STUDY OF LONGITUDINAL VIBRATIONS OF A COMPOSITE ROD USING SPLINE FUNCTIONS

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Abstract

In this work, we introduce a method for studying the longitudinal vibrations of a composite rod. Using a discretisation procedure with respect to the spatial variable, and the spline functions we state a finite element type method to aproximate the solution. Besides the general advantages of the methods based on finite elements, this approach has a concise statement and requires a small computational effort.

AMS Subject Classification: 35A35, 65M06.
Key words: mixed problem, spline functions, numerical modelling.

1 Preliminaires

Let be given the operator $\mathcal{A}: C^2([0,+\infty[\times[0,\ell];\mathbf{R}) \to C^0([0,+\infty[\times[0,\ell];\mathbf{R})$, defined as

$$(\mathcal{A}u)(t,x) = \frac{\partial^2 u}{\partial t^2}(t,x) - c^2\frac{\partial^2 u}{\partial x^2}(t,x) - f(t,x,u(t,x)),$$

where c is a constant and $f: [0,+\infty[\times[0,\ell]\times\mathbf{R} \to \mathbf{R}$ is a continuous function with the property that $f(\cdot,x,u(x,\cdot)): [0,\alpha] \to \mathbf{R}$ is integrable for all $\alpha \in]0,+\infty[, \forall x \in [0,\ell]$ and satisfies the condition of Lipschitz with respect to u, namely exists $M > 0$ such that

$$|f(t,x,u) - f(t,x,\tilde{u})| \leq M \cdot |u - \tilde{u}|, \quad \forall (t,x) \in [0,+\infty[\times[0,\ell] \text{ and } \forall u,\tilde{u} \in \mathbf{R}.$$

In [6], is stated the result in the following.

Theorem 1.1 *Let u be the solution of the mixed values problem*

$$\begin{aligned}
&\mathcal{A}u = 0,\\
&u(t,0) = u_0(t),\ \ u(t,\ell) = u_1(t),\\
&u(0,x) = u_2(x),\ \ \frac{\partial u}{\partial t}(0,x) = u_3(x), \quad \forall x \in [0,\ell],
\end{aligned}$$

Editor Gr.Tsagas *Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, 1997*, 83-89

and $v \in C^2([0,+\infty[\times[0,\ell];\mathbf{R})$ *a function such that*

$$|(\mathcal{A}v)(t,x)| \le \delta(t,x),$$
$$|u_1(t)-v_1(t)| < \varepsilon,\ t\in[0,\alpha],\ \text{where } v(t,\ell)=v_1(t),$$
$$\left|\frac{\partial u}{\partial t}(0,x)-\frac{\partial v}{\partial t}(0,x)\right| < \varepsilon_1,\quad \varepsilon>0,\ \varepsilon_1>0.$$

Then, we have

$$|u(t,x)-v(t,x)| \le Ae^{t\sqrt{M}}+Be^{-t\sqrt{M}}-\frac{\delta}{M},\quad \forall x\in[0,\ell],$$

where

$$A=\frac{1}{2}\left(\varepsilon+\frac{\delta+\varepsilon_1}{M}\right),\ B=\frac{1}{2}\left(\varepsilon+\frac{\delta-\varepsilon_1}{M}\right)$$

Moreover, if $M\to 0$ *then*

$$|u(t,x)-v(t,x)| \le \varepsilon+\varepsilon_1 t+\frac{\delta}{2}t^2,\quad \forall x\in[0,\ell].$$

Now, let be given V a composite rod of length $\ell>0$ with cross-sectional area S. If $F\colon[0,+\infty[\times[0,\ell]\to\mathbf{R}$ is a continuous function representing the perturbation force distributed along the rod, E the elasticity constant, and ρ the density, then the equation of longitudinal vibrations of the rod is given as ([4]):

$$-SE\frac{\partial^2 u}{\partial x^2}(t,x)+\rho S\frac{\partial^2 u}{\partial t^2}=F(t,x),\quad x\in[0,\ell],\ t\in[0,\alpha]. \tag{1}$$

If denote

$$f(t,x,u(t,x))=\frac{1}{\rho S}F(t,x) \text{ and } c^2=\frac{E}{\rho},$$

then the equation (1) can be written as

$$\frac{\partial^2 u}{\partial t^2}(t,x)-c^2\frac{\partial^2 u}{\partial x^2}(t,x)-f(t,x,u(t,x))=0 \text{ or } \mathcal{A}u=0. \tag{2}$$

If consider the rod fixed at its ends, then we get the boundary conditions

$$u(t,0)=u(t,\ell)=0,\quad \forall t\in[0,\alpha], \tag{3}$$

and the initial conditions

$$\begin{cases} u(0,x)=\varphi(x), \\ \dfrac{\partial u}{\partial t}(0,x)=g(x), \end{cases} \quad x\in[0,\ell], \tag{4}$$

that is, at the moment $t=0$, is given the amplitude of each point and its velocity. Suppose that $\varphi, g\in C^0([0,\ell];\mathbf{R})$ and $\varphi(0)=\varphi(\ell)=0$. Hence, we obtained a mixed values problem (2), (3), (4).

In the following, we shall transform this problem in n numerical problems using spline functions.

2 Main result

Consider $n \in \mathbf{N}$, $n \geq 2$ and put $h = \dfrac{\ell}{n}$. Let $\mathcal{P}(t, x)$ be the cubic spline function [3] which approximate the function u on the rectangle $[0, \ell] \times [0, \alpha]$. Denote

$$X_n = \{x_i / x_i = ih, \quad i = 0, 1, \ldots, n\},$$

and

$$\begin{cases} U_i(t) = \dfrac{\partial^2 \mathcal{P}}{\partial x^2}(t, x_i), \\ U_i(t) = u(t, x_i), \end{cases} \quad x_i \in X_n, \quad i = 0, 1, \ldots, n.$$

We consider the functions $\varphi_i \in C^2([0, \alpha]; \mathbf{R})$ solutions of the differential system

$$\frac{d^2 \varphi_i}{dt^2}(t) = c^2 U_i(t) + f(t, x_i), \quad t \in [0, \alpha],\ i = 1, \ldots, n-1, \tag{5}$$

where U_i depends on φ_i, and suppose the following initial conditions hold

$$\begin{cases} U_i(0) = \varphi(x_i), \\ \dfrac{d\varphi_i}{dt}(0) = g(x_i), \end{cases} \quad i = 1, \ldots, n-1. \tag{6}$$

We obtained "$n-1$" numerical problems (5)-(6), which make use of spline functions to approximate the solution of the problem (2)-(4).

This background allows to state our result given in

Theorem 2.1 *If $u(t, x)$ is the solution of the problem $(2)-(4)$, and φ_i are solutions for $(5)-(6)$, then*

$$\lim_{n \to \infty} |u(t, x_i) - \varphi_i(t)| = 0.$$

Proof. For any interval $[\xi_i, x_i]$, and $\xi_i \in]x_{i-1}, x_i]$, we shall apply the result stated in theorem 1.1 to the functions $u(t, x)$ and $\psi_i(t, x)$, where $\psi_i \colon [0, \alpha] \times [\xi_i, x_i] \to \mathbf{R}$ is defined as

$$\psi_i(t, x) = \frac{1}{2}(x - x_i)^2 U_i(t) + \varphi_i(t) \tag{7}$$

and $\varphi_i(t)$ are solutions of (5)-(6). It is obvious that $\psi_i \in C^2$ on the whole definition domain. Taking into account this property, we infer that there exists functions $(t, x) \to a_i(t, x)$ such that

$$\begin{cases} \dfrac{\partial^2 \psi_i}{\partial t^2}(t, x) = \dfrac{\partial^2 \psi_i}{\partial t^2}(t, x_i) + a_i(t, x), \\ \lim\limits_{x \to x_i} a_i(t, x) = 0. \end{cases}$$

According to (7), $\psi_i(t, x_i) = \varphi_i(t)$, it follows that $\dfrac{\partial^2 \psi_i}{\partial t^2}(t, x_i) = \dfrac{d^2 \varphi_i}{dt^2}(t)$. But φ_i is the solution of (5)-(6), hence

$$\frac{d^2 \varphi_i}{dt^2}(t) = c^2 \cdot U_i(t) + f(t, x_i).$$

It follows that

$$\frac{\partial^2 \psi_i}{\partial t^2}(t,x) = c^2 \cdot U_i(t) + f(t,x_i) + a_i(t,x_i).$$

Using again (7), we obtain

$$\frac{\partial^2 \psi_i}{\partial x^2}(t,x) = U_i(t),$$

therefore

$$(\mathcal{A}\psi_i)(t,x) = f(t,x) - f(t,x_i) + a_i(t,x).$$

If we put

$$\delta_i = \sup_{\substack{t\in[0,a] \\ x\in[\xi_i,x_i]}} |f(t,x) - f(t,x_i) + a_i(t,x)|,$$

we find

$$|(\mathcal{A}\psi_i)(t,x)| \leq \delta_i.$$

Since the function f is a continuous one, we have

$$\lim_{x\to x_i} |f(t,x) - f(t,x_i)| = 0$$

and

$$\lim_{x\to x_i} \delta_i = 0.$$

Because $\psi_i \in C^2$, it follows that the function $b_i(x) = \psi_i(0,x) - \psi_i(0,x_i)$ satisfies the condition $\lim\limits_{x\to x_i} b_i(x) = 0$. But

$$\psi_i(0,x_i) = \varphi_i(0) = \varphi(x_i) \text{ and } u(0,x) = \varphi(x),$$

hence

$$|u(0,x) - \psi_i(0,x)| = |\varphi(x) - \varphi(x_i) - b_i(x)| = |c_i(x) - b_i(x)|,$$

where $c_i(x) = \varphi(x) - \varphi(x_i)$. Since φ is continuous, we have $\lim\limits_{x\to x_i} c_i(x) = 0$. If we put $\varepsilon_i = \sup\limits_{x\in[\xi_i,x_i]} |c_i(x) - b_i(x)|$, we get

$$|u(0,x) - \psi_i(0,x)| \leq \varepsilon_i, \quad \forall x \in [0,\ell].$$

To apply the result stated in Theorem 1.1, we have to evaluate the expression

$$\left|\frac{\partial u}{\partial t}(0,x) - \frac{\partial \psi_i}{\partial t}(0,x)\right|.$$

Since the function ψ_i is of C^2 class, we infer that the function $\frac{\partial \psi_i}{\partial t}$ is continuous, hence $d_i(x) = \frac{\partial \psi_i}{\partial t}(0,x) - \frac{\partial \psi_i}{\partial t}(0,x_i)$ satisfies the condition $\lim\limits_{x\to x_i} d_i(x) = 0$.

But, from $\dfrac{\partial \psi_i}{\partial t}(0, x_i) = \dfrac{d\varphi_i}{dt}(0) = g(x_i)$, it follows $\dfrac{\partial \psi_i}{\partial t}(0, x) = g(x_i) + d_i(x)$, so

$$\left|\frac{\partial u}{\partial t}(0, x) - \frac{\partial \psi_i}{\partial t}(0, x)\right| = |g(x) - g(x_i) - d_i(x)|.$$

If denote $e_i(x) = g(x) - g(x_i)$ and take into account that g is continuous function, we obtain $\lim\limits_{x \to x_i} e_i(x) = 0$. Therefore,

$$\left|\frac{\partial u}{\partial t}(0, x) - \frac{\partial \psi_i}{\partial t}(0, x)\right| = |e(x_i) - d_i(x)|.$$

In addition,

$$\varepsilon_i^1 = \sup_{x \in [\xi_i, x_i]} |e_i(x) - d_i(x)|,$$

satisfies

$$\left|\frac{\partial u}{\partial t}(0, x) - \frac{\partial \psi_i}{\partial t}(0, x)\right| \leq \varepsilon_i^1, \quad \forall x \in [0, \ell].$$

Finally, if the conditions of Theorem 1.1 are satisfied, we obtain

$$|u(t, x) - \psi_i(t, x)| \leq \varepsilon_i + t\varepsilon_i^1 + t^2 \cdot \frac{\delta_i}{2}, \quad \forall x \in [\xi_i, x_i] \text{ şi } \forall i = 1, \dots, n-1. \tag{8}$$

If denote

$$\varepsilon = \max\{\varepsilon_i;\ i = 1, \dots, n-1\},$$
$$\varepsilon^1 = \max\{\varepsilon_i^1;\ i = 1, \dots, n-1\},$$
$$\delta = \max\{\delta_i;\ i = 1, \dots, n-1\}$$

from (8), for $x = x_i$, we have

$$|u(t, x_i) - \varphi_i(t)| \leq \varepsilon + t\varepsilon_1 + t^2 \cdot \frac{\delta}{2}.$$

Now, since $h \to 0$ when $n \to +\infty$, and $x \to x_i$ if $x \in [\xi_i, x_i]$, we infer that

$$\lim_{n \to \infty} |u(t, x_i) - \varphi_i(t)| = 0$$

and the theorem is proved.

Remark 2.1 *Besides the general advantages of the methods based on finite elements, this approach has a concise statement and requires a small computational effort. Moreover, using splins we have all the numerical values of the unknown function in the interval* $[0, \ell]$.

3 Numerical modelling

In this section, we discuss a way to obtain a numerical model using finite differences to replace the derivatives. We use superscripts to denote time and subscripts to denote position. For the equation (2), we deduce

$$\frac{u_j^{i+1} - 2u_j^i + u_j^{i-1}}{(\Delta t)^2} - c^2 \frac{u_{j+1}^i - 2u_j^i + u_{j-1}^i}{(\Delta x)^2} = f(t_i, x_j, u_j^i),$$

where $\Delta t = k = \dfrac{\alpha}{n}$, $\Delta x = h = \dfrac{\ell}{n}$, $x_j = jh$, $t_i = ik$ and $u_j^i = u(t_i, x_j)$. Now, we write the displacement at the end of the current interval, that is at $t = t_{i+1}$. We have

$$u_j^{i+1} - 2u_j^i + u_j^{i-1} - (c \cdot \Delta t)^2 \frac{u_{j+1}^i - 2u_j^i + u_{j-1}^i}{(\Delta x)^2} = (\Delta t)^2 f(t_i, x_i, u_j^i),$$

and we get

$$u_j^{i+1} = \left(\frac{c \cdot \Delta t}{\Delta x}\right)^2 (u_{j+1}^i + u_{j-1}^i) - u_j^{i-1} + 2\left(1 - \left(\frac{c \cdot \Delta t}{\Delta x}\right)^2\right) u_j^i + (\Delta t)^2 f(t_i, x_j, u_j^i).$$

We select the discretization intervals such that $\left(\dfrac{c \cdot \Delta t}{\Delta x}\right)^2 = 1$. This choice leads to smaller computational time and ensure sufficient conditions for convergence. We remark also that u_j^{i+1} depends on initial conditions at the moments t_i and t_{i-1}. It follows the finite diference equation

$$u_j^{i+1} = u_{j+1}^i + u_{j-1}^i - u_j^{i-1} + (\Delta t)^2 f(t_i, x_i, u_j^i). \tag{9}$$

The equation (9) gives the model with differences and allows the numerical solving of (2). It is interesing a discussion about the choice of the discretiation steps, as above. If make use of the method of characteristics, we can prove that if the ratio given above is greater than 1, then the method is not convergent. Also, this condition is required to have the stability of the method. Moreover, if the ratio is less than 1, then the results are less accurate.

Now we have to transform the initial conditions and the boundary conditions. We know the function u at $t = t_0 = 0$ from the initial conditions, but to calculate u at $t = \Delta t = t_1$, we need the values at $t = t_{-1}$. So we need the values of the displacements before to start the iterative process (9). In this respect, we shall define the function $u(t, x)$ backward in time, such that the term t_{-1} makes good sense. An usual method is to consider fictitious values which come from the initial velocity $\dfrac{\partial u}{\partial t}(0, x) = g(x)$, using central differences for them. We have

$$\frac{\partial u}{\partial t}(0, x_j) = \frac{u_j^1 - u_j^{-1}}{2\Delta t} = g(x_j),$$

and

$$u_j^{-1} = u_j^1 - 2g(x_j)\Delta t. \tag{10}$$

Equation (10) is valid only at $t = 0$. By equation (9) for $i = 0$, and equation (10), we obtain

$$\begin{cases} u_j^1 = u_{j+1}^0 + u_{j-1}^0 - u_j^{-1} + (\Delta t)^2 f(t_i, x_j, u_j^i) \\ u_j^{-1} = u_j^1 - 2g(x_j)\Delta t. \end{cases}$$

Finally, we get

$$u_j^1 = \frac{1}{2}\left(u_{j+1}^0 + u_{j-1}^0\right) + g(x_j)\Delta t + \frac{1}{2}(\Delta t)^2 f(t_i, x_j, u_j^i). \tag{11}$$

For the first step, use (11), and then use (9). This approach allows to find the displacement of each knot. Then using splines we can draw the solution.

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