SPECTRA AND FOLIATION

Gr. Tsagas

Abstract

Let (M, g) be a compact Riemannian manifold. On (M, g) we consider a Riemannian foliation. There exists two categories of elliptic operators: the Laplacian and the Jacobian. These give spectra. The aim of the present paper is to study the influence of these spectra on the Riemannian foliation.

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1 Introduction

Let (M, g) be a compact Riemannian manifold of dimension n. From M we obtain the vector spaces $\Lambda^q(M, \mathbb{R})$, $q = 0, 1, ..., n$ of exterior q-forms on M. On these we can apply different elliptic differential operators and therefore we have different spectra. There is the following basic problem. Do these spectra determine some topological or geometric structures on M? This problem has been solved is some cases, but it still remain some open cases ([1], [2], [3], [8], [9]).

Let F be a Riemannian foliation on the compact Riemannian manifold (M, g).

Now, we can put the problem: *do some spectra determine the Riemannian foliation on* (M, g)? The aim of the present paper is to study this problem.

The whole paper contains four sections. Each of them is analyzed as follows. The first section is the introduction. Basic elements of the Riemannian foliation are contained in the second section. The third section includes basic properties of two elliptic differential operators. The relation between spectra and Riemannian foliation are included in the last section.

2 Basic elements of the Riemannian foliations

Let (M, g) be a compact Riemannian manifold of dimension n. We consider the integral distribution D on M, that is

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$$D = \{X \in D^1(M) /\, if X, Y \in D, \text{then}\, [X, Y] \in D\} \tag{2.1}$$

This distribution gives a foliation on M.

Let N be a submanifold of M through a point P of M, such that $T_P(N) = D_P$. These N are called leaves of M by the foliation of D.

Essential D can be considered as the tangent bundle of the leave N. There is a vector subbundle Q of $T(M)$, such that:

$$D \oplus Q = T(M) \tag{2.2}$$

which is called normal bundle with respect to the foliation D of M. The metric is said to be a bundle-like metric, if the induced metric $\widetilde{g} = g/L$ on the normal bundle satisfies the holonomy condition

$$L_X g = 0, \, \forall X \in T(L). \tag{2.3}$$

For distinguished chart $U \subset M$ the leaves of F in U are given as the fields of a Riemannian submersion $f : U \to V \subset N$ onto an open subset V of a model Riemannian manifold N.

From the above we concluded that, if L is the tangent bundle of the leaf N and Q the normal bundle of the foliation, then we have:

$$T(M) = L \oplus Q. \tag{2.4}$$

If $p = \dim L$ and $q = \dim Q$, then $n = \dim M = p + q$ (sometimes we denote Q by L^+).

Let $\overline{\bigtriangledown}$ be the Riemannian connection of the normal bundle $Q = T(M)/L$. It can be easily seen that $\overline{\bigtriangledown}$ is seen as the torsion. From $\bigtriangledown$ which is free of torsion we obtain the curvature tensor field $R_{\overline{\bigtriangledown}}$, the Ricci tensor field $\rho_{\overline{\bigtriangledown}}$ and the scalar curvature $T_{\overline{\bigtriangledown}}$.

For overlapping charts U_α and U_β, that is $U_\alpha \cap U_\beta \neq \varnothing$, the corresponding local transitions $\gamma_{\alpha\beta} = f_\alpha \circ f_\beta^{-1}$ on N are isometries.

It is known that on the manifold M we have the Laplace operator, which is an elliptic operator. Applied on the functions from $C^\infty(M)$ it has the form:

$$\Delta = \delta d : C^\infty(M) \to C^\infty(M), \tag{2.5}$$

Now, we give some examples of foliations.

Example 2.1 Let M be a differentiable manifold of dimension n. Let X be a vector field on M without singularities. Each maximal integral curve C_ℓ determined by X is one-dimensional connected submanifold of M. These curves have the properties $(i) \cup C_\ell = M$, $(ii)\, C_{\ell_1} \cap C_{\ell_2} = \emptyset$, $\ell_1 \neq \ell_2$. Denote by E_x that tangent space of C_ℓ at the point $x \in M$. Then

$$E = \bigcup_{x \in M} E_x$$

is a smooth subbundle of rank one of $T(M)$. All these curves C_ℓ define a foliation of dimension 1 or co-dimension $n - 1$. The leaves of this foliation are all the curves.

The above example can be generalized as follows.

Example 2.2 We consider a differentiable manifold M of dimension n and $\{N_\ell\}$ a family of connected submanifolds of dimension q of M having the following properties:

(i) $M = \underset{\ell}{\cup} N_\ell$.

(ii) $N_{\ell_1} \cap N_{\ell_2} = \emptyset$, $\ell_1 \neq \ell_2$.

(iii) If F_x is the tangent space of N_ℓ at the point x, then

$$F = \underset{x \in M}{\cup} F_x$$

is a smooth subbundle of rank q of $T(M)$.

Remark 2.3 The example 2.2 is another definition of a foliation on the manifold. The family (N_ℓ) is a foliation on M and N_ℓ is a leaf of the foliation (N_ℓ).

Example 2.4 We consider the manifold $M = Q \times T$. The family $(Q \times \{t\})_{t \notin T}$ is (if Q is connected) a foliation on M.

Example 2.5 Let $p : M^m \to N^n$ be a submersion of M^m into N^n, $m \geq n$. The family of the connected components of the inverse images $p^{-1}(x)$, $x \in N$ is a foliation of co-dimension n of M.

Remark 2.6 If $L = (N_\ell)$ is a foliation of co-dimension one on the differentiable manifold M, then its tangent subbundle $T(L)$ can be locally defined by a pfaffian which is a differential 1-form ω without singularities:

$$L_x = \{e \in T_x(M) / \langle e, \omega \rangle = 0\}.$$

3 Basic Properties

From the normal bundle Q of the foliation L we obtain the vector space ΓQ of the cross sections on Q. Now, we define the Jacobi operator J_∇ on ΓQ as follows:

$$J_\nabla : \Gamma Q \to \Gamma Q,\ J_\nabla V = (\Delta_\nabla - p_\nabla)V,\ \forall V \in \Gamma Q, \tag{3.1}$$

where Δ_∇ is the Laplacian acting on the sections of ΓQ. Thus J_∇ is an elliptic operator having discrete spectrum and each eigenvalue has finite multiplicity . This spectrum is denoted by:

$$Sp(F, J_\nabla) = \{0 \leq \lambda_1 = \lambda_1 = ... < \lambda_2 = \lambda_2 = ... < \infty\}. \tag{3.2}$$

We assume that the foliation F on M is a transversally orientable of codimension $q = 1$. Then the transversally Ricci operator p_∇ vanishes. From this we conclude that sections of Q can be identified with functions on M. From this we obtain that an eingensection of the Jacobi operator on ΓQ corresponds to an eigenfuction of the Laplace operator on M associated to the same eigenvalue.

Therefore we have:

$$Sp(M, g) \neq Sp(J_\nabla, F), q \geq 2\,. \tag{3.3}$$

The relation (3.3) has meaning under the condition that:

$$codimension(F) = q \geq 2,$$

which implies that:

$$\dim F = \dim N \leq n - 2, \tag{3.4}$$

which means that the leaves N of D must not be hypersufaces of M.

From the spectrum (3.2) we construct the function $f(t)$ defined by:

$$f(t) = \sum_{m=0}^{\infty} m_i e^{-\lambda_i t} \approx \underset{t \to 0+1}{(4\pi t)}^{-n/2} \{\alpha_0 + \alpha_1 t + \alpha_2 t^2 + ...\}, \tag{3.5}$$

where the second member is the asymptotic expansion of the $f(t)$ and $\alpha_i, i = 0, 1, ...$ are given by:

$$\alpha_0 = q\,Vol(M), \tag{3.6}$$

$$\alpha_1 = q\beta_1 + \int\limits_M T_\nabla dM, \tag{3.7}$$

$$\alpha_2 = q\beta_2 + \frac{1}{12} \int\limits_M \left[2T_M T_\nabla + 6\,|p_\nabla|^2 - |R_\nabla|^2\right] dM, \tag{3.8}$$

where dM is the volume element of M, β_1, β_2 are the first coefficients of the following asymptotic expansion

$$\varphi(t) = \sum_{i=0}^{\infty} v_i e^{-\mu_i t} \cong (4\pi t)^{-n/2} \left\{\beta_0 + \beta_1 t + \beta_2 t^2 + ...\right\}, \tag{3.9}$$

where μ_i, $i = 0, 1, ...$ are the eigenvalues with multiplicity v_i of the Laplace operator on the differential functions $C^\infty(M)$ on M, which form the spectrum

$$Sp(M, g) = \{0 = \mu_0 < \mu_1 = \underset{\overleftrightarrow{v_1}}{\ldots} = \mu_1 < \mu_2 = \underset{\overleftrightarrow{v_2}}{\ldots} = \mu_2 < \mu_3 \ldots < \infty\}. \tag{3.10}$$

It is known that β_0, β_1, $\beta_2, \ldots$, are given by:

$$\beta_0 = VolM,\ \beta_1 = \frac{1}{6} \int\limits_M T_M dM,$$

$$\beta_2 = \frac{1}{360} \int\limits_M \left(2\,|R_M|^2 - 2\,|p_M|^2 + 5T_M^2\right) dM. \tag{3.11}$$

We consider a foliation F on the compact Riemannian manifold (M, g) which is bundle-like metric. From now on this foliation is denoted by:

$$(M, g, F). \tag{3.12}$$

Definition 3.1 Two Riemannian foliations (M,g,F) and (M_0,g_0,F_0) are called *isospectral*, if

$$Sp(M,g)=Sp(M_0,g_0)\,,\ \text{and}\ Sp(F,J_\nabla)=Sp(F_0,J_\nabla). \tag{3.13}$$

Now, we formulate the following basic problems:

1^{st} PROBLEM Let F be a given foliation on a compact Riemannian manifold (M,g), which is bundle-like metric. Do the spectra of $Sp(M,g)$ and $Sp(F,J_\nabla)$ determine completely this foliation on (M,g)?

2^{nd} PROBLEM For a given bundle-like metric foliation F on a fixed compact Riemannian manifold (M,g), determine the spectra of $Sp(M,g)$ and $Sp(F,J_\nabla)$.

3^{rd} PROBLEM The same problem as the first, but substituting $Sp(M,g)$ by the $Sp^k(M,g)$, where $k=1,2,...,n-1$.

Example 3.2 We consider the standard sphere

$$(M,g)=(S^3(1),g)$$

and the foliation F defined by the Hopf fibration

$$S^1(1)\to S^3(1)\to S^2(1/2)\cong P^1(C).$$

For this Hopf fibration we have:

$$Sp(S^1,g_0(r))=\{\lambda_0(S^1(r))=1,\ \lambda_k(S^1(r))=k^2/r^2:k=1,2,...\},$$

with multiplicity

$$m_0(S^1)=1,\ m_k(S^1)=2,\ k=1,2,...,$$

where $S^1(r)$ is the circle of radius r.

$$Sp(S^3(1))=\{\lambda_k(S^3(1))=k(k+6):k=0,1,...\}$$

with multiplicity

$$m_k(S^3(1))=\frac{2(k+3)}{k}\prod_{v=1}^{k-1}\frac{6+v}{v}.$$

We denote by $S^3(r_1)$ the sphere of radius r_1 and dimension 3.
For the $P^1(C)$ we have (see [10]):

$$Sp(P^1(C),g)=\{\lambda_k((P^1(C),g))=\frac{k(k+1)}{2}:k=0,1,2,...\}$$

with multiplicity

$$m_k(P^1(C),g)=(1+2k)\prod_{v=0}^{k-1}\left(\frac{1-v}{v+1}\right)^2\frac{k(k+1)}{2}\ ;\ k=0,\,1,2,\ldots,$$

where the Riemannian metric g on $P^1(C)$ is induced by the negative of the Killing-Cartan form on the Lie algebra of $SU(4)$.

This is a Riemannian foliation on M and the metric g is bundle-like for F. For the Jacobian operator $J_\bigtriangledown$ for this foliation we have:

$$p_\bigtriangledown = 4I, \qquad J_\bigtriangledown = \Delta - 4I,$$

where I is the identity map.

The spectrum of this Jacobian operator has the form

$$Sp(F, J_\bigtriangledown) = \left\{ \begin{array}{l} \frac{1}{2}k(k+1) + i \text{ with multiplicity } 2k+1 \\ \frac{1}{2}k(k+1) - i \text{ with multiplicity } 2k+1 \end{array} \right\},$$

where $k \in \frac{1}{2}Z$ and $i \in \{-k, -k+1, ..., k-1, k\}$.

4 Main results

Now, we consider the Riemannian foliations (M, g, F) and $(M^{'}, g^{'}, F^{'})$ on the compact Riemannian manifolds (M, g) and $(M^{'}, g^{'})$ respectively. We assume that they are isospectral in the previous meaning, that is:

$$Sp(M, g) = Sp(M^{'}, g^{'}), \quad Sp(F, J_\bigtriangledown) = Sp(F^{'}, J^{'}_\bigtriangledown). \tag{4.1}$$

The relation (4.1) by means of

$$\alpha_0 = \alpha^{'}_0, \ \alpha_1 = \alpha^{'}_1, \ \alpha_2 = \alpha^{'}_2, \tag{4.2}$$

$$\beta_0 = \beta^{'}_0, \ \beta_1 = \beta^{'}_1, \ \beta_2 = \beta^{'}_2, \tag{4.3}$$

which by means of (3.6), (3.7), (3.8) and (3.13) imply

$$qVolM = qVolM_0, \, q\beta_1 + \int\limits_M T_\bigtriangledown dM = q\beta^{'}_1 + \int\limits_M T^{'}_\bigtriangledown dM^{'}, \tag{4.4}$$

$$\begin{aligned} q\beta^{'}_2 + \frac{1}{12} \int\limits_M \left[2T_M T_\bigtriangledown + 6 \left| p_\bigtriangledown \right|^2 - \left| R_\bigtriangledown \right|^2 \right] dM = \\ = q\beta^{'}_2 + \frac{1}{12} \int\limits_{M'} \left[2T^{'}_M T^{'}_\bigtriangledown + 6 \left| p^{'}_\bigtriangledown \right|^2 - \left| R^{'}_\bigtriangledown \right|^2 \right] dM^{'}. \end{aligned} \tag{4.5}$$

The relation (4.4) and (4.5) by mean of (4.3) imply:

$$\int\limits_M T_\bigtriangledown dM = \int\limits_{M'} T^{'}_\bigtriangledown dM^{'}, \tag{4.6}$$

$$\int\limits_{M}\left[2T_M T_\nabla + 6\left|p_\nabla\right|^2 - \left|R_\nabla\right|^2\right] dM = \int\limits_{M'}\left[2T_M^{'} T_\nabla^{'} + 6\left|p_\nabla^{'}\right|^2 - \left|R_\nabla^{'}\right|^2\right] dM^{'}. \tag{4.7}$$

In order to find the relation between the spectra of the operator Δ and J_∇ and the foliation F of the Riemannian manifold (M, g) we use the formulas (3.6), (3.7), (3.8) and (4.7).

The following results have been proved ([6]).

Theorem 4.1 *Let* (M, g, F) *and* (M_0, g_0, F_0) *be two isospectral Riemannian foliations. Then we have: (i)* $\dim M = \dim M_0$*, (ii)* $VolM = VolM_0$*, (iii)* (M, g) *and* (M_0, g_0) *have equal total scalar curvature, (iv)* $co\dim F = co\dim F_0 = q$ *and hence* F *and* F_0 *have the same energy that is* $E = \frac{1}{2}qVolM = \frac{1}{2}qVolM_0$*, (v)* F *and* F_0 *have equal total transversal scalar curvature.*

Proof. We explain some notions in this theorem.

The total scalar curvature is:

$$\int\limits_{M} T_M dM = \int\limits_{M_0} T_{M_0} dM_0. \tag{4.8}$$

The total transversal scalar curvature is:

$$\int\limits_{M} T_\nabla dM = \int\limits_{M_0} T_{\nabla_0} dM_0. \tag{4.9}$$

Remark 4.2 We can use Jacobi operator with minimal submanifolds to obtain similar results. The same result is valid, if we use Jacobi operator with harmonic maps.

Remark 4.3 It is obvious that when we have isometric property (i.e. congruent data), then it implies isospectrality.

Conversely the property is not true, because it is possible to have two Riemannian manifolds which are not isometric, but isospectral in the sense of Riemannian foliation. Below is given an example.

Example 4.4 Let $\mathbb{R}^{16}$ be the Euclidean space of sixteen dimension. There are two lattices L_1 and L_2, such that $(\mathbb{R}^{16}/L_1, g_0/L_1)$, $(\mathbb{R}^{16}/L_2, g_0/L_2)$ are isospectral, but not isometric. These two manifolds carry trivial Riemannian hypersurface foliation which are obtained by projecting the lattices onto the line generated by the first of sixteen independent vectors of the lattice.

This implies the following problem.

PROBLEM 4.5 Construct two Riemannian foliations, which are isospectral, but not isometric.

Theorem 4.6 *Let* (M, g, F) *and* $(M^{'}, g^{'}, F^{'})$ *be two Riemannian foliations. We assume that* $Sp^0(M, g) = Sp^0(N, h)$*,* $Sp(M, g) = Sp^1(M^{'}, g^{'})$ *and* $Sp(F, J_\nabla) = Sp(J^{'}, J_\nabla^{'})$ *and these have the same total scalar curvature and the foliation have the same total* *scalar curvature. Then (M, g) is an Einstein manifold if and only if, $(M^{'}, g^{'})$ is an Einstein manifold, and L^{+} is integrable if and only if, $L^{'+}$ is integrable.*

Proof. We obtain firstly the relations:

$$Sp^{0}(M, g) = Sp^{0}(M^{'}, g^{'}),\ Sp^{1}(M, g) = Sp^{1}(M^{'}, g^{'}). \tag{4.10}$$

From the second of the relations (4.10) we conclude the equations:

$$\alpha_{1,0} = \alpha^{'}_{1,0},\ \alpha_{1,1} = \alpha^{'}_{1,1},\ \alpha_{2,2} = \alpha^{'}_{2,2}, \tag{4.11}$$

which by means of ([8]) yield:

$$nVolM = n\int\limits_{M} dM = nVolM^{'} = n^{'}\int\limits_{M'} dM^{'}, \tag{4.12}$$

$$\frac{n-6}{6}\int\limits_{M} T_{M}dM = \frac{n-6}{6}\int\limits_{M'} T^{'}_{M}dM^{'}, \tag{4.13}$$

$$\frac{1}{360}\int\limits_{M}\left[2(n-15)\left|R_{M}\right|^{2} - 2(90-n)\left|p_{M}\right|^{2} + 5(n-12)T_{M}^{2}\right]dM =$$

$$\frac{1}{360}\int\limits_{M'}\left[2(n-15)\left|R^{'}_{M'}\right|^{2} - 2(90-n)\left|p^{'}_{M'}\right|^{2} + 5(n-12)T^{'2}_{M'}\right]dM^{'}. \tag{4.14}$$

If we compare the formulas (3.11), (4.12), (4.13) and (4.14) we obtain the result that (M, g) is an Einstein manifold if and only if , $(M^{'}, g^{'})$ is an Einstein manifold.

From this and [6] we obtain that L^{+} is integrable, if and only if, L^{+} is integrable.
□

From the above we have the following result.

Theorem 4.7 *We consider two foliations (M, g, F) and $(M^{'}, g^{'}, F^{'})$ which are isospectral and $Sp^{1}(M, g) = Sp^{1}(M^{'}, g^{'})$. If (M, g) is an Einstein manifold and F consists of totally geodesic submanifolds, then so is $F^{'}$ and L^{+} is integrable if and only if $L^{'+}$ is integrable.*

Proof. From the relations:

$$Sp^{0}(M, g) = Sp^{0}(M^{'}, g^{'})\, and\, Sp^{1}(M, g) = Sp^{1}(M^{'}, g^{'}) \tag{4.15}$$

and (M, g) is an Einstein manifold we conclude that $(M^{'}, g^{'})$ is an Einstein manifold. Therefore $L^{'+}$ is integrable, since L^{+} is integrable.

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Author's address:

Gr. Tsagas
Division of Mathematics
Department of Mathematics and Physics
Faculty of Technology
Aristotle University of Thessaloniki
Thessaloniki 54006 GREECE