

A GENERALIZATION OF THE GILKEY–BRANSON–FULLING FORMULA

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Abstract

A generalization of the Gilkey–Branson–Fulling formula is given.

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1 Introduction

Let M be a closed, n -dimensional, orientable, Riemannian manifold and h a pointwise non-singular $(1,1)$ tensor field on M with vanishing Nijenhuis tensor. The harmonic theory due to W.V.D. Hodge ([4]) was extended by P.R. Eisenman and A.P. Stone ([2]) to include the influence of such tensor fields on M . More precisely, for such an h there is an associated exterior derivative d_h that also has an adjoint δ_h with respect to the usual global inner product given by

$$(\alpha, \beta) = \int_M \alpha \wedge * \beta,$$

where α and β are p -forms and $*$ denotes the Hodge star operator. Thus one can define a generalized Laplace–Beltrami operator $\Delta_h^{(p)}$ on $A^p(M)$ – the space of p -forms on M by setting

$$\Delta_h^{(p)} = d_h \delta_h + \delta_h d_h.$$

This construction leads to a generalization of the classical Hodge decomposition theorem ([2], [6]) and of the classical de Rham theorem ([2]). Also spectral properties of the operator $\Delta_h^{(p)}$ can be found in [1], [5] and [6].

The main goal of this paper is to present a generalization of the Gilkey–Branson–Fulling formula ([3]) in this framework.

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2 The main results

Let $D_h^{(p)}$ be the natural second-order differential operator on $A^p(M)$ given by

$$D_h^{(p)} = a^2 d_h \delta_h + b^2 \delta_h d_h,$$

where $a, b \in \mathbf{R}$. It is known that $D_h^{(p)}$ is a *self-adjoint*, positive and elliptic operator with positive definite leading symbol.

Let $\{\lambda_m\}$ be the eigenvalues of $D_h^{(p)}$. Then the (global) trace of the fundamental solution of the corresponding heat equation is given by

$$\text{Tr}(e^{-tD_h^{(p)}}) = \sum_{m=0}^{\infty} e^{-t\lambda_m}.$$

As $t \rightarrow 0^+$, there is an asymptotic expansion of the form:

$$\text{Tr}(e^{-tD_h^{(p)}}) \sim (4\pi)^{-\frac{n}{2}} \sum_{m=0}^{\infty} a_m(D_h^{(p)}) t^{(2m-2n)/2},$$

where $a_m(D_h^{(p)})$ are local invariants of the differential operator $D_h^{(p)}$. Now we have:

Theorem 2.1. *Let $D_h^{(p)} = a^2 d_h \delta_h + b^2 \delta_h d_h$ and $\Delta_h^{(p)} = d_h \delta_h + \delta_h d_h$ acting on $A^p(M)$. Then*

$$a_m^{(p)}(D_h^{(p)}) = b^{2m-n} a_m^{(p)}(\Delta_h^{(p)}) + (b^{2m-n} - a^{2m-n}) \sum_{j \leq p} (-1)^{j-p} a_m^{(j)}(\Delta_h^{(p)}),$$

for each $p \in \mathbf{N}$, $0 \leq p \leq n$.

Corollary 2.1. *In the particular case when h is the identity we find again the formula of Gilkey–Branson–Filling ([3]).*

Proof. (of Theorem 2.1)

Let

$$\begin{aligned} \Delta_{h,d_h}^{(p)} &= \Delta_h^{(p)}|_{\text{Im } d_h}, \\ \Delta_{h,\delta_h}^{(p)} &= \Delta_h^{(p)}|_{\text{Im } \delta_h} \end{aligned}$$

and for each $t > 0$

$$\begin{aligned} f(t, d_h^{(p)}) &= \text{Tr}(e^{-t\Delta_{h,d_h}^{(p)}}), \\ f(t, \delta_h^{(p)}) &= \text{Tr}(e^{-t\Delta_{h,\delta_h}^{(p)}}), \\ f(t, \Delta_h^{(p)}) &= \text{Tr}(e^{-t\Delta_h^{(p)}}), \\ \beta_p &= \dim \ker(\Delta_h^{(p)}) = \text{the Betti number of } M. \end{aligned}$$

Then we have via the Hodge decomposition theorem:

$$f(t, \Delta_h^{(p)}) = \beta_p + f(t, d_h^{(p)}) + f(t, \delta_h^{(p)}). \quad (3.1)$$

Since

$$d_h \Delta_{h, \delta_h}^{(p-1)} = \Delta_{h, \delta_h}^{(p-1)} d_h$$

and

$$d_h \Delta_{h, d_h}^{(p)} = \Delta_{h, d_h}^{(p)} d_h$$

it follows that

$$f(t, d_h^{(p)}) = f(t, \delta_h^{(p-1)}). \quad (3.2)$$

Now we can write successively:

$$\begin{aligned} f(t, \delta_h^{(p)}) &\stackrel{(3.1)}{=} f(t, \Delta_h^{(p)}) - \beta_p - f(t, d_h^{(p)}) \\ &\stackrel{(3.2)}{=} f(t, \Delta_h^{(p)}) - \beta_p - f(t, \delta_h^{(p-1)}) \\ &\stackrel{(3.1)}{=} f(t, \Delta_h^{(p)}) - \beta_p - [f(t, \Delta_h^{(p-1)}) - \beta_{p-1}] \\ &\quad + f(t, \Delta_h^{(p-2)}) - \beta_{p-2} - f(t, \delta_h^{(p-3)}) \\ &= \dots \\ &= f(t, \Delta_h^{(p)}) - \beta_p [f(t, \Delta_h^{(p-1)}) - \beta_{p-1}] [f(t, \Delta_h^{(p-2)}) - \beta_{p-2}] \\ &\quad + \dots + f(t, \Delta_h^{(0)}) - \beta_0 \\ &= \sum_{j \leq p} (-1)^{p-j} [f(t, \Delta_h^{(j)}) - \beta_j]. \end{aligned}$$

On the other hand

$$\begin{aligned} f(t, d_h^{(p)}) &= f(t, \delta_h^{(p-1)}) \\ &= \sum_{j \leq p-1} (-1)^{p-j} [f(t, \Delta_h^{(j)}) - \beta_j]. \end{aligned}$$

If we make now the same computations for the operator $D_h^{(p)}$ we obtain:

$$\begin{aligned} f(t, D_h^{(p)}) &= \text{Tr}(e^{-tD_h^{(p)}}) \\ &= \beta_p + f(a^2 t, d_h^{(p)}) + f(b^2 t, \delta_h^{(p)}) \\ &= \beta_p + \sum_{j \leq p} (-1)^{p-j} [f(b^2 t, \Delta_h^{(j)}) - \beta_j] \\ &\quad - \sum_{j < p} (-1)^{p-j} [f(a^2 t, \Delta_h^{(j)}) - \beta_j] \\ &= f(b^2 t, \Delta_h^{(p)}) + \sum_{j < p} (-1)^{p-j} [f(b^2 t, \Delta_h^{(j)}) - \beta_j] \\ &\quad - f(a^2 t, \Delta_h^{(j)}). \end{aligned}$$

Using now the asymptotic development of Minakshisundaram–Pleijel type we have:

$$f(t, D_h^{(p)}) = (4\pi t)^{-\frac{n}{2}} \sum_{k=0}^{\infty} a_k^{(p)}(D_h^{(p)}) t^k,$$

$$f(b^2 t, \Delta_h^{(j)}) = (4\pi t)^{-\frac{n}{2}} b^{-n} \sum_{k=0}^{\infty} a_k^{(p)}(\Delta_h^{(p)}) b^{2k} t^k,$$

$$f(a^2 t, \Delta_h^{(j)}) = (4\pi t)^{-\frac{n}{2}} a^{-n} \sum_{k=0}^{\infty} a_k^{(p)}(\Delta_h^{(p)}) a^{2k} t^k$$

and then we can conclude that:

$$a_m^{(p)}(D_h^{(p)}) = b^{2m-n} a_m^{(p)}(\Delta_h^{(p)})$$

$$+ (b^{2m-n} - a^{2m-n}) \sum_{j < p} (-1)^{j-p} a_p^{(j)}(\Delta_h^{(j)}),$$

as desired. \square

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