STRUCTURE OF THE ISOUNITS

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Abstract

The isounit plays an important role in the theory of isostructures. The aim of the present paper is to study special cases of isounity.

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1 Introduction

We consider a non empty set U. We assume that there is an element $\widehat{I}$ such that the product $\widehat{u} = \widehat{I}u$ can be defined for every element $u \in U$. Therefore we obtain the set which is defined by

$$\widehat{U} = \left\{ \widehat{u} = u\widehat{I} \,/\, u \in U \right\}. \tag{1.1}$$

The set $\widehat{U}$ is called isotopic of U by mean of $\widehat{I}$. The lifting which transfers the U into $\widehat{U}$ is called isotopy. The element $\widehat{I}$ is called isounit. We assume that the isounit $\widehat{I}$ comes from a set S endowed with an interior law $\Box$ with unit e. Therefore, to the isounit $\widehat{I}$ corresponds an element T with the property $\widehat{I} \Box T = e$. The element T is called isotopic element. In some cases $\widehat{U}$ can be defined as follows:

$$\widehat{U} = \left\{ \widehat{u} = T^{-1}u \,/\, u \in U \right\}. \tag{1.2}$$

If U has some algebraic structures, then these structures can be transferred in $\widehat{U}$.

The aim of the present paper is to study the structure and some properties of the isounit $\widehat{I}$.

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2 The structure of the isounit

We consider the set U endowed with a law of composition denoted by ,,$\bullet$". We call this *multiplication* and we denote it also by:

$$\bullet : U \times U \to U,\ \bullet(\alpha, \beta) \to \alpha \bullet \beta = \alpha\beta \tag{2.1}$$

U can be consisting of numbers (functions or operators).

We try the generalization of the current simplest possible multiplication "$\alpha\beta$", which can be also denoted by ,,$\alpha \times \beta$" for convenience into the isotopic multiplication or isomultiplication. This isomultiplication is denoted by ,,$*$", which has the meaning:

$$\widehat{\alpha} * \widehat{\beta} = \widehat{\alpha}T\widehat{\beta} = \alpha\widehat{I}T\beta\widehat{I} = \alpha\beta\widehat{I} = \widehat{\alpha \times \beta}. \tag{2.2}$$

This operation $*$ can be written:

$$* = \times T \times, \tag{2.3}$$

where T is a fixed, invertible quantity for all possible elements α, β which is called isotopic element.

We must notice that the lifting (2.3) is isotopic, because (for nondegenerate element T) it preserves all the original operators. We assume that on U there is a unit e with respect to the multiplication ,,$\times$", that means

$$a \times e = e \times a = a, \qquad (\forall) a \in U. \tag{2.4}$$

We list (2.4) by the isotopy, then we obtain

$$\widehat{\alpha} \times \widehat{I} = \widehat{I} * \widehat{\alpha} = \widehat{\alpha}\,, \qquad (\forall)\alpha \in U, \tag{2.5}$$

where $\widehat{I} = T^{-1}$. The element $\widehat{I}$ is called *isounit* and T is called *isotopic element*.

Assuming that $\widehat{I}$ preserves all the properties of the unit e, we obtain that the lifting

$$e \to \widehat{I} = e\widehat{I} = \widehat{I}e \tag{2.6}$$

is an isotopy, that is the unit e and the isounit $\widehat{I}$ as well products $\widehat{\alpha} * \widehat{\beta} = \widehat{\alpha} \times \widehat{\beta}$ coincide at the abstract level by cooperation.

We introduce the isonumbers as the generalization of conventional numbers.

The isounit $\widehat{I}$, in the general case, must not be constant but it depends on different factors. In some problems in Physics it has the form:

$$\widehat{I} = \widehat{I}\left(t,\ x,\ \dot{x},\ \ddot{x},\ \Psi,\ \Psi 1,\ d\Psi \uparrow,\ \mu,\ r,\ u \ldots\right).$$

In the classical quantum mechanics we have assumed that Plank's constant $h = 1$. New experimental results on the motion of quantum require to consider that the unit $I = h$ is not constant but it has the form:

$$\widehat{h} = \widehat{I}\left(t,\ x,\ \dot{x},\ \ddot{x},\ \Psi,\ \Psi \uparrow,\ \mu,\ r,\ u \ldots\right),$$

where t is the time, $x = (x_1, x_2, x_3)$ the coordinate of the quantum, $\dot{x}$, $\ddot{x}$ the derivatives of x, Ψ, $\Psi \uparrow$ the wavefunctions, $d\Psi$, $d\Psi \uparrow$ their derivatives, μ the density, r the temperature, n the index reflection, etc.

3 Special constant isounits

In this section we give some isounits which are constant. In order to understand the notion of the isounit, we give some examples.

PROBLEM 3.1. *We consider the ring of the integers $\mathbb{Z}$. Construct an isotopic $\widehat{\mathbb{Z}}$ of $\widehat{I}$ using an element $a \neq 1$ different that the multiplication unit 1 in $\mathbb{Z}$. Then $\widehat{\mathbb{Z}}$ is an isoring with constant isounit.*

Solution. In order to construct the isotopic set $\widehat{U}$ of $\mathbb{Z}$ we must consider that $\alpha \in \mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers, because $\alpha \neq 1$ does not have inverse element in $\mathbb{Z}$ but it has in $\mathbb{Q}$ and it is known that $\mathbb{Z} \subseteq \mathbb{Q}$. The isotopic set $\widehat{\mathbb{Z}}$ is defined by

$$\widehat{\mathbb{Z}} = \left\{ \widehat{\beta} = \beta * \alpha \,/\, \beta \in \mathbb{Z} \right\}.$$

The composition law on $\widehat{\mathbb{Z}}$ is defined by

$$\widehat{\beta} * \widehat{\gamma} = \widehat{\beta}\frac{1}{\alpha}\widehat{\gamma} = \beta\alpha\frac{1}{\alpha}\gamma\alpha = \beta\gamma\alpha = \beta\widehat{\gamma}$$

The isotopic element is $\frac{1}{\alpha}$. The lifting

$$U \to \widehat{U}$$

is an isotopy. The isotopic set $\widehat{\mathbb{Z}}$ becomes an isoring.

PROBLEM 3.2. *Let $\mathbb{R}$ be the field of real numbers. We construct the isotopic set $\widehat{\mathbb{R}}$ which is defined by*

$$\widehat{\mathbb{R}} = \{\widehat{\alpha} = \alpha A \,/\, \text{where } A = \begin{pmatrix} 1 & 0 & . & . & 0 \\ 0 & 2 & . & . & 0 \\ 0 & 0 & 3 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & n \end{pmatrix}, \alpha \in \mathbb{R}\}.$$

Then $\mathbb{R}$ is an isofield with constant isounit.

Solution. In this case we have the isounit

$$\widehat{I} = A = \begin{pmatrix} 1 & 0 & . & . & 0 \\ 0 & 2 & . & . & 0 \\ 0 & 0 & 3 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & n \end{pmatrix}.$$

Therefore $\widehat{\alpha} = \alpha\widehat{I} = \begin{pmatrix} 1\alpha & 0 & . & . & 0 \\ 0 & 2\alpha & . & . & 0 \\ 0 & 0 & 3\alpha & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & n\alpha \end{pmatrix}.$

The isotopic element $T$is the following:

$$T = \widehat{I}^{-1} = A^{-1} = \begin{pmatrix} 1 & 0 & . & . & 0 \\ 0 & \frac{1}{2} & . & . & 0 \\ 0 & 0 & \frac{1}{3} & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & \frac{1}{n} \end{pmatrix}.$$

It can easily be proved that $\mathbb{R}$ is an isofield.

4 Special non-constant units

We shall study isounits which are not constant but they depend on one variable.

PROBLEM 4.1. *We construct the isotopic set* $\widehat{\mathbb{R}} = \{\widehat{x} = x \cdot e^x \,/ x \in \mathbb{R}\}$. *Prove that* $\widehat{\mathbb{R}}$ *is an isofield with isounit* $\widehat{I} = e^x$.

Solution. From this operation we have:

$$\widehat{x} * \widehat{y} = \widehat{x} \cdot T \cdot \widehat{y} = (\widehat{x} \cdot T^{-1}) \cdot T \cdot (y \cdot T^{-1}) = x \cdot y \cdot T^{-1} = \widehat{xy}. \quad (Definition)$$

Therefore the isotopic element is

$$T = e^{xy-x-y}.$$

We can prove that all the other properties for an isofield are valid. These properties are as follows:

$$A_1) \quad \left.\begin{array}{l} (\widehat{x}\widehat{+}\widehat{y})\widehat{+}\widehat{z} \overset{(II)}{=} (x+y) \cdot e^{x+y} \widehat{+} z \cdot e^z \overset{(II)}{=} [(x+y)+z] \cdot e^{(x+y)+z} \\ \widehat{x}\widehat{+}(\widehat{y}\widehat{+}\widehat{z}) \overset{(II)}{=} x \cdot e^x \widehat{+} (y+z) \cdot e^{y+z} \overset{(II)}{=} [x+(y+z)] \cdot e^{x+(y+z)} \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\widehat{x}\widehat{+}\widehat{y})\widehat{+}\widehat{z} = \widehat{x}\widehat{+}(\widehat{y}\widehat{+}\widehat{z}).$$

$$A_2) \quad \left.\begin{array}{l} \widehat{x}\widehat{+}\widehat{0} \overset{(II)}{=} x \cdot e^x \widehat{+} 0 \cdot e^0 \overset{(II)}{=} (x+0) \cdot e^{x+0} = x \cdot e^x = \widehat{x} \\ \widehat{0}\widehat{+}\widehat{x} \overset{(II)}{=} 0 \cdot e^0 \widehat{+} x \cdot e^x \overset{(II)}{=} (0+x) \cdot e^{0+x} = x \cdot e^x = \widehat{x} \end{array}\right\} \Rightarrow \widehat{x}\widehat{+}\widehat{0} = \widehat{0}\widehat{+}\widehat{x} = \widehat{x}.$$

$$A_3) \quad \left.\begin{array}{l} \widehat{x}\widehat{+}(\widehat{-x}) \overset{(II)}{=} x \cdot e^x \widehat{+} (-x \cdot e^{-x}) \overset{(II)}{=} (x-x) \cdot e^{x-x} = 0 \cdot e^0 = 0 \\ (\widehat{-x})\widehat{+}\widehat{x} \overset{(II)}{=} (-x \cdot e^{-x}) \widehat{+} x \cdot e^x \overset{(II)}{=} (-x+x) \cdot e^{-x+x} = e^0 \cdot 0 = 0 \end{array}\right\} \Rightarrow$$

$$\widehat{x}\widehat{+}(\widehat{-x}) = (\widehat{-x})\widehat{+}\widehat{x} = 0.$$

$$A_4) \quad \left.\begin{array}{l} \widehat{x}\widehat{+}\widehat{y} \overset{(II)}{=} x \cdot e^x \widehat{+} y \cdot e^y \overset{(II)}{=} (x+y) \cdot e^{x+y} \\ \widehat{y}\widehat{+}\widehat{x} \overset{(II)}{=} y \cdot e^y \widehat{+} x \cdot e^x \overset{(II)}{=} (y+x) \cdot e^{y+x} \end{array}\right\} \Rightarrow \widehat{x}\widehat{+}\widehat{y} = \widehat{y}\widehat{+}\widehat{x}.$$

Therefore, $\left(\widehat{\mathbb{R}}, \widehat{+}\right)$ is an Abelian group.

$$B_1)\quad \left.\begin{array}{r} (\widehat{x} * \widehat{y}) * \widehat{z} \overset{(I)}{=} x \cdot y \cdot e^{x \cdot y} * z \cdot e^{z} \overset{(I)}{=} (x \cdot y) \cdot z \cdot e^{(x \cdot y) \cdot z} \\ \widehat{x} * (\widehat{y} * \widehat{z}) \overset{(I)}{=} x \cdot e^{x} * (y \cdot z \cdot e^{y \cdot z}) \overset{(I)}{=} x \cdot (y \cdot z) \cdot e^{x \cdot (y \cdot z)} \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\widehat{x} * \widehat{y}) * \widehat{z} = \widehat{x} * (\widehat{y} * \widehat{z}).$$

$$B_2)\quad \left.\begin{array}{r} \widehat{x} * \widehat{I} \overset{\text{by def}}{=} \widehat{x} \cdot T \cdot \widehat{I} \overset{\widehat{I}=T^{-1}}{=} \widehat{x} \\ \widehat{I} * \widehat{x} \overset{\text{by def}}{=} \widehat{I} \cdot T \cdot \widehat{x} \overset{\widehat{I}=T^{-1}}{=} \widehat{x} \end{array}\right\} \Rightarrow \widehat{x} * \widehat{I} = \widehat{I} * \widehat{x} = \widehat{x}.$$

$B_3)$ $\forall \widehat{x} \in \left(\widehat{\mathbb{R}} - \{0\}\right)$ we have:

$$\left.\begin{array}{r} \widehat{x} * \widehat{x}^{-1} \overset{\text{by def}}{=} x \cdot e^{x} \cdot T \cdot x^{-1} e^{x^{-1}} = x \cdot e^{x} \cdot e^{xx^{-1} - x - x^{-1}} \cdot x^{-1} \cdot e^{x^{-1}} = x \cdot x^{-1} = \widehat{1} = \widehat{I} \\ \widehat{x}^{-1} * \widehat{x} \overset{\text{by def}}{=} x^{-1} \cdot e^{x^{-1}} \cdot T \cdot x \cdot e^{x} = x^{-1} \cdot e^{x^{-1}} \cdot e^{xx^{-1} - x - x^{-1}} \cdot x \cdot e^{x} = x^{-1} \cdot x = \widehat{1} = \widehat{I} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \widehat{x} * \widehat{x}^{-1} = \widehat{x}^{-1} * \widehat{x} = \widehat{I}.$$

$$B_4)\quad \left.\begin{array}{r} \widehat{x} * \widehat{y} \overset{(I)}{=} x \cdot y \cdot e^{x \cdot y} \\ \widehat{y} * \widehat{x} \overset{(I)}{=} y \cdot x \cdot e^{y \cdot x} \end{array}\right\} \Rightarrow \widehat{x} * \widehat{y} = \widehat{y} * \widehat{x}.$$

$$C_1)\ \widehat{x} * (\widehat{y} + \widehat{z}) \overset{(II)}{=} \widehat{x} \cdot (\widehat{y + z}) \overset{(II)}{=} x \cdot (y + z) \cdot e^{x \cdot (y+z)} \overset{(II)}{=} (x \cdot y + x \cdot z) \cdot e^{(x \cdot y + x \cdot z)} \overset{(II)}{=}$$

$$\overset{(II)}{=} x \cdot y \cdot e^{x \cdot y} \widehat{+} x \cdot z \cdot e^{x \cdot z} \overset{(I)}{=} \widehat{x} * \widehat{y} \widehat{+} \widehat{x} * \widehat{z} \Rightarrow \widehat{x} * (\widehat{y} + \widehat{z}) = \widehat{x} * \widehat{y} \widehat{+} \widehat{x} * \widehat{z}$$

$$(\widehat{x} \widehat{+} \widehat{y}) * \widehat{z} \overset{(II)}{=} (\widehat{x + y}) * \widehat{z} \overset{(I)}{=} (x + y) \cdot z \cdot e^{(x+y) \cdot z} = (x \cdot z + y \cdot z) e^{(x \cdot z + y \cdot z)} \overset{(II)}{=}$$

$$\overset{(II)}{=} x \cdot z \cdot e^{x \cdot z} \widehat{+} y \cdot z \cdot e^{y \cdot z} \overset{(I)}{=} \widehat{x} * \widehat{z} \widehat{+} \widehat{y} * \widehat{z} \Rightarrow (\widehat{x} \widehat{+} \widehat{y}) * \widehat{z} = \widehat{x} * \widehat{z} \widehat{+} \widehat{y} * \widehat{z}.$$

Therefore, $\left(\widehat{\mathbb{R}}, \widehat{+}, *\right)$ is an isofield.

PROBLEM 4.2 *We consider the isotopic set of* $\widehat{\mathbb{R}}$ *defined by*

$$\widehat{\mathbb{R}} := \{\widehat{x} = a \cdot x \cdot e^{\gamma x} / \, a, \gamma, x \in \widehat{\mathbb{R}}, \cap a \neq 0\}.$$

Solution. From the definition we have:

$$\widehat{x} * \widehat{y} = \widehat{x} \cdot T \cdot \widehat{y} = (x \cdot a \cdot e^{\gamma x}) \cdot T \cdot (y \cdot a \cdot e^{\gamma y}) = a \cdot x \cdot e^{\gamma x} \frac{e^{\gamma xy - \gamma x - \gamma y}}{a} y \cdot a \cdot e^{\gamma y} =$$

$$= a \cdot x \cdot y \cdot e^{\gamma xy} \, (I).$$

Therefore the isotopic element is $T = e^{\gamma(xy - x - y)}$.

Now we can prove with the same method as in problem 4.1 that $\widehat{\mathbb{R}}$ is an isofield with isounit $T = e^{\gamma(xy - x - y)}$.

PROBLEM 4.3 *Let* $\widehat{\mathbb{R}} = \{\widehat{x} = x \cdot \widehat{f}(\widehat{x}) \,/\, x \in \mathbb{R}$ *and the isofunction* $\widehat{f}(\widehat{x}) = \varphi(x)\,\}$ *be isotopic set of* R*. Then* $\widehat{\mathbb{R}}$ *is an isofield.*

Solution.

$$\widehat{x} * \widehat{y} \overset{\text{by def}}{=} \widehat{x} \cdot T \cdot \widehat{y} = [x \cdot \varphi\,(x)] \cdot T \cdot [y \cdot \varphi\,(y)] =$$

$$= x \cdot \varphi(x) \cdot \frac{\varphi(x \cdot y)}{\varphi(x) \cdot \varphi(y)} \cdot y \cdot \varphi(y) = x \cdot y \cdot \varphi(x \cdot y) \qquad (I)$$

$$\widehat{x} \widehat{+} \widehat{y} \overset{\text{by def}}{=} x \cdot \varphi(x) \widehat{+} y \cdot \varphi(y) \overset{\text{we require}}{=} (x+y) \cdot \varphi(x+y) = x + y \;\; (II)$$

The isoset $\widehat{\mathbb{R}}$ is closed according to the operations $*$ and $\widehat{+}$, defined by (I) and (II) respectively.

We will prove the following properties:

$$A_1)\quad \left.\begin{array}{l} (\widehat{x}\widehat{+}\widehat{y})\widehat{+}\widehat{z} \overset{(II)}{=} (x+y) \cdot \varphi\,(x+y)\,\widehat{+} z \cdot \varphi\,(z) \overset{(II)}{=} [(x+y)+z] \cdot \varphi[(x+y)+z] \\ \widehat{x}\widehat{+}(\widehat{y}\widehat{+}\widehat{z}) \overset{(II)}{=} x \cdot \varphi(x) \widehat{+} (y+z) \cdot \varphi(y+z) \overset{(II)}{=} [x+(y+z)] \cdot \varphi[x+(y+z)] \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\widehat{x}\widehat{+}\widehat{y})\widehat{+}\widehat{z} = \widehat{x}\widehat{+}(\widehat{y}\widehat{+}\widehat{z})$$

$$A_2)\quad \left.\begin{array}{l} \widehat{x}\widehat{+}\widehat{0} \overset{(II)}{=} x \cdot \varphi(x) \widehat{+} 0 \cdot \varphi(0) \overset{(II)}{=} (x+0) \cdot \varphi(x+0) = x \cdot \varphi(x) = \widehat{x} \\ \widehat{0}\widehat{+}\widehat{x} \overset{(II)}{=} 0 \cdot \varphi(0) \widehat{+} x \cdot \varphi(x) \overset{(II)}{=} (0+x) \cdot \varphi(0+x) = x \cdot \varphi(x) = \widehat{x} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \widehat{x}\widehat{+}\widehat{0} = \widehat{0}\widehat{+}\widehat{x} = \widehat{x}$$

$$A_3)\quad \left.\begin{array}{l} \widehat{x}\widehat{+}(\widehat{-x}) \overset{(II)}{=} x \cdot \varphi(x) \widehat{+} (-x) \cdot \varphi(-x) \overset{(II)}{=} [x+(-x)] \cdot \varphi[x+(-x)] = 0 \cdot \varphi(0) = 0 \\ (\widehat{-x})\widehat{+}\widehat{x} \overset{(II)}{=} (-x) \cdot \varphi(-x) \widehat{+} x \cdot \varphi(x) \overset{(II)}{=} [(-x)+x] \cdot \varphi[(-x)+x] = \varphi(0) \cdot 0 = 0 \end{array}\right\} \Rightarrow$$

$$\widehat{x}\widehat{+}(\widehat{-x}) = (\widehat{-x})\widehat{+}\widehat{x} = 0.$$

$$A_4)\quad \left.\begin{array}{l} \widehat{x}\widehat{+}\widehat{y} \overset{(II)}{=} x \cdot \varphi(x) \widehat{+} y \cdot \varphi\,(y) \overset{(II)}{=} (x+y) \cdot \varphi(x+y) \\ \widehat{y}\widehat{+}\widehat{x} \overset{(II)}{=} y \cdot \varphi\,(y)\,\widehat{+} x \cdot \varphi(x) \overset{(II)}{=} (y+x) \cdot \varphi(y+x) \end{array}\right\} \Rightarrow \widehat{x}\widehat{+}\widehat{y} = \widehat{y}\widehat{+}\widehat{x}.$$

Therefore, $\left(\widehat{\mathbb{R}}, \widehat{+}, *\right)$ is an Abelian group.

$$B_1)\quad \left.\begin{array}{l} (\widehat{x} * \widehat{y}) * \widehat{z} \overset{(I)}{=} x \cdot y \cdot \varphi(x \cdot y) * z \cdot \varphi\,(z) \overset{(I)}{=} [(x \cdot y) \cdot z] \cdot \varphi[(x \cdot y) \cdot z] \\ \widehat{x} * (\widehat{y} * \widehat{z}) \overset{(I)}{=} x \cdot \varphi(x) * (y \cdot z) \cdot \varphi(y \cdot z) \overset{(I)}{=} [x \cdot (y \cdot z)] \cdot \varphi[x \cdot (y \cdot z)] \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\widehat{x} * \widehat{y}) * \widehat{z} = \widehat{x} * (\widehat{y} * \widehat{z}).$$

$$B_2)\quad \left.\begin{array}{l} \widehat{x} * \widehat{I} \overset{\text{by def}}{=} \widehat{x} \cdot T \cdot \widehat{I} \overset{\widehat{I}=T^{-1}}{=} \widehat{x} \\ \widehat{I} * \widehat{x} \overset{\text{by def}}{=} \widehat{I} \cdot T \cdot \widehat{x} \overset{\widehat{I}=T^{-1}}{=} \widehat{x} \end{array}\right\} \Rightarrow \widehat{x} * \widehat{I} = \widehat{I} * \widehat{x} = \widehat{x}$$

$$B_3)\, \forall \widehat{x} \in \left(\widehat{\mathbb{R}} - \{0\}\right) \Leftrightarrow x \cdot \varphi(x) \neq 0 \overset{x \neq 0}{\Leftrightarrow} \varphi(x) \neq 0 \text{ we have }:$$

$$\left.\begin{array}{c}\widehat{x} * \widehat{x}^{-1} \overset{\text{by def}}{=} x \cdot \varphi(x) \cdot T \cdot x^{-1} \cdot \varphi(x^{-1}) = \\ x \cdot \varphi(x) \cdot \dfrac{\varphi(x \cdot x^{-1})}{\varphi(x) \cdot \varphi(x^{-1})} \cdot x^{-1} \cdot \varphi(x^{-1}) = (x \cdot x^{-1}) \cdot \varphi(x \cdot x^{-1}) = \widehat{I} \\ \widehat{x}^{-1} * \widehat{x} \overset{\text{by def}}{=} x^{-1} \cdot \varphi(x^{-1}) \cdot T \cdot x \cdot \varphi(x) = \\ x^{-1} \cdot \varphi(x^{-1}) \cdot \dfrac{\varphi(x \cdot x^{-1})}{\varphi(x) \cdot \varphi(x^{-1})} \cdot x \cdot \varphi(x) = (x^{-1} \cdot x)\varphi(x^{-1} \cdot x) = \widehat{I}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \widehat{x} * \widehat{x}^{-1} = \widehat{x}^{-1} * \widehat{x} = \widehat{I}.$$

$$B_4) \quad \left.\begin{array}{l}\widehat{x} * \widehat{y} \overset{(I)}{=} (x \cdot y) \cdot \varphi(x \cdot y) \\ \widehat{y} * \widehat{x} \overset{(I)}{=} (y \cdot x) \cdot \varphi(y \cdot x)\end{array}\right\} \Rightarrow \widehat{x} * \widehat{y} = \widehat{y} * \widehat{x}.$$

$C_1)$ $\widehat{x}*(\widehat{y}+\widehat{z}) \overset{(II)}{=} \widehat{x}\cdot(\widehat{y+z}) \overset{(II)}{=} [x\cdot(y+z)]\cdot\varphi[x\cdot(y+z)] \overset{(II)}{=} (x\cdot y+x\cdot z)\cdot\varphi(x\cdot y+x\cdot z) \overset{(II)}{=}$

$\overset{(II)}{=} (x \cdot y) \cdot \varphi(x \cdot y) \widehat{+} (x \cdot z) \cdot \varphi(x \cdot z) \overset{(I)}{=} \widehat{x} * \widehat{y} \widehat{+} \widehat{x} * \widehat{z} \Rightarrow \widehat{x} * (\widehat{y} + \widehat{z}) = \widehat{x} * \widehat{y} \widehat{+} \widehat{x} * \widehat{z}$

$(\widehat{x} \widehat{+} \widehat{y}) * \widehat{z} \overset{(II)}{=} (\widehat{x+y}) * \widehat{z} \overset{(I)}{=} [(x+y)\cdot z]\cdot\varphi[(x+y)\cdot z] = (x\cdot z+y\cdot z)\cdot\varphi(x\cdot z+y\cdot z) \overset{(II)}{=}$

$\overset{(II)}{=} (x \cdot z) \cdot \varphi(x \cdot z) \widehat{+} (y \cdot z) \cdot \varphi(y \cdot z) \overset{(I)}{=} \widehat{x} * \widehat{z} \widehat{+} \widehat{y} * \widehat{z} \Rightarrow (\widehat{x} \widehat{+} \widehat{y}) * \widehat{z} = \widehat{x} * \widehat{z} \widehat{+} \widehat{y} * \widehat{z}.$

Therefore, $\left(\widehat{\mathbb{R}}, \widehat{+}, *\right)$ is an isofield, on condition that $\widehat{f}(\widehat{x}) = \varphi(x) \neq 0$. Thus we require $\widehat{f}(\widehat{x})$ to be defined on $\widehat{A} \cap \mathit{supp}\left(\widehat{f}\right)$, where $\widehat{A}$ is the domain of $\widehat{f}(\widehat{x})$, when there is $\left[\widehat{f}(\widehat{x})\right]^{-1} \left[\sup p\left(\widehat{f}\right), \text{on } \widehat{A}\right]$.

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