# CLASSIFICATION OF NILPOTENT LIE ALGEBRAS OF MAXIMAL RANK AND GIVEN TYPE

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**Abstract**

One of the basic problems of Lie algebras is the classification. There are different methods to approach this problem. One method is to use Kac-Moody Lie algebras. The aim of the present paper is to determine Nilpotent Lie algebras of maximal rank and given type. Special case is of type $E_6$.

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**Key words:** Lie algebra, Cartan algebra Kac-Moody Lie algebra, rank of an algebra.

## 1 Introduction.

Let $g$ be a Nilpotent Lie algebra of dimension $n$ over a field $K$ of characteristic zero. The classification of Nilpotent Lie algebras is a difficult problem.

There are three different methods to solve this problem.

One method is based on the maximal abelian ideal of a Nilpotent Lie algebra.

The Jordan's representation of a matrix is used for the classification of Nilpotent Lie algebra. This is the second method.

The last method uses Kac-Moody Lie algebras for determination of a class of Nilpotent Lie algebras.

The aim of the present paper is to determine a special class of Nilpotent Lie algebras by means of Kac-Moody Lie algebras.

The whole paper contains five paragraphs. Each of them is analyzed as follows.

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## 2 Basic results

Let $g$ be a Lie algebra over a field $K$, of characteristic zero ($K = \mathbb{C}$ or $\mathbb{R}$) and of dimension $n$ ($\dim g = n$).

One of the basic unsolved problem of Lie algebras is their classification. It is known that any Lie algebra can be written as:

$$g = g_1 \oplus g_2 \,(\text{Levis theorem}),$$

where $g_1$ is a semisimple Lie algebra and $g_2$ is a solvable one.

Any semisimple Lie algebras $g_1$ can be analyzed as follows:

$$g_1 = t_1 \oplus t_2 \oplus \ldots \oplus t_k,$$

where $t_i$ $(i = 1, 2, \ldots, k)$ are simple Lie algebras.

The simple Lie algebras are the following:
(i) $A_n$, $B_n$, $C_n$, and $D_n$ classical,
(ii) $G_2$, $F_4$, $E_6$, $E_7$, $E_8$ exceptional.

The classification of the simple Lie algebras over $\mathbb{C}$ has been done by Killing and Cartan, using a special subalgebra, the Cartan Subalgebra and the Killing-Cartan form.

A subalgebra $b$ of a Lie algebra $g$ is a *Cartan Subalgebra*, if it satisfies the following conditions:
(i)$\underbrace{[[h, h], h, \ldots, h]}_{r} = 0$, for some $r \in \mathbb{N}$,
(ii) if $[x, h] \in b$, $\forall h \in b$, then $x \in b$.

The dimension of the Cartan subalgebra of a Lie algebra $g$ is called *rank* of $g$ (rank $g$). The Killing-Cartan form $v$ on a Lie algebra $g$ is defined by:

$$v : g \times g \to k,\; v : (x, y) \to v(x, y) = Tr(adx \circ ady).$$

The problem of the classification of the solvable Lie algebras has been transformed to classify the Nilpotent Lie algebras (by Malev's theorem).

One method to classify the Nilpotent Lie algebras is to use the maximal abelian ideal (Dixmier, Morozov, Goze, Tsagas, Kobotis,...). An other method is to use the Kac-Moody algebras.

Till now, the following problems of the classification of Nilpotent Lie algebras have been solved: for those of maximal rank and $A_n$, $B_n$, $C_n$, $D_n$ by Louis Sautharoubaue-G. Favre (1994), of type $G_2$ by L. Sautharoubaue (1982) and of type $F_4$ by G. Favre -Gr. Tsagas (1993).

In 1997, the problem of the Classification of Nilpotent Lie algebras of maximal rank and of type $E_6$ and $E_7$ has been solved by Gr. Tsagas-X. Agrafiotou using the method of the Kac-Moody algebra. The classification of Nilpotent Lie algebras of maximal rank and of type $E_8$ is under investigation (Gr. Tsagas and X. Agrafiotou). Now, we describe the method by using Kac-Moody algebras.

Let $A(a_{ij})$, $1 \leq i,j \leq n$ be a square matrix with entries from $\mathbb{Z}$ having the following properties $(i = 1,2,...,n)$:

(i) $a_{ii} = 2$

(ii) $a_{ij} \leq 0$, if $i \neq j$

(iii) if $a_{ij} = 0$, then $a_{ji} = 0$, $i \neq j$.

Then, this square matrix $A(a_{ij})$, $1 \leq i,j \leq n$ is called *Generalized Cartan Matrix* (G. C. M.).

Two G. C. M.'s $A(a_{ij})$, $1 \leq i,j \leq n$ and $B(b_{ij})$, $1 \leq i,j \leq n$ are called *equivalent* if $\exists G_\sigma \in G_n$ ( $G_n$ is the set of the permutations of $\{1,...,n\}$) such that:

$$b_{ij} = a_{\sigma i,\sigma j},\ \forall i,j = 1,...n.$$

Let $n$ be a Lie algebra with $3n$ generators $x_i$, $h_i$, $y_i$ $(i = 1,...n)$ and $A(a_{ij})$, $1 \leq i,j \leq n$ be a G. C. M. associated to the Lie algebras $n$, satisfying the relation:

$$\left.\begin{array}{ll} (i)\,[h_i,h_j] = 0, & [x_i,x_j] = \delta_{ij}h\,, \\ (ii)\ [h_i,x_i] = a_{ij}x_j\,, & [h_i,y_j] = -a_{ij}y_j\,, \\ (iii)\ (adx_i)^{1-a_{ij}}x_j = 0, & (ady_i)^{1-a_{ij}}y_j = 0. \end{array}\right\} i,j = 1,...,n.$$

Then the Lie algebra $L$ is called *Kac-Moody algebra.*

It is known that:

$$L = L_+ \oplus H \oplus L,$$

where

$$L_+ = \underset{a\in\triangle_+}{\oplus} L_a,\ L_- = \underset{\alpha\in\triangle_+}{\oplus} L_\alpha,\ L_0 = H = \overset{n}{\underset{i=1}{\oplus}} kh_i,$$

$$\begin{array}{l} \triangle = \{a \in \mathbb{Z}^n /\, a \neq 0,\ n_a \neq (0)\}, \\ \triangle_+ = \{a \in \mathbb{N}^n /\, a \neq 0,\ n_a \neq (0)\}, \\ \triangle_- = \{-a \in \mathbb{N}^n /\, a \neq 0,\ n_a \neq (0)\} \end{array}$$

and

$$\begin{array}{l} \triangle = \triangle_+ \cup \triangle_- \cup \{0\}\,, \\ \alpha : H \to \mathbb{C}\,(a \in H^*), \\ L_\alpha = \{x \in \eta :\ [h,x] = \alpha(h)x,\ \forall h \in H\}, \\ L_\alpha \neq \{0\},\ L_0 = H,\ [L_\alpha, L_\beta] \subset L_{\alpha+\beta},\ \forall \alpha,\beta \in \triangle. \end{array}$$

The $\dim L_\alpha$ is called *multiplicity* of $L$.

Let $g$ be a Nilpotent Lie algebra. The Nilpotent Lie algebra $g$ is called of *type* $n$ if:

$$n = \dim g/[g,g].$$

Let $\dim g = k$, then $k \leq n$. If $k = n$, then $g$ is called of *maximal rank.*

**Theorem 1** *(L. Sautharoubaue). The mapping:*

$$Gl(g).I \to \eta_+(g)/ \underset{\alpha\in I}{\oplus} g_\alpha$$

*is a bijection from the set of all $Gl(g)$-orbits of all the sets of the ideals of $g$ onto $Nilp(\max, g)$.*

(Where $I \subset \triangle_+(g)$, that means $\forall\alpha \in I \to \alpha + \alpha_i \in I$, $\forall i < 1, ..., n$ and $\alpha_i$ an element of the base of $\triangle$, $\alpha \in \triangle$ and $\alpha + \alpha_i \in \triangle_+(g)$ and $Nilp(\max, g)$ is the set of all the Nilpotent Lie Algebras $g$ of maximal rank).

# 3 Some results concerning Kac-Moody Lie algebras

We consider a generalized Cartan Matrix $A$ defined by:

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{pmatrix}.$$

Let $L$ be a Kac-Moody algebra of dimension $3n$ with following generators:

$$\{X_1,\, X_2,\, ...,\, X_n\},$$
$$\{H_1,\, H_2,\, ...,\, H_n\},$$
$$\{Y_1,\, Y_2,\, ...,\, Y_n\}.$$

It is known that the Lie brackets of the generators are known. From the G.C.M. and Lie brackets we can determine chain conditions. From the nature of roots we can find their relations:

$$X_r = X_\sigma + X_i,\ i = 1, ..., n.$$

Those permits to determine all the ideal of $\triangle_+$ and therefore all the vector spaces which are formed by the vector $\{X_1, ..., X_n\}$.

All these vectors form Nilpotent Lie algebras of maximal rank and type $A$.

The Lie brackets which correspond to each Lie algebra, are coming from the relations:

$$(adX_i)^{1-\alpha_{ij}} X_j = 0$$

and using Chevalley-triple.

# 4 The Lie algebras of maximal rank and of type $E_6$

We study the Nilpotent Lie algebras of maximal rank and of type $E_6$.

$$\overline{E}_6 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The positive root system $\triangle_+ = \{\alpha_1, ..., \alpha_{36}\}$ whose base is $\{\alpha_1, ..., \alpha_6\}$.
Chain condition: $N = \{\alpha_1, ..., \alpha_6, \alpha_1 + \alpha_3, \alpha_2 + \alpha_6, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6\}$.
This chain condition comes from the formula:

$$\alpha_i - a_{ij}\alpha_j \; (i < j).$$

The highest root is $\alpha_{36} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ and the height of $\alpha_{36} = 11$.

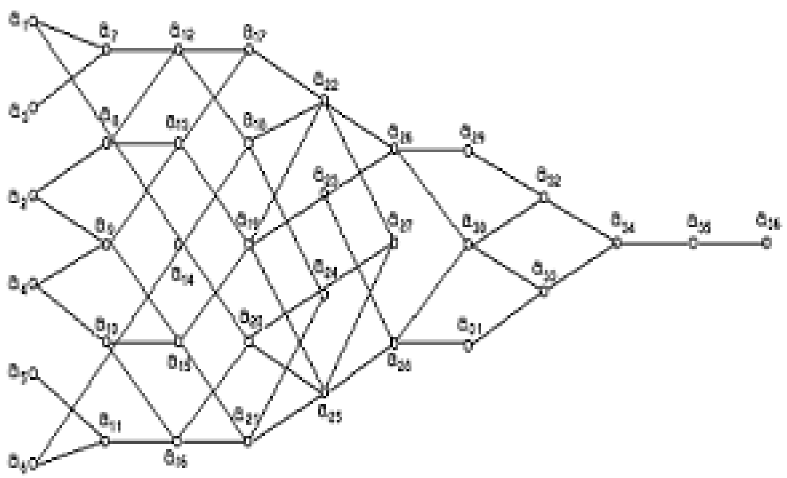

Figure 1.

Number of N.L.A. of maximal rank non-isomorphic is 149.
Dynkin diagram :

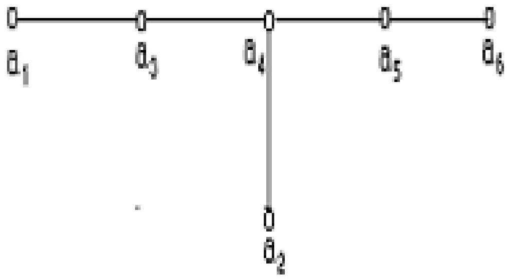


Figure 2.

The positive root system $\triangle_+$ of $E_6$ has the form:

$\triangle_+ = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_1+\alpha_3,\ \alpha_2+\alpha_4,\ \alpha_4+\alpha_5,\ \alpha_5+\alpha_6,\ \alpha_1+\alpha_3+\alpha_4,\ \alpha_2+\alpha_3+\alpha_4,\ \alpha_2+\alpha_4+\alpha_5, \alpha_3+\alpha_4+\alpha_5,\ \alpha_4+\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\ \alpha_1+\alpha_3+\alpha_4+\alpha_5,\ \alpha_2+\alpha_3+\alpha_4+\alpha_5,\ \alpha_2+\alpha_4+\alpha_5+\alpha_6,\ \alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5,\ \alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6,\ \alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6, \alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5,\ \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6,\ \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6,\ \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\}$,

where $\alpha_1, \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5$ and $\alpha_6$ is the canonical base of $Z_6$ and having the following form:

$\{\alpha_1 = (1,0,0,0,0,0),\ \alpha_2 = (0,1,0,0,0,0), \alpha_3 = (0,0,1,0,0,0), \alpha_4 = (0,0,0,1,0,0), \alpha_5 = (0,0,0,0,1,0), \alpha_6 = (0,0,0,0,0,1)\ \}$.

The positive roots $\triangle_+$ can be written, using another notation, as follows:

$$\alpha_1 = (1,0,0,0,0,0),\ \alpha_2 = (0,1,0,0,0,0),\ \alpha_3 = (0,0,1,0,0,0),$$

$$\alpha_4 = (0,0,0,1,0,0),\ \alpha_5 = (0,0,0,0,1,0),\ \alpha_6 = (0,0,0,0,0,1)\,,$$

$$\alpha_7 = (1,0,1,0,0,0),\ \alpha_8 = (0,1,0,1,0,0),\ \alpha_9 = (0,0,1,1,0,0),$$

$$\alpha_{10} = (0,0,0,1,1,0),\ \alpha_{11} = (0,0,0,0,1,1),\ \alpha_{12} = (1,0,1,1,0,0)\,,$$

$$\alpha_{13} = (0,0,1,1,1,0),\ \alpha_{14} = (0,1,0,1,1,0),\ \alpha_{15} = (0,0,1,1,1,0),$$

$$\alpha_{16} = (0,0,0,1,1,1),\ \alpha_{17} = (1,1,1,1,0,0),\ \alpha_{18} = (1,0,1,1,1,0),$$

$$\alpha_{19} = (0,1,1,1,1,0),\ \alpha_{20} = (0,1,0,1,1,1),\ \alpha_{21} = (0,0,1,1,1,1),$$

$$\alpha_{22} = (1,1,1,1,1,0),\ \alpha_{23} = (0,1,1,2,1,0),\ \alpha_{24} = (1,0,1,1,1,1),$$

$$\alpha_{25} = (0,1,1,1,1,1),\ \alpha_{26} = (1,1,1,2,1,0),\ \alpha_{27} = (1,1,1,1,1,1),$$

$$\alpha_{28} = (0,1,1,2,1,1),\ \alpha_{29} = (1,1,2,2,1,0),\ \alpha_{30} = (1,1,1,2,1,1),$$

$$\alpha_{31} = (0,1,1,2,2,1),\ \alpha_{32} = (1,1,2,2,1,1),\ \alpha_{33} = (1,1,1,2,2,1),$$

$$\alpha_{34} = (1,1,2,2,2,1),\ \alpha_{35} = (1,1,2,3,2,1),\ \alpha_{36} = (1,2,2,3,2,1),$$

where:

$\alpha_7 + \alpha_4 = \alpha_{12}$ , $\alpha_{13} + \alpha_1 = \alpha_{17}$, $\alpha_{19} + \alpha_6 = \alpha_{25}$ , $\alpha_{28} + \alpha_1 = \alpha_{30}$, $\alpha_9 + \alpha_3 = \alpha_{13}$, $\alpha_{14} + \alpha_3 = \alpha_{19}$, $\alpha_{21} + \alpha_3 = \alpha_{25}$, $\alpha_{28} + \alpha_1 = \alpha_{30}$, $\alpha_9 + \alpha_5 = \alpha_{15}$, $\alpha_{15} + \alpha_6 = \alpha_{21}$, $\alpha_{20} + \alpha_2 = \alpha_{25}$, $\alpha_{28} + \alpha_5 = \alpha_{31}$, $\alpha_8 + \alpha_1 = \alpha_{12}$, $\alpha_{14} + \alpha_1 = \alpha_{18}$, $\alpha_{22} + \alpha_4 = \alpha_{26}$, $\alpha_{29} + \alpha_6 = \alpha_{32}$, $\alpha_8 + \alpha_2 = \alpha_{13}$, $\alpha_{14} + \alpha_2 = \alpha_{19}$, $\alpha_{22} + \alpha_6 = \alpha_7$, $\alpha_{30} + \alpha_3 = \alpha_{32}$, $\alpha_8 + \alpha_5 = \alpha_{14}$, $\alpha_{14} + \alpha_6 = \alpha_{20}$, $\alpha_{23} + \alpha_1 = \alpha_{26}$, $\alpha_{30} + \alpha_5 = \alpha_{33}$, $\alpha_{10} + \alpha_2 = \alpha_{15}$, $\alpha_{16} + \alpha_2 = \alpha_{21}$, $\alpha_{23} + \alpha_6 = \alpha_{28}$, $\alpha_{31} + \alpha_1 = \alpha_{33}$, $\alpha_{10} + \alpha_3 = \alpha_{14}$, $\alpha_{16} + \alpha_3 = \alpha_{20}$, $\alpha_{24} + \alpha_2 = \alpha_{27}$, $\alpha_{32} + \alpha_5 = \alpha_{34}$, $\alpha_{10} + \alpha_6 = \alpha_{16}$, $\alpha_{17} + \alpha_5 = \alpha_{22}$, $\alpha_{25} + \alpha_1 = \alpha_{27}$, $\alpha_{33} + \alpha_3 = \alpha_{34}$, $\alpha_{11} + \alpha_4 = \alpha_{16}$, $\alpha_{18} + \alpha_2 = \alpha_{22}$, $\alpha_{25} + \alpha_4 = \alpha_{28}$, $\alpha_{34} + \alpha_4 = \alpha_{35}$, $\alpha_{12} + \alpha_2 = \alpha_{17}$, $\alpha_{18} + \alpha_6 = \alpha_{24}$, $\alpha_{26} + \alpha_3 = \alpha_{29}$, $\alpha_{15} + \alpha_3 = \alpha_{19}$, $\alpha_{13} + \alpha_1 = \alpha_{17}$, $\alpha_{19} + \alpha_4 = \alpha_{23}$, $\alpha_{27} + \alpha_4 = \alpha_{30}$, $\alpha_{20} + \alpha_1 = \alpha_{24}$, $\alpha_{13} + \alpha_5 = \alpha_{19}$, $\alpha_{19} + \alpha_6 = \alpha_{25}$, $\alpha_{21} + \alpha_1 = \alpha_{24}$, $\alpha_{21} + \alpha_3 = \alpha_{25}$, $\alpha_7 + \alpha_1 = \alpha_3$, $\alpha_8 + \alpha_3 = \alpha_4$, $\alpha_9 + \alpha_2 = \alpha_4$, $\alpha_{10} + \alpha_4 = \alpha_5$, $\alpha_{11} + \alpha_5 = \alpha_6$ ,

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$\alpha_{18} = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5$, $\alpha_{26} = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$, $\alpha_{34} = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$,

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$\alpha_{24} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$, $\alpha_{32} = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$.

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