GENERALISED ALGEBROIDS AND SECOND ORDER GEOMETRY

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Abstract

The aim of this paper is to give three new examples of generalised algebroids and groupoid-like structures, defind in the previous paper [7]. The generalised algebroids include the known definitions of Lie algebroid, prealgebroid and Courant algebroid and the new definition of a generalised prealgebroid. The non-trivial examples that are given in the paper extend those considered in the previous paper [7].

1 Relative Tangent Spaces and Almost Lie Structures

Let (θ, D) be an anchored vector bundle (AVB) (or a relative tangent space, defined in [3]), where $\theta = (R, q, M)$ is a vector bundle and is $D : \theta \to \tau M$ a vector bundle morphism of θ, called an *anchor* (an arrow, or a tangent map). Notice that we denoted as $\tau M = (TM, p, M)$ the tangent bundle of M.

If $\xi = (E, \pi, M)$ is an other vector bundle on the same base M, then consider the fibered product $RE = TE \times_{TM} R = \{(x, y) \in TE \times R : \pi_*(x) = D(y)\}$ of the differential $\pi_* : TE \to TM$ and of the given anchor $D : R \to \tau M$. Let $\Delta : RE \to TE$ be the canonical projection, $\bar{\pi}$ the canonical projection of the tangent bundle τE and $t = \bar{\pi} \circ \Delta$. Then the fibered manifold $R\xi = (RE, t, E)$ is a vector bundle, called the R-anchored vector bundle of ξ, and that the diagram:

$$\begin{array}{ccc} R & \stackrel{q}{\longrightarrow} & M \\ \uparrow s & & \uparrow \pi \\ RE & \stackrel{t}{\longrightarrow} & E \end{array} \tag{1}$$

commutes. Notice also that the above diagram has all the arrows as projections of vector bundles, the arrow t is a π-morphism and the arrow s is a q-morphism of vector bundles. The couple $(R\xi, \Delta)$ is an AVB.

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Following [5], for every morphism (f,g) of the AVB's (θ',D') and (θ,D) (i.e. $f:M'\to M$, $g:R'\to R$ such that (f,g) is a morphism of vector bundles and $D\circ g=f_*\circ D'$) , one can consider the R-differential of an $f-$vector bundle morphism $h:\xi'\to\xi$, denoted by $h^T:R\xi'\to R\xi$ and given by $h^T_{e'}(X',Y')=(h_{*,x'}X',\ g_{x'}(Y'))$, $(\forall)e'\in E'$, $\pi'(e')=x'$, $X'\in T'_{e'}E'$, $Y'\in R'_{x'}$, $\pi'_{*,e'}(X')=D_{x'}(Y')$.

Notice that the above construction depends on the AVB (θ,D). The *R-vertical* bundle of ξ related to the AVB (θ,D) is the vector subbundle $\ker t\overset{not}{=}VR\xi\subset R\xi$. If we denote the inclusion morphism as $i:VR\xi\to R\xi$, then a *nonlinear R-connection* on ξ related to the AVB (θ,D) is a left splitting of the inclusion i, i.e. is a vector bundle morphism $C:R\xi\to VR\xi$ such that $C\circ i=id_{VR\xi}$.

Notice also that the diagram (1) is a natural generalization of the diagram:

$$\begin{array}{ccc} TM & \overset{p}{\longrightarrow} & M \\ \uparrow\pi_* & & \uparrow\pi \\ TE & \overset{p'}{\longrightarrow} & E, \end{array} \tag{2}$$

where the left column is replaced in (1) by AVB's. This allows us to reconsider most of the formal constructions related to tangent bundles.

A *bracket* (or a Lie map) on an AVB (θ,D) is a map $[\cdot,\cdot]:\Gamma(\theta)\times\Gamma(\theta)\to\Gamma(\theta)$ which is bilinear over $I\!R$. An almost Lie structure (ALS) is a triple $(\theta,D,[\cdot,\cdot])$, where $[\cdot,\cdot]$ is a skew symmetric bracket such that $[X,fY]=(DX)(f)Y+f[X,Y]$, $(\forall)X,Y\in\Gamma(\theta)$ and $f\in\mathcal{F}(M)$.

In [4] it is indicated how to construct a differential calculus using an ALS and it is proved that there is a one to one correspondence between the ALS's on a vector bundle θ and the 1-degree derivations of the exterior algebra of θ^*.

An *algebroid* is an ALS $(\theta,D,[\cdot,\cdot])$ which enjoys the property that $[DX,DY]=D([X,Y])$, $(\forall)X,Y\in\Gamma(\theta)$, where the first bracket is the Lie bracket on $\mathcal{X}(M)$. A *Lie algebroid* is an algebroid $(\theta,D,[\cdot,\cdot])$ which has a null Jacobiator, i.e. $\mathcal{J}(X,Y,Z)\equiv\sum_{\text{cicl.}}[[X,Y],Z]=0$, $(\forall)$ $X,Y,Z\in\Gamma(\theta)$. Notice that an algebroid it characterized by the fact that its 1-derivation d enjoys the property $d^2f=0$, $(\forall)f\in\mathcal{F}(M)$, while a Lie algebroid it characterized by the fact that $d^2f=0$ and $d^2\omega=0$, $(\forall)f\in\mathcal{F}(M)$, $\omega\in\mathcal{A}^1(\theta)$, which mean $d^2=0$.

2 Generalized algebroids and groupoid-like structures

We exhibit now some related topics on generalized algebroids from [7]. For an AVB (θ,D), a *bracket* on θ is in the sequel an $I\!R$-linear map $[\cdot,\cdot]_\theta:\Gamma(\theta)\times\Gamma(\theta)\to\Gamma(\theta)$. We say that an $\mathcal{F}(M)$-submodule $\mathcal{M}\subset\Gamma(\theta)$ is *closed* if $[X,Y]_\theta\in\mathcal{M}$, whenever X or $Y\in\mathcal{M}$. We call the *derived module* of $\mathcal{M}\subset\Gamma(\theta)$ as being the $\mathcal{F}(M)$-module $\mathcal{D}er(\mathcal{M})\subset\Gamma(\theta)$ which is the intersection of all closed submodules of $\Gamma(\theta)$ which contain $\mathcal{M}$.

Definition 1 *Let (θ, D) be an AVB, $[\cdot,\cdot]_\theta$ be a bracket and $\mathcal{S} \subset \Gamma(\theta)$ be an $\mathcal{F}(M)$-submodule such that $D(X) = 0$, $(\forall)X \in \mathcal{S}$.*

*We say that $(\theta, D, [\cdot,\cdot]_\theta)$ is an $\mathcal{S}$-*algebroid *(or a* generalized algebroid *if no confusion arises) if the following properties are satisfied:*

(GA1) $\mathcal{J}(X,Y,Z) \in \mathcal{S}$, $(\forall)X,Y,Z \in \Gamma(\theta)$;

(GA2) $[X, f\cdot Y]_\theta - f\cdot[X,Y]_\theta - D(X)(f)Y \in \mathcal{S}$, $[f\cdot X, Y]_\theta - f\cdot[X,Y]_\theta + D(X)(f)Y \in \mathcal{S}$, $(\forall)X,Y \in \Gamma(\theta)$, $f \in \mathcal{F}(M)$;

(GA3) $[D(X), D(Y)] = D([X,Y]_\theta)$, $(\forall)X,Y \in \Gamma(\theta)$;

(GA4) $[X,Z]_\theta \in \mathcal{S}$, *whenever X or Z are in $\mathcal{S}$.*

where $\mathcal{J}(X,Y,Z) \overset{not}{=} [[X,Y]_\theta, Z]_\theta + [[Y,Z]_\theta, X]_\theta + [[Z,X]_\theta, Y]_\theta$, $(\forall)X,Y,Z \in \Gamma(\theta)$ *is the Jacobiator of the bracket.*

Notice that the bracket $[\cdot,\cdot]_\theta$ need not to be skew-symmetric.

It is easy to see that the conditions *(GA2)*, *(GA4)* and the fact that $\mathcal{S}$ is an $\mathcal{F}(M)$-module implies that $\mathcal{S}$ is closed.

For an $\mathcal{S}$-algebroid, denote as

$$\mathcal{M}_{\mathcal{J}} = \{\mathcal{J}(X,Y,Z) : X,Y,Z \in \Gamma(\theta)\} \tag{3}$$

and

$$\mathcal{M}_{\mathcal{L}} = [\{[X, fY]_\theta - f\,[X,Y]_\theta - D(X)(f)Y, [f\cdot X, Y]_\theta - f\cdot[X,Y]_\theta + D(X)(f)Y \mid (\forall)X,Y \in \Gamma(\theta),\ f \in \mathcal{F}(M)\}\,. \tag{4}$$

It is easy to see that $\mathcal{D}er(\mathcal{M}_{\mathcal{J}})$, $\mathcal{D}er(\mathcal{M}_{\mathcal{L}}) \subset \mathcal{D}er(\mathcal{M}_{\mathcal{J}} \cup \mathcal{M}_{\mathcal{L}}) \subset \mathcal{S}$.

If $\mathcal{S} = \{0\}$ we obtain the classical definition of a Lie algebroid.

We consider now the algebroid defined by J. Pradines in [8]:

Definition 2 *Let θ be a vector bundle, $D : \theta \to \tau M$ be an anchor and $[\cdot,\cdot]_\theta$ be a bracket on θ. We say that the triple $(\theta, D, [\cdot,\cdot]_\theta)$ is an* algebroid *provided that:*

(PA0) the bracket is skew-symmetric;

(PA1) $[X, f\cdot Y]_\theta = f\cdot[X,Y]_\theta + D(X)(f)\cdot Y$, $(\forall)X,Y \in \Gamma(\theta)$, $f \in \mathcal{F}(M)$;

(PA2) $[D(X), D(Y)] = D([X,Y]_\theta)$, $(\forall)X,Y \in \Gamma(\theta)$.

We say that $(\theta, D, [\cdot,\cdot]_\theta)$ is a right *or a* left generalized algebroid *if the condition* (PA0) *is removed and the condition* (PA1) *is replaced by:*

(PA1L) $[X, f\cdot Y]_\theta = f\cdot[X,Y]_\theta + D(X)(f)\cdot Y$, $(\forall)X,Y \in \Gamma(\theta)$, $f \in \mathcal{F}(M)$,

respectively by:

(PA1R) $[f \cdot X, Y]_\theta = f \cdot [X, Y]_\theta - D(Y)(f) \cdot X$, $(\forall) X, Y \in \Gamma(\theta)$, $f \in \mathcal{F}(M)$.

A simultaneous left and right generalized algebroid is just an algebroid. Notice that for an algebroid it follows that $\mathcal{M}_\mathcal{L} = \{0\}$.

The following results from [7] give characterizations for generalized algebroids, easily to handle.

Proposition 1 *Let θ be a vector bundle, $D : \theta \to \tau M$ be an anchor and $[\cdot, \cdot]$ be a bracket on θ which has the property $D([X, Y]_\theta) = [D(X), D(Y)]$, $(\forall) X, Y \in \Gamma(\theta)$.*

Then there is an $\mathcal{S}$-algebroid structure $(\theta, D, [\cdot, \cdot]_\theta)$, with $\mathcal{S} = \mathcal{D}er(\mathcal{M}_\mathcal{J} \cup \mathcal{M}_\mathcal{L})$, where $\mathcal{M}_\mathcal{J}$ and $\mathcal{M}_\mathcal{L}$ are given by the formulas (3) and (4) respectively.

Corollary 1 *If $(\theta, D, [\cdot, \cdot]_\theta)$ is an $\mathcal{S}$-algebroid then $(\theta, D, [\cdot, \cdot]_\theta)$ is a $\mathcal{D}er(\mathcal{M}_\mathcal{J} \cup \mathcal{M}_\mathcal{L})$-algebroid and $\mathcal{D}er(\mathcal{M}_\mathcal{J} \cup \mathcal{M}_\mathcal{L})$ is the minimal $\mathcal{S}$.*

Corollary 2 *An algebroid $(\theta, D, [\cdot, \cdot]_\theta)$ is an $\mathcal{S}$ -algebroid, with the minimal $\mathcal{S} = \mathcal{D}er(\mathcal{M}_\mathcal{J})$.*

Other algebroids are the Courant algebroids, which are defined by Liu-Weinstein-Xu in [1]. A Courant algebroid is an $\mathcal{S}$ -algebroid in three ways: one has an antisymmetric bracket, a second is a generalized right algebroid, and a third is a generalized left algebroid. All the above algebroids have the minimal $\mathcal{S} = \mathcal{D}er(\mathcal{M}_\mathcal{C})$, where $\mathcal{M}_\mathcal{C} = \{\mathcal{D}(f) \mid f \in \mathcal{F}(M)\}$ (see [1] and [7] for more details).

Notice that the generalized algebroid is a progress for the "Open Problem 3" from [1]: "*What is the geometric meaning of such asymmetric bracket, satisfying most of the axioms of a Lie algebroid ?*"

Starting from the "Open problem 5" proposed in [1] for Courant algebroids, the following problem can be suggested for generalized algebroids: "*Find the global groupoid-like object corresponding to a generalized algebroid.*"

A possible answer to this problem is given in [7], considering generalized algebroids which can be associated with groupoid-like objects. In fact, the construction of the Lie algebroid of a differentiable groupoid is extended to the construction of a generalized algebroid which can be defined using a surjective map and a fibered manifold with some groupoid-like properties. We present below this construction made in [7].

Definition 3 *We say that $G \overset{\alpha}{\underset{\beta}{\rightrightarrows}} G_0$ is a* groupoid-like *structure provided that the following properties are satisfied:*

(GL1) *A surjective map $\alpha : G \to G_0$ and a fibered manifold $\beta : G \to G_0$ (i.e. β is a surjective submersion) are given, denoted as $G \overset{\alpha}{\underset{\beta}{\rightrightarrows}} G_0$.*

(GL2) *G_0 is a submanifold of G and the inclusion map $i : G_0 \to G$ is a right inverse of both α and β, (i.e. $\alpha \circ i = \beta \circ i = id_{G_0}$).*

(GL3) *Consider the β-vertical vector bundle $\ker\beta_*$ and the induced vector bundle $\mu = i^*\ker\beta_*$, which is the restriction of $\ker\beta_*$ to G_0. Suppose that for every section $s \in \Gamma(\mu)$ there is a β-vertical section denoted as $\alpha^*s \in \Gamma(\ker\beta_*)$, such that $\alpha_*(\alpha^*s)$ is a field on G_0 and $s \to .\alpha_*(\alpha^*s)$ is an anchor on μ.*

(GL4) *For every $s,t \in \Gamma(\mu)$, consider the sections $U = [\alpha^*s, \alpha^*t] \in \Gamma(\ker\beta_*)$ and $u = i^*U \in \Gamma(\mu)$ and suppose that $\alpha_*(U - \alpha^*u) = 0$.*

An important particular case is when $\Gamma(\mu) \ni s \to \alpha^*s \in \Gamma(\ker\beta_*)$ defines an i-morphism of vector bundles μ and $\ker\beta_*$. It is not the only possible case, as we shall see later, but it is the case when $G \overset{\alpha}{\underset{\beta}{\rightrightarrows}} G_0$ is a differentiable groupoid (see [2] for more details).

Thus a differentiable groupoid is a groupoid-like structure. A differentiable groupoid defines canonically a Lie algebroid. For a groupoid-like structure we can prove a similar result:

Proposition 2 *Every groupoid-like structure $G \overset{\alpha}{\underset{\beta}{\rightrightarrows}} G_0$ defines canonically a generalized algebroid with an antisymmetric bracket.*

Proof. The bracket is defined by the formula:

$$[s,t]_\mu = i^*[\alpha^*s, \alpha^*t], \tag{5}$$

thus the bracket is antisymmetric. From *(GL3)*, (*GL4)* and the definition of D it follows that the condition *(GA3)* holds true. We can use now proposition 1 and the conclusion follows.□

In order to get a bracket which is not necessary antisymmetric, the condition *(GL4)* from Definition 3 can be replaced with the condition:

(GL4') For every $s,t \in \Gamma(\mu)$, consider the sections $U = [\alpha^*s, \alpha^*t] \in \Gamma(\ker\beta_*)$ and $u = i^*(\alpha^*(i^*U)) \in \Gamma(\mu)$ and suppose that $\alpha_*(U - \alpha^*u) = 0$.

In this case the proposition 2 can be reformulated as follows: "*Every groupoid-like structure $G \overset{\alpha}{\underset{\beta}{\rightrightarrows}} G_0$ defines canonically a generalized algebroid.*" We do not deal with this case in the sequel .

3 Some examples of generalized groupoids and algebroids

We give in that follows three non-trivial examples of groupoid-like objects, which are not groupoids in the usual sense, and we construct their generalized algebroids given by proposition 2. These constructions extend those from [7].

Consider a smooth manifold M, an AVB (D,θ), $\theta = (R,\pi,M)$, the R-tangent bundles $R\theta = (RR,p,R)$ and $RR\theta = (RRR,q,RR)$. We say that $S : R \to RR$ is an R-semispray on θ if S is simultaneously a section in the vector bundles $R\theta$ (the R-tangent bundle) and $R\theta = (RR, \Pi, R)$, where Π is the canonical projection on R in the diagram

$$\begin{array}{ccc} R & \xrightarrow{\pi} & M \\ \uparrow \Pi & & \uparrow \pi \\ RR & \xrightarrow{p} & R\,. \end{array} \tag{6}$$

Thus we can consider that S is the inclusion of R in RR. We are going to show that $RR \overset{\alpha}{\underset{\beta}{\rightrightarrows}} R$ is a groupoid-like structure, in three suitable manners, extending the case $\theta = \tau M$ considered in [7]. In order to construct these structures, we use local coordinates. .

Let (x^i) be local coordinates on M, which change by $x^{i'} = x^{i'}(x^i)$. We can assume that these coordinates induce:

the vectorial coordinates (x^i, y^α) on R, which change following the formulas:

$$x^{i'} = x^{i'}(x^i), y^{\alpha'} = g_\alpha^{\alpha'} y^\alpha; \tag{7}$$

the vectorial coordinates $(x^i, y^\alpha, X^\beta, Y^\gamma)$ on RR, which change following the formulas (7) and

$$X^{\beta'} = g_\beta^{\beta'} X^\beta, Y^{\gamma'} = D_\gamma^i g_{\beta,i}^{\gamma'} y^\beta X^\gamma + g_\gamma^{\gamma'} Y^\gamma; \tag{8}$$

the vectorial coordinates $(x^i, y^\alpha, X^\beta, Y^\gamma, A^\delta, B^\eta, C^\nu, D^\mu)$ on RRR, which change following the formulas (7), (8) and

$$A^{\delta'} = g_\delta^{\delta'} A^\delta, B^{\eta'} = D_\eta^i g_{\beta,i}^{\eta'} y^\beta A^\eta + g_\eta^{\eta'} B^\eta, C^{\nu'} = D_\nu^i g_{\beta,i}^{\nu'} X^\beta A^\nu + g_\nu^{\nu'} C^\nu,$$

$$D^{\mu'} = \left(D_\delta^i \left(D_\beta^j g_{\alpha,j}^{\mu'} \right)_{,i} y^\alpha X^\beta + D_\delta^i g_{\gamma,i}^{\mu'} Y^\gamma \right) A^\delta + D_\beta^i g_{\eta,i}^{\mu'} X^\beta B^\eta +$$
$$D_\nu^i g_{\alpha,i}^{\nu'} y^\alpha C^\nu + g_\mu^{\mu'} D^\mu. \tag{9}$$

Following [4, 5], the R-differential of an $f-$vector bundle morphism $h : \xi' \to \xi$, denoted by $h^T : R\xi' \to R\xi$ is given in local coordinates by:

$$h^T(x^{i'}, y^{a'}, X^{\beta'}, Y^{b'}) = \left(f^i(x^{i'}), h_{a'}^a y^{a'}, g_{\alpha'}^\alpha X^{\alpha''}, \frac{\partial h_{c'}^b}{\partial x^{j'}} D_{\mu'}^{j'} X^{\mu'} + h_{b'}^b Y^{b'} \right).$$

The local form of the R-semispray S is $(x^i, y^\alpha) \to (x^i, y^\alpha, y^\alpha, S^\beta(x^i, y^\alpha))$, so it is the local form of the inclusion of R in RR, considered here. The local coordinates for $\ker(\Pi)^T$ are $(x^i, y^\alpha, X^\beta, Y^\gamma, B^\eta, D^\mu)$. The local coordinates on the fibres of the vector bundle $\ker(\Pi)^T$ are (B^η, D^μ), which change according the formulas:

$$B^{\eta'} = g_\eta^{\eta'} B^\eta,\ D^{\mu'} = D_\beta^i g_{\eta,i}^{\mu'} X^\beta B^\eta + g_\mu^{\mu'} D^\mu. \tag{10}$$

Restricting to R the above formula, we get:

$$B^{\eta''} = g^{\eta'}_{\eta} B^{\eta},\ D^{\mu'} = D^i_{\beta} g^{\mu'}_{\eta,i} y^{\beta} B^{\eta} + g^{\mu'}_{\mu} D^{\mu}, \tag{11}$$

thus it follows that the restriction of the vector bundle $\ker(\Pi)^T$ to R is a vector bundle μ which is isomorphic with the vector bundle $R\theta$, $\Phi : \mu \to R\theta$. Let us denote by $\{s'_{\gamma}, \dfrac{\partial'}{\partial Y^l}\}$ the local base of the sections of this vector bundle. Notice that Φ sends isomorphic the restriction of $\xi = \ker p^T \cap \ker(\Pi)^T$ to R into the vertical bundle $VR\theta = \ker \Pi$ of $R\theta$.

Notice that ξ is a vector subbundle of the vector bundles $\ker p^T$ and $\ker(\Pi)^T$, ξ has the base R and the local coordinates (D^{μ}) on the fibres. Consider a left splitting $P : \ker(\Pi)^T \to \xi$ of the inclusion $I : \xi \to \ker(\Pi)^T$ and the reduction of $\ker(\Pi)^T$ as the Whitney sum $\ker P \oplus \operatorname{Im} P$. The local form on fibres of the splitting P is $(B^{\eta}, D^{\mu}) \to (D^{\mu} + P^{\mu}_{\eta}(x^i, y^{\alpha}, X^{\beta}, Y^{\gamma})B^{\eta})$. Consider the local base of sections on $\Gamma(\ker(\Pi)^T)$, adapted to this decomposition:

$$\left\{\frac{\delta}{\delta y^{\alpha}} = \frac{\partial}{\partial y^{\alpha}} - P^{\beta}_{\alpha}\frac{\partial}{\partial Y^{\beta}}, \frac{\partial}{\partial Y^{\nu}}\right\} \tag{12}$$

We say that the local functions $\{P^{\beta}_{\alpha}\}$ are the local components of the splitting P. They change according to the formula:

$$P^{\alpha'}_{\beta'} g^{\beta'}_{\alpha} = P^{\gamma}_{\alpha} g^{\alpha'}_{\gamma} - g^{\alpha'}_{\beta,i} D^i_{\alpha} X^{\beta}.$$

Restricted to R, the above formula becomes:

$$\bar{P}^{\alpha'}_{\beta'} g^{\beta'}_{\alpha} = \bar{P}^{\gamma}_{\alpha} g^{\alpha'}_{\gamma} - g^{\alpha'}_{\beta,i} D^i_{\alpha} y^{\beta},$$

where $\bar{P}^{\alpha'}_{\beta'} = P^{\alpha'}_{\beta'}(x^{i'}, y^{\gamma'}, y^{\gamma'}, S^{\delta'}(x^{i'}, y^{\gamma'}))$ and $\bar{P}^{\alpha}_{\beta} = P^{\alpha}_{\beta}(x^i, y^{\gamma}, y^{\gamma}, S^{\delta}(x^i, y^{\gamma}))$, thus the local functions $\{\bar{P}^{\alpha}_{\beta}\}$ are the local components of a non-linear R-connection on the vector bundle θ, defined by a left splitting $\bar{P}' : R\theta \to \ker \Pi$ of the canonical inclusion of the vertical bundle $i : \ker \Pi \to R\theta$. Taking into account the isomorphism $\Phi : \mu \to R\theta$, which sends $\xi_{|R}$ isomorphic onto $\ker \Pi$, it follows that a left splitting $\bar{P} : \mu \to \xi_{|R}$ of the canonical inclusion is induced. The splitting $\bar{P}$ defines a local base of sections adapted to the reduction of μ as the Whitney sum $\ker \bar{P} \oplus \operatorname{Im} \bar{P}$:

$$\left\{\frac{\delta'}{\delta y^{\alpha}} = \frac{\partial'}{\partial y^{\alpha}} - \bar{P}^{\beta}_{\alpha}\frac{\partial'}{\partial Y^{\beta}}, \frac{\partial'}{\partial Y^{\eta}}\right\}. \tag{13}$$

where $\dfrac{\partial'}{\partial y^{\alpha}} = S^*\dfrac{\partial}{\partial y^{\alpha}}$ and $\dfrac{\partial'}{\partial Y^{\eta}} = S^*\dfrac{\partial}{\partial Y^{\eta}}$ are sections induced by S. Consider the induced vector bundle $(\Pi)^*\mu$ and denote as $\dfrac{\bar{\delta}}{\delta y^{\alpha}} = (\Pi)^*\dfrac{\delta'}{\delta y^{\alpha}}$ and $\dfrac{\bar{\partial}}{\partial Y^{\eta}} = (\Pi)^*\dfrac{\partial'}{\partial Y^{\eta}}$.

We can construct now an isomorphism of the vector bundles $\ker(\Pi)^T$ and $(\Pi)^*\mu$, making the natural associations:

$$\frac{\delta}{\delta y^{\alpha}} \to \frac{\bar{\delta}}{\delta y^{\alpha}}\ ,\ \frac{\partial}{\partial Y^{\eta}} \to \frac{\bar{\partial}}{\partial Y^{\eta}}\ . \tag{14}$$

According to this isomorphism, for a section $s \in \Gamma(\mu)$, which has the local form

$$s = a^{\alpha}(x^i, y^{\beta}) \frac{\bar{\delta}}{\delta y^{\alpha}} + b^{\eta}(x^i, y^{\beta}) \frac{\bar{\partial}}{\partial Y^{\eta}}, \tag{15}$$

the corresponding section $(\Pi)^T s \in \Gamma(\ker(\Pi)^T)$ has the local form

$$(\Pi)^T s = a^j(x^i, y^j) \frac{\delta}{\delta y^j} + b^l(x^i, y^j) \frac{\partial}{\partial Y^l} \ .$$

It is easy to see that a (global) isomorphism $\alpha^* \mu \to \ker(\Pi)^T$ is defined in this way. Since: $p^T(\frac{\delta}{\delta y^{\alpha}}) = \frac{\partial}{\partial y^{\alpha}}$ and $p^T(\frac{\partial}{\partial Y^{\eta}}) = 0$, it follows that

$$p^T\left((\Pi)^* s\right) = a^{\beta} \frac{\partial}{\partial y^{\beta}} \ . \tag{16}$$

The conditions *(GL1)* and *(GL2)* are fulfilled taking $RR \overset{p}{\underset{\pi_*}{\rightrightarrows}} R$ and the inclusion given by the R-semispray S. The above construction shows that a splitting P of the inclusion $I : \ker p^T \cap \ker(\Pi)^T \to \ker(\Pi)^T$ makes possible the condition *(GL3)*. In order to verify the condition *(GL4)*, take $s, t \in \Gamma(RR)$, which has the local form (15) for s and the local form

$$t = c^{\gamma}(x^i, y^{\beta}) \frac{\bar{\delta}}{\delta y^{\gamma}} + d^{\nu}(x^i, y^{\beta}) \frac{\bar{\partial}}{\partial Y^{\nu}} \tag{17}$$

for t. Then:

$$[(\Pi)^* s, (\Pi)^* t] =$$

$$\left[a^{\alpha}(x^i, y^{\beta}) \frac{\delta}{\delta y^{\alpha}} + b^{\eta}(x^i, y^{\beta}) \frac{\partial}{\partial Y^{\eta}}, c^{\gamma}(x^i, y^{\beta}) \frac{\delta}{\delta y^p} + d^{\nu}(x^i, y^{\beta}) \frac{\partial}{\partial Y^{\nu}} \right] = \tag{18}$$

$$\left(a^{\alpha} \frac{\partial d^{\eta}}{\partial y^{\alpha}} - c^{\alpha} \frac{\partial b^{\eta}}{\partial y^{\alpha}} + a^{\alpha} c^{\beta} \Omega^{\eta}_{\alpha\beta} \right) \frac{\partial}{\partial Y^{\eta}} + \left(a^{\alpha} \frac{\partial c^{\gamma}}{\partial y^{\alpha}} - c^{\alpha} \frac{\partial a^{\gamma}}{\partial y^{\alpha}} \right) \frac{\delta}{\delta y^{\gamma}},$$

where:

$$\Omega^{\eta}_{\alpha\beta} \frac{\partial}{\partial Y^{\eta}} = \left[\frac{\delta}{\delta y^{\alpha}}, \frac{\delta}{\delta y^{\beta}} \right].$$

It is easy to see that the condition *(GL4)* is fulfilled. Thus a splitting P of the inclusion $I : \ker p^T \cap \ker(\Pi)^T \to \ker(\Pi)^T$ and an R-semispray S define canonically a groupoid-like structure on $RR \overset{p}{\underset{\pi_*}{\rightrightarrows}} R$, where the inclusion is S.

Let us construct the corresponding generalized algebroid. For $s, t \in \Gamma(\mu)$, which has the local forms (15) and (17) respectively, we have:

the local form for the anchor

$$D(s) = a^{\alpha} \frac{\partial}{\partial y^{\alpha}} \tag{19}$$

from (16) and

the local form for the bracket

$$[s,t] = \left(a^\alpha \frac{\partial d^\eta}{\partial y^\alpha} - c^\alpha \frac{\partial b^\eta}{\partial y^\alpha} + a^\alpha c^\beta \tilde{\Omega}^\eta_{\alpha\beta}\right) \frac{\partial'}{\partial Y^\eta} + \left(a^\alpha \frac{\partial c^\gamma}{\partial y^\alpha} - c^\alpha \frac{\partial a^\gamma}{\partial y^\alpha}\right) \frac{\delta'}{\delta y^\gamma}$$

from (18), where: $\tilde{\Omega}^\eta_{\alpha\beta}(x^i, y^\gamma) = \Omega^\eta_{\alpha\beta}(x^i, y^\gamma, y^\gamma, S^l(x^i, y^\gamma))$. It is easy to see that the generalized algebroid is in fact an algebroid with an antisymmetric bracket.

Let us return to the general case and use the notations from Definition 3. In [7] there is proved the following result:

Proposition 3 *If* $G \overset{\alpha}{\underset{\beta}{\rightrightarrows}} G_0$ *is a groupoid-like structure,* $\Gamma(\mu) \ni s \to \alpha^* s \in \Gamma(\ker \beta_*)$ *defines an i-morphism of vector bundles* μ *and* $\ker \beta_*$ *and*

$$i^*(\alpha^* t) = t, (\forall) t \in \Gamma(\mu), \tag{20}$$

then the generalized algebroid given by proposition 2 is an algebroid with an antisymmetric bracket.

Notice that in the case when a vector bundle morphism $\mu \to \ker \beta^T$ is induced, the relation (20) must fail, in order to get to a generalized algebroid which is not an algebroid.

We construct now the second example, which illustrates the above assertion. Taking instead of (14) a new isomorphism given by

$$\frac{\delta}{\delta y^\alpha} \to \frac{\bar{\delta}}{\delta y^\alpha}\,,\ \frac{\partial}{\partial Y^\eta} \to -\frac{\bar{\partial}}{\partial Y^\eta}\,. \tag{21}$$

then the conditions *(GL1)-(GL4)* are fulfilled. We get the formula (19) for the anchor and the following formula:

$$[s,t] = \left(-a^\alpha \frac{\partial d^\eta}{\partial y^\alpha} + c^\alpha \frac{\partial b^\eta}{\partial y^\alpha} + a^\alpha c^\beta \tilde{\Omega}^\eta_{\alpha\beta}\right) \frac{\partial'}{\partial Y^\eta} + \left(a^\alpha \frac{\partial c^\beta}{\partial y^\alpha} - c^\alpha \frac{\partial a^\beta}{\partial y^\alpha}\right) \frac{\delta'}{\delta y^\beta}.$$

for the bracket. It is easy to see that the generalized algebroid is not an algebroid, since the condition *(PA1)* does not hold true.

We give now a third example, when the association $\Gamma(\mu) \ni s \to \alpha^* s \in \Gamma(\ker \beta_*)$ is not a vector bundle morphism, but an affine morphism of vector bundles.

Consider the section $X_0 \in \Gamma(\xi)$ given by $X_0 = y^\alpha \dfrac{\partial}{\partial Y^\alpha}$. It is easy to see that it is globally defined. For $s \in \Gamma(\mu)$, given locally by the formula (15), we define $(\Pi)^* s \in \Gamma(\ker \beta_*)$ using the formula:

$$(\Pi)^* s = X_0 + a^\alpha(x^i, y^\beta) \frac{\delta}{\delta y^\alpha} + b^\eta(x^i, y^\beta) \frac{\partial}{\partial Y^\eta}\,.$$

It is easy to see that $(\Pi)^*$ is an affine morphism on fibres and the conditions *(GL1)-(GL4)* are fulfilled. We get the formula (19) for the anchor and the following formula :

$$[s,t] = \left(a^\eta - c^\eta + a^\alpha \frac{\partial d^\eta}{\partial y^\alpha} - c^\alpha \frac{\partial b^\eta}{\partial y^\alpha} + a^\alpha c^\beta \tilde{\Omega}^\eta_{\alpha\beta}\right) \frac{\partial'}{\partial Y^p} + \left(a^\alpha \frac{\partial c^\gamma}{\partial y^\alpha} - c^\alpha \frac{\partial a^\gamma}{\partial y^\alpha}\right) \frac{\delta'}{\delta y^\gamma}.$$

for the bracket. It is easy to see that the generalized algebroid is not an algebroid, since the condition *(PA1)* does not hold true, but the bracket is antisymmetric.

An open problem is to find a Courant algebroid defined by a groupoid-like object. We think that the setting taken in the above examples can be used in order to construct a Courant algebroid which is not a double of a Lie bialgebroid (this problem was proposed in [1] as 'Open Problem 3').

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