)$.
(b) $\mathcal{F}E(A)\mathcal{F}^{-1}\hat{x} = \widehat{\chi}_A \star \hat{x}$, *for every* $A \in \mathrm{Bor}(G)$ *and* $\hat{x} \in L^2(\Gamma, K)$; *here* $\widehat{\chi}_A$ *is the Fourier transform of* χ_A *and the convolution is defined by* $(\widehat{\chi}_A \star \hat{x})(\gamma) = \int_\Gamma \widehat{\chi}_A(\gamma - \theta)\hat{x}(\theta)d\theta$.
(c) $(\widehat{U}_\gamma x)(t) = \gamma(-t)x(t)$, *for every* $\gamma \in \Gamma$, $x \in L^2(G, K)$ *and* $t \in G$.
(d) $\widehat{E}(B)x = (\mathcal{F}^{-1}\chi_B) \star x$, *for every* $B \in \mathrm{Bor}(\Gamma)$ *and* $x \in L^2(G, K)$.
(e)$W_\gamma = \mathcal{F}\widehat{U}_\gamma\mathcal{F}^{-1}$, *for every* $\gamma \in \Gamma$.
(f)$F(B) = \mathcal{F}\widehat{E}(B)\mathcal{F}^{-1}$, *for every* $B \in \mathrm{Bor}(\Gamma)$.
(g)$\mathcal{F}U_s\mathcal{F}^{-1} = \widehat{W}_s$.
(h) $\mathcal{F}E(A)\mathcal{F}^{-1} = \widehat{F}(A)$.

14. Theorem *Let* (H, G, P, V) *be a system of imprimitivity and let* $(H, \Gamma, \widehat{P}, \widehat{V})$ *be the dual system. Then there is a Hilbert space* K *and* $\Omega : H \to L^2(G, K)$, $\widetilde{\Omega} : H \to L^2(\Gamma, K)$, *such that:*
(a) Ω *and* $\widetilde{\Omega}$ *are Hilbert space isomorphisms.*
(b) $\Omega P(A)\Omega^{-1} = E(A)$ *and* $\Omega V_t\Omega^{-1} = U_t$, *for every* $A \in \mathrm{Bor}(G)$ *and* $t \in G$.
(c) $\Omega\widehat{P}(B)\Omega^{-1} = \widehat{E}(B)$ *and* $\Omega\widehat{V}_\gamma\Omega^{-1} = \widehat{U}_\gamma$, *for every* $B \in \mathrm{Bor}(\Gamma)$ *and* $\gamma \in \Gamma$.
(d)$\widetilde{\Omega}\widehat{P}(B)\widetilde{\Omega}^{-1} = F(B)$ *and* $\widetilde{\Omega}\widehat{V}_\gamma\widetilde{\Omega}^{-1} = W_\gamma$, *for every* $B \in \mathrm{Bor}(\Gamma)$ *and* $\gamma \in \Gamma$.
(e) $\widetilde{\Omega}P(A)\widetilde{\Omega}^{-1} = \widehat{F}(A)$ *and* $\widetilde{\Omega}V_t\widetilde{\Omega}^{-1} = \widehat{W_t}$, *for every* $A \in \mathrm{Bor}(G)$ *and* $t \in G$.

Proof. We apply Mackey's Theorem ([6]) for V and $\widehat{V}$; it results that there is an at most countable index set J, a family of Hilbert spaces $\{K_n\}_{n\in J}$ and unitary operators $S_n : K_n \to L^2(G)$, such that $\bigoplus_{n\in J} K_n = H$ and
(i) K_n is an invariant subspace for V and $\widehat{V}$, for every $n \in J$.
(ii) $(S_nV_tS_n^{-1}f)(s) = f(s-t)$, for every $n \in J$, $t, s \in G$, $\gamma \in \Gamma$ and $f \in L^2(G)$.
(iii) $(S_n\widehat{V}_\gamma S_n^{-1}f)(t) = \gamma(-t)f(t)$, for every $n \in J$, $t \in G$, $\gamma \in \Gamma$ and $f \in L^2(G)$.
Let $\Psi : H \to \bigoplus_{n\in J} L^2(G)$, $\Psi = \bigoplus_{n\in J} S_n$ and let K be a Hilbert space of appropriate dimension, such that there is an unitary operator $\Phi : \bigoplus_{n\in J} L^2(G) \to L^2(G.\ K)$.
We define $\Omega : H \to L^2(G, K)$ by $\Omega = \Phi \circ \Psi$. The above properties (i), (ii) imply the second part of (b) and (i), (iii) imply the second part of (c). We now prove the first part of (b). For every $\gamma \in \Gamma$, we have:

$$\Omega\widehat{V}_\gamma\Omega^{-1} = \Omega[\int_G \gamma(-t)dP(t)]\Omega^{-1} = \widehat{U}^\gamma = \int_G \gamma(-t)dE(t).$$

Stone's Theorem and the second part of (b) prove the first part of (c). Let $\widetilde{\Omega} = \mathcal{F} \circ \Omega$. Then (d) and (e) are consequences of Lemma 13 and relations (b) and (c). □

15. Definition Let (H, G, P, V)be a system of imprimitivity. An operator $T \in$

$\mathcal{L}(H)$ is called *time-invariant* if $TV^t = V^tT$, for every $t \in G$, ([4],[5],[10],[12]).

The convolution operator U_μ is a time-invariant one.

An operator T is called *diagonal* if $TP(A) = P(A)T$, for every $A \in \mathrm{Bor}(G)$.

16. Theorem *Let (H, G, P, V) be a system of imprimitivity, let $(H, \Gamma, \widehat{P}, \widehat{V})$ be its dual and let $T \in \mathcal{L}(H)$.*
(a) If T is a diagonal operator, then there is a Hilbert space K and $\phi \in L^\infty(G, \mathcal{L}(K))$, such that T is unitarily-equivalent with the multiplication operator M_ϕ (on the space $L^2(G, K)$); moreover, if there is $\mu \in M(\Gamma, \mathcal{L}(K))$, such that $\widehat{\mu} = \phi$, then T is unitarily-equivalent with the convolution operator W_μ (on the space $L^2(\Gamma\, K)$).
(b) If T is a time-invariant operator, (i.e. $TV^t = V^tT$), then there is a Hilbert space K and $\phi \in L^\infty(\Gamma, \mathcal{L}(K))$, such that T is unitarily-equivalent with the multiplication operator M_ϕ (on the space $L^2(\Gamma, K)$); moreover, if there is $\mu \in M(G, \mathcal{L}(K))$, such that $\widehat{\mu} = \phi$, then T is unitarily-equivalent with the convolution operator U_μ (on the space $L^2(G, K)$).

Proof. (a) Let $T \in \mathcal{L}(H)$, such that $TP(A) = P(A)T$, for every $A \in \mathrm{Bor}(G)$. By applying Theorem 14 it results that there is a Hilbert space K and there are unitary operators $\Omega : H \to L^2(G, K)$ and $\widetilde{\Omega} : H \to L^2(\Gamma, K)$, such that $\widetilde{\Omega} = \Im \circ \Omega$; moreover, we have ((b) and (f) from Theorem 14):
(i) $\Omega\, P(A)\, \Omega^{-1} = E(A)$, for every $A \in \mathrm{Bor}(G)$.
(ii) $\widetilde{\Omega}\, \widehat{P}(B)\, \widetilde{\Omega}^{-1} = F(B)$, for every $B \in \mathrm{Bor}(\Gamma)$.
From (i) it results that the operator $\Omega\, T\, \Omega^{-1}$ commutes with the canonical spectral measure E on $L^2(G, K)$:

$$\Omega\, T\, \Omega^{-1}\, E(A) = \Omega\, T\, P(A)\, \Omega^{-1} = \Omega\, P(A)\, T\, \Omega^{-1} = E(A)\, \Omega\, T\, \Omega^{-1}.$$

By applying Theorem 4.6, p. 230 from ([7]) to the operator $\Omega\, T\, \Omega^{-1}$ it results that there is $\phi \in L^\infty(G, \mathcal{L}(K))$, such that $\Omega\, T\, \Omega^{-1} = M_\phi$. If there is $\mu \in M(\Gamma, \mathcal{L}(K))$, such that $\widehat{\mu} = \phi$, then Theorem 13(a) gives $M_{\widehat{\mu}} = \Im\, W_\mu\, \Im^{-1}$ (here W_μ denotes the convolution operator on $L^2(\Gamma, K)$), which ends the proof of (a)..
(b) Analogously, the operator $\widetilde{\Omega}\, T\, \widetilde{\Omega}^{-1}$ commutes with the canonical spectral measure F on $L^2(\Gamma, K)$. $\square$

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