UNIFICATION OF THE NILPOTENT LIE ALGEBRAS OF DIMENSION THREE AND FOUR

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Abstract

It is known that there are two Nilpotent Lie algebras of dimension three and three Nilpotent Lie algebras of dimension four. The aim of the present paper is to prove that each of these Nilpotent Lie algebras can be obtained by some vector spaces, using Santilli's theory.

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1 Introduction.

Let g be a Nilpotent Lie algebra over a field K of characteristic zero ($K = \mathbb{R}$ or $K = \mathbb{C}$). It is known that the classification of these Lie algebras is an unsolved problem. It has been solved, if the dimension n of g, that means $\dim g = n$, is less or equal that eight (see [3]).

Santilli's theory is an extension of the theory of Lie algebras. Now, we can ask if the Nilpotent Lie algebras can be obtained by only one vector space using different isounits, or by isomorphic vector spaces.

2 Basic elements

Let g be a Lie algebra of dimension n, that is $\dim g = n$, over the field $K(K = \mathbb{R}$ or $\mathbb{C})$. We form the following central descending sequence:

$$C^0 g = g,\ C^1 g = [C^0 g, g],\ C^2 g = [C^1 g, g],\ ...,\ C^k g = [C^{k-1} g, g].$$

If $C^k g = 0$ for $k \geq 2$, then g is called Nilpotent of Nilpotency k.

The classification of Nilpotent Lie algebras remains a difficult open problem.

There are many methods to face this problem.

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3 Basic elements for algebraic isostructures

Let g be a Lie algebra over a field K of characteristic zero.

It is known on g there is the Lie bracket $[\cdot,\cdot]$, with the properties:

$$[\cdot,\cdot]: g \times g \to g,\ [\cdot,\cdot]: (X,Y) \to [X,Y]\,, \tag{3.1}$$

$$[X,Y] = -[Y,X] \text{ (anticommutative law)}, \tag{3.2}$$

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \text{ (Jacobi's identity)}. \tag{3.3}$$

If on the vector space g, there is another law $,,\circ''$, then we define a Lie bracket $[\cdot,\cdot]$ as follows:

$$[X,Y] = X \circ Y - Y \circ X.$$

Since g is a vector space, we can define a new law of decomposition by means of an element T, which is called Lie-Santilli bracket denoted by $[\cdot,\cdot]_T$, defined by:

$$[\,]_T : g \times g \to g\,, \tag{3.4}$$

$$[\cdot,\cdot]_T : (X,Y) \to [X,Y]_T = XTY - YTX\ . \tag{3.5}$$

It can be easily proved that this law satisfies the conditions:

$$[X,Y]_T = XTY - YTX\ = -[Y,X]_T, \tag{3.6}$$

$$[X,[Y,Z]_T]_T + [Z,[X,Y]_T]_T + [Y,[Z,X]_T]_T = 0. \tag{3.7}$$

It is obvious that $[X,Y]_T$ depends on T. Therefore for each $T \in g$ we obtain an interior law on g. It can be easily seen that:

$$[X,Y] = \ [X,Y]_I,$$

where I is the identity element of g with respect to $,,\circ''$.

From now, we consider the Lie algebra g consists of matrices, that is:

$$g = \{A/\ A \text{ special matrix}\}.$$

The Lie bracket $[\cdot,\cdot]$ on g is defined by:

$$[A_1, A_2] = A_1A_2 - A_2A_1$$

where A_1A_2 is the product of two matrices.

In some cases we can start from a vector space g which is not Lie algebra with respect to the Lie bracket. However it can be Lie-Santilli algebra with respect to the Lie-Santilli bracket:

$$[X, Y]_T = XTY - YTX$$

with proper T.

The following theorem has been proved in [3].

Theorem 1 *Let g be a Lie algebra of dimension n over a field of characteristic zero. Let $\widehat{g}$ be the isotopic vector space of g with isounit I. Then $\widehat{g}$ is a Lie-Santilli algebra with respect to Lie-Santilli brackets.*

There is the following problem. We assume that g is a vector space over a field F, which is not a Lie algebra with Lie brackets

$$[X, Y] \notin g,\ (\forall) X, Y \in g.$$

It is possible to find Lie-Santilli bracket $[\cdot, \cdot]^{'}$ on g such that g with this bracket $[\cdot, \cdot]^{'}$ becomes a Lie-Santilli algebra.

We assume that there is a set $V \supseteq g$ with law of composition $\boxdot$ having the identity element e. Let I be an element of V having the inverse element T, that is

$$I \boxdot T = T \boxdot I = e\,.$$

We suppose we can construct the set

$$\widehat{g} = \{\widehat{X} = XI \,/\, X \in g\}.$$

It has been proved that $\widehat{g}$ is a vector space over F.

We assume that we can define the following bracket laws:

$$[X, Y] = XY - YX\,,\ [X, Y]_I = XIY - YIX\,,\ [X, Y]_T = XTY - YTX.$$

We know that $[X, Y]$ is not a law of composition on g. It is however possible that $[X, Y]_I$ and $[X, Y]_T$ to be laws of composition on g.

We also assume the laws

$$[\widehat{X}, \widehat{Y}] = \widehat{X}\widehat{Y} - \widehat{Y}\widehat{X}\,,\ [\widehat{X}, \widehat{Y}]_I = \widehat{X}I\widehat{Y} - \widehat{Y}I\widehat{X}\,,\ [\widehat{X}, \widehat{Y}]_T = \widehat{X}T\widehat{Y} - \widehat{Y}T\widehat{X}.$$

It is possible that some of them define a law of composition on $\widehat{g}$.

4 Nilpotent Lie algebras of dimension three

Let g be a Nilpotent Lie algebra of dimension three. There are two such Nilpotent Lie algebras:

a) Abelian $g_{3,1}$ with base $\{e_1, e_2, e_3\}$ having the property:

$$[e_i, e_j] = 0,\ i \neq j,\ i, j = 1, 2, 3.$$

b) No-decomposable $g_{2,3}$ with base $\{e_1, e_2, e_3\}$ having the property:

$$[e_1, e_2] = e_3, [e_1, e_3] = 0, [e_2, e_3] = 0.$$

Now, we can state the following theorem.

Theorem 2 *There is a vector space V and two matrices T_1 and T_2 such that $V \circ T_1 = g_{3,1}$ and $V \circ T_2 = g_{3,2}$.*

Proof. We consider the vector space V defined by:

$$V = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \alpha \\ 0 & \beta+\gamma & \gamma \end{pmatrix} /\alpha, \beta, \gamma \in R \right\}.$$

We obtain as T_1 and T_2 the following matrices:

$$T_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix} \text{ and } T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

From the above we obtain:

$$V \circ T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \alpha \\ 0 & \beta+\gamma & \gamma \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta & \beta & 0 \end{pmatrix} = g_{3,1},$$

$$V \circ T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \alpha \\ 0 & \beta+\gamma & \gamma \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \gamma & \beta & 0 \end{pmatrix} = g_{3,2}.$$

Hence the theorem has been proved. □

5 Nilpotent Lie algebras of dimension four

We consider a Nilpotent Lie algebra of dimension four. There are three such Nilpotent Lie algebras:

a) Abelian $g_{4,1}$ with base $\{e_1, e_2, e_3, e_4\}$ having the property:

$$[e_i, e_j] = 0, i \neq j, i, j = 1, 2, 3, 4.$$

b) Decomposable $g_{4,2}$ with base $\{e_1, e_2, e_3, e_4\}$ having the property:

$$[e_1, e_2] = e_3, [e_i, e_j] = 0, i \neq j, i \neq 1, j \neq 2, \ i, j = 1, 2, 3, 4.$$

c) Non-decomposable $g_{4,3}$ with base $\{e_1, e_2, e_3, e_4\}$ having the property:

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4 \text{ and all other Lie brackets are } 0.$$

Now, we can state the following theorem.

Theorem 3 *: There are vector spaces V_1, V_2 and V_3 isomorphic and matrices T_1, T_2 and T_3 such that $V_1 \circ T_1 = g_{4,1}$ and $V_2 \circ T_2 = g_{4,2}$ and $V_3 \circ T_3 = g_{4,3}$.*

Proof. : We obtain

$$V_1 = \begin{pmatrix} \beta & \alpha & 0 & 0 \\ \delta & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, V_2 = \begin{pmatrix} \beta & \alpha & 0 & 0 \\ \gamma & 0 & 0 & 0 \\ \delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{and } V_3 = \begin{pmatrix} \beta & -\gamma & \alpha & 0 \\ \gamma & \delta & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$T_1 = T_2 = T_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then it can be proved that:

$$V_1 \circ T = g_{4,1}, \; V_2 \circ T = g_{4,2}, \; V_3 \circ T = g_{4,3}.$$

□

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