MULTI-TIME DYNAMICS INDUCED BY 1-FORMS AND METRICS

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Abstract

The main idea of this paper is to show that a pair of semi-Riemann manifolds and a given 1-form generate a multi-time dynamics via a second-order Lagrangian that is linear with respect to partial accelerations.

Section 1 and 2 recall the notions of jet bundles of order one and order two. Section 3 builds some first-order and second-order Lagrangians induced by 1-forms and metrics, and shows that the corresponding PDEs of extremals are determined by the Otsuki connection obtained from Maxwell (helicity) tensor field and the associated Christoffel symbols of first kind. Section 4 emphasizes the possibility of introducing an electric or magnetic multi-parameter dynamics.

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1 First-order jet bundle

Let (T, h) and (M, g) be semi-Riemann manifolds of dimensions p and n. Local coordinates on the manifolds T, and M will be written

$$t = (t^\alpha), \quad \alpha = 1, \ldots, p$$

$$x = (x^i), \quad i = 1, \ldots, n.$$

The components of the metrics h and g, and the associated Christoffel symbols will be denoted respectively by

$$h_{\alpha\beta}, g_{ij}, H^\alpha_{\beta\gamma}, G^i_{jk}.$$

We use the product bundle $(T \times M, \pi, T)$ whose shorthand is π and we recall some basic notions of the geometry of jet bundles [4].

A map $o : W \subset T \to T \times M$ is called a *local section* of the projection π if it satisfies the condition $\pi \circ o = id_W$. Of course, a section is just the graph of a function from

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the domain W to $T \times M$. If $t \in T$, then the set of all sections of π, whose domains contain the point t, will be denoted $\Gamma_t(\pi)$.

Let us describe a local section in the sense of coordinates. If $\phi \in \Gamma_t(\pi)$ and (t^α, x^i), $\alpha = 1, \ldots, p$; $i = 1, \ldots, n$, are coordinate functions around the point $\phi(t) \in T \times M$, then

$$t^\alpha(\phi(t)) = t^\alpha(\pi\phi(t)) = t^\alpha(t),$$
$$x^i(\phi(t)) = \phi^i(t).$$

Consequently only the last n coordinates $\phi^i = x^i \circ \phi$ are of interest for describing a local section ϕ.

Suppose $\phi, \psi \in \Gamma_t(\pi)$ satisfy $\phi(t) = \psi(t)$. Let (t^α, x^i) and $(t^{\alpha'}, x^{i'})$ be two adapted coordinate systems around the point $\phi(t)$. If

$$\frac{\partial \phi^i}{\partial t^\alpha}(t) = \frac{\partial \psi^i}{\partial t^\alpha}(t),$$

then

$$\frac{\partial \phi^{i'}}{\partial t^{\alpha'}}(t) = \frac{\partial \psi^{i'}}{\partial t^{\alpha'}}(t).$$

This remark justifies the following

Definition. Two local sections $\phi, \psi \in \Gamma_t(\pi)$ are called 1-*equivalent* at t if

$$\phi(t) = \psi(t), \quad \frac{\partial \phi^i}{\partial t^\alpha}(t) = \frac{\partial \psi^i}{\partial t^\alpha}(t).$$

The equivalence class containing ϕ is called the 1-*jet* of ϕ at t and is denoted by $j_t^1\phi$.

Let us show that the set of all the 1-jets of local sections of π has a natural structure as a differentiable manifold. The atlas which describes this structure is constructed from an atlas of adapted coordinate charts on the total space $T \times M$.

Definition. The set

$$J^1\pi = \{j_t^1\phi | t \in T, \ \phi \in \Gamma_t(\pi)\}$$

is called *first jet manifold (bundle)*.

Definition. Let (U, u), $u = (t^\alpha, x^i)$ be an adapted coordinate system on $T \times M$. The *induced coordinate system* (U^1, u^1) on $J^1\pi$ is defined by

$$U^1 = \{j_t^1\phi | \phi(t) \in U\}, \quad u^1 = (t^\alpha, x^i, x^i_\alpha),$$

where

$$t^\alpha(j_t^1\phi) = t^\alpha(t), \quad x^i(j_t^1\phi) = x^i(\phi(t)), \quad x^i_\alpha(j_t^1\phi) = \frac{\partial \phi^i}{\partial t^\alpha}(t).$$

$x^i_\alpha : U^1 \to R$ are called *derivative coordinates* on U^1.

Proposition. *Given an atlas of adapted charts (U, u) on $T \times M$, the corresponding collection of charts (U^1, u^1) is a finite-dimensional C^∞ atlas on $J^1\pi$.*

A local changing of coordinates $(t^\alpha, x^i, x^i_\alpha) \to (t^{\alpha'}, x^{i'}, x^{i'}_{\alpha'})$ is given by

$$t^{\alpha'} = t^{\alpha'}(t^\alpha), \quad x^{i'} = x^{i'}(x^i), \quad x^{i'}_{\alpha'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial t^\alpha}{\partial t^{\alpha'}}x^i_\alpha.$$

where

$$\det\left(\frac{\partial t^{\alpha'}}{\partial t^{\alpha}}\right) \neq 0, \quad \det\left(\frac{\partial x^{i'}}{\partial x^{i}}\right) \neq 0.$$

The expression of the Jacobian matrix of this local diffeomorphism shows that $J^1\pi$ is always orientable.

Definition. A *1-rst order Lagrangian density of energy* on π is a function $L \in C^\infty(J^1\pi)$.

Now we suppose that the manifold T is orientable. A density of energy L produces the *Lagrangian*

$$\mathcal{L} = L\sqrt{|h|}$$

and the *total energy*

$$E(\phi, W) = \int_W L(j_t^1\phi)dv_h,$$

where $dv_h = \sqrt{|h|}dt^1 \wedge \ldots \wedge dt^p$ denotes the volume element induced by the semi-Riemann metric h, and W is a relatively compact domain.

Generally, we look for the critical points of the functional E, i.e., the extremals of the Lagrangian $\mathcal{L}$.

The natural dual bases

$$\left(\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i_\alpha}\right), \ (dt^\alpha, dx^i, dx^i_\alpha)$$

are not suitable for the geometry of $J^1\pi$, inducing complicated formulas for changing of components of geometrical objects under a change of coordinates. For that reason they are replaced by the adapted dual bases

$$\left(\frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} + H^\gamma_{\alpha\beta}x^i_\gamma\frac{\partial}{\partial x^i_\beta}, \ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^h_{ij}x^k_\alpha\frac{\partial}{\partial x^h_\alpha}, \ \frac{\partial}{\partial x^i_\alpha}\right).$$

$$\left(dt^\beta, dx^j, \delta x^j_\beta = dx^j_\beta - H^\gamma_{\beta\lambda}x^j_\gamma dt^\lambda + G^j_{hk}x^h_\beta dx^k\right).$$

Using these frames we define on $J^1\pi$ the induced Sasaki-like semi-Riemann metric [7]

$$S_1 = h_{\alpha\beta}dt^\alpha \otimes dt^\beta + g_{ij}dx^i \otimes dx^j + h^{\alpha\beta}g_{ij}\delta x^i_\alpha \otimes \delta x^j_\beta,$$

$$S_1^{-1} = h^{\alpha\beta}\frac{\delta}{\delta t^\alpha} \otimes \frac{\delta}{dt^\beta} + g^{ij}\frac{\delta}{\delta x^i} \otimes \frac{\delta}{\delta x^j} + h_{\alpha\beta}g^{ij}\frac{\partial}{\partial x^i_\alpha} \otimes \frac{\partial}{\partial x^j_\beta}.$$

2 Second-order jet bundle

Now we define the second jet manifold $J^2\pi$ whose elements are 2-jets $j_t^2\phi$ of local sections $\phi \in \Gamma_t(\pi)$. A 2-jet is an equivalence class containing those local sections with the same value and same first two derivatives at t.

Suppose $\phi, \psi \in \Gamma_t(\pi)$ satisfy $\phi(t) = \psi(t)$. Let (t^α, x^i) and $(t^{\alpha'}, x^{i'})$ be two adapted coördinate systems around the point $\phi(t)$. If

$$\frac{\partial \phi^i}{\partial t^\alpha}(t) = \frac{\partial \psi^i}{\partial t^\alpha}(t), \quad \frac{\partial^2 \phi^i}{\partial t^\alpha \partial t^\beta}(t) = \frac{\partial^2 \psi^i}{\partial t^\alpha \partial t^\beta}(t),$$

then

$$\frac{\partial \phi^{i'}}{\partial t^{\alpha'}}(t) = \frac{\partial \psi^{i'}}{\partial t^{\alpha'}}(t), \quad \frac{\partial^2 \phi^{i'}}{\partial t^{\alpha'} \partial t^{\beta'}}(t) = \frac{\partial^2 \psi^{i'}}{\partial t^{\alpha'} \partial t^{\beta'}}(t).$$

Definition. Two local sections $\phi, \psi \in \Gamma_t(\pi)$ are called *2-equivalent* at t if

$$\phi(t) = \psi(t), \ \frac{\partial \phi^i}{\partial t^\alpha}(t) = \frac{\partial \psi^i}{\partial t^\alpha}(t), \ \frac{\partial^2 \phi^i}{\partial t^\alpha \partial t^\beta}(t) = \frac{\partial^2 \psi^i}{\partial t^\alpha \partial t^\beta}(t).$$

The equivalence class containing ϕ is called *2-jet* of ϕ at t and is denoted by $j_t^2\phi$.

Definition. The set

$$J^2\pi = \{j_t^2\phi | t \in T, \ \phi \in \Gamma_t(\pi)\}$$

is called *second jet manifold (bundle)*.

Definition. Let (U, u), $u = (t^\alpha, x^i)$ be an adapted coordinate system on $T \times M$. The *induced coordinate system* (U^2, u^2) on $J^2\pi$ is defined by

$$U^2 = \{j_t^2\phi | \phi(t) \in U\}, \ u^2 = (t^\alpha, x^i, x^i_\alpha, x^i_{\alpha\beta}),$$

where

$$t^\alpha(j_t^2\phi) = t^\alpha(t), \ x^i(j_t^2\phi) = x^i(\phi(t)),$$

$$x^i_\alpha(j_t^2\phi) = x^i_\alpha(j_t^1\phi), \quad x^i_{\alpha\beta}(j_t^2\phi) = \frac{\partial^2 \phi^i}{\partial t^\alpha \partial t^\beta}(t).$$

The pn functions $x^i_\alpha : U^2 \to R$, and the $\frac{1}{2}np(p+1)$ functions $x^i_{\alpha\beta} : U^2 \to R$ are called *derivative coordinates*.

Proposition. *Given an atlas of adapted charts (U, u) on $T \times M$, the corresponding collection of charts (U^2, u^2) is a finite-dimensional C^∞ atlas on $J^2\pi$.*

A local changing of coordinates

$$(t^\alpha, x^i, x^i_\alpha, x^i_{\alpha\beta}) \to (t^{\alpha'}, x^{i'}, x^{i'}_{\alpha'}, x^{i'}_{\alpha'\beta'})$$

is given by

$$t^{\alpha'} = t^{\alpha'}(t^\alpha), \quad x^{i'} = x^{i'}(x^i),$$

$$x^{i'}_{\alpha'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial t^\alpha}{\partial t^{\alpha'}} x^i_\alpha$$

$$x^{i'}_{\alpha'\beta'} = \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j} \frac{\partial t^\alpha}{\partial t^{\alpha'}} \frac{\partial t^\beta}{\partial t^{\beta'}} x^i_\alpha x^j_\beta + \frac{\partial x^{i'}}{\partial x^i} \frac{\partial^2 t^\alpha}{\partial t^{\alpha'} \partial t^{\beta'}} x^i_\alpha + \frac{\partial x^{i'}}{\partial x^i} \frac{\partial t^\alpha}{\partial t^{\alpha'}} \frac{\partial t^\beta}{\partial t^{\beta'}} x^i_{\alpha\beta}$$

where

$$\det\left(\frac{\partial t^{\alpha'}}{\partial t^{\alpha}}\right)\neq 0,\quad \det\left(\frac{\partial x^{i'}}{\partial x^{i}}\right)\neq 0.$$

The expression of the Jacobian matrix of this local diffeomorphism shows that $J^2\pi$ is orientable iff the manifolds T and M are orientable.

Definition. A *2-nd order Lagrangian density of energy* on π is a function $L\in C^\infty(J^2\pi)$.

The density of energy L produces the Lagrangian $\mathcal{L}=L\sqrt{|h|}$ and the *total energy*

$$E(\phi,W)=\int_W L(j_t^2\phi)dv_h,$$

where $dv_h=\sqrt{|h|}dt^1\wedge\ldots\wedge dt^p$ denotes the volume element induced by the semi-Riemann metric h, and W is a relatively compact domain in T.

Generally, we look for the critical points of the functional E, i.e., the extremals of the Lagrangian $\mathcal{L}$.

Open problems. 1) The natural dual bases

$$\left(\frac{\partial}{\partial t^\alpha},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^i_\alpha},\frac{\partial}{\partial x^i_{\alpha\beta}}\right),\ (dt^\alpha,dx^i,dx^i_\alpha,dx^i_{\alpha\beta})$$

are not suitable for the geometry of $J^2\pi$ since they induce complicated formulas.

Let

$$\frac{\delta}{\delta t^\alpha}=\frac{\partial}{\partial t^\alpha}+A_\alpha{}^i\frac{\partial}{\partial x^i}+A_\alpha\binom{i}{\beta}\frac{\partial}{\partial x^i_\beta}+A_\alpha\binom{i}{\beta\gamma}\frac{\partial}{\partial x^i_{\beta\gamma}}$$

$$\frac{\delta}{\delta x^i}=A_i{}^\alpha\frac{\partial}{\partial t^\alpha}+\frac{\partial}{\partial x^i}+A_i\binom{j}{\beta}\frac{\partial}{\partial x^j_\beta}+A_i\binom{j}{\beta\gamma}\frac{\partial}{\partial x^j_{\beta\gamma}}$$

$$\frac{\delta}{\delta x^i_\alpha}=A\binom{i}{\alpha}^\beta\frac{\partial}{\partial t^\beta}+A\binom{i}{\alpha}^j\frac{\partial}{\partial x^j}+\frac{\partial}{\partial x^i_\alpha}+A\binom{i}{\alpha}\binom{j}{\beta\gamma}\frac{\partial}{\partial x^j_{\beta\gamma}}$$

$$\frac{\delta}{\delta x^i_{\alpha\beta}}=A\binom{i}{\alpha\beta}^\gamma\frac{\partial}{\partial t^\gamma}+A\binom{i}{\alpha\beta}^j\frac{\partial}{\partial x^j}+A\binom{i}{\alpha\beta}\binom{j}{\gamma}\frac{\partial}{\partial x^j_\gamma}+\frac{\partial}{\partial x^i_{\alpha\beta}}$$

$$\delta t^\beta=dt^\beta+B_j{}^\beta dx^j+B\binom{\gamma}{j}^\beta dx^j_\gamma+B\binom{\gamma\delta}{j}^\beta dx^j_{\gamma\delta}$$

$$\delta x^j=B_\gamma{}^j dt^\gamma+dx^j+B\binom{\gamma}{k}^j dx^k_\gamma+B\binom{\gamma\delta}{k}^j dx^k_{\gamma\delta}$$

$$\delta x^j_\beta=B_\gamma\binom{j}{\beta}dt^\gamma+B_k\binom{j}{\beta}dx^k+dx^j_\beta+B\binom{\gamma\delta}{k}^j_\beta dx^k_{\gamma\delta}$$

$$\delta x^j_{\beta\gamma}=B_\delta\binom{j}{\beta\gamma}dt^\delta+B_k\binom{j}{\beta\gamma}dx^k+B\binom{\delta}{k}\binom{j}{\beta\gamma}dx^k_\delta+dx^j_{\beta\gamma}$$

Find the components A,B such that

$$\left(\frac{\delta}{\delta t^\alpha},\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^i_\alpha},\frac{\delta}{\delta x^i_{\alpha\beta}}\right),\ (\delta t^\alpha,\delta x^i,\delta x^i_\alpha,\delta x^i_{\alpha\beta})$$

be dual bases.

2) Using the previous dual frames, study the Sasaki-like semi-Riemann metric

$$S_2 = h_{\alpha\beta}\delta t^\alpha \otimes \delta t^\beta + g_{ij}\delta x^i \otimes \delta x^j + h^{\alpha\beta} g_{ij}\delta x^i_\alpha \otimes \delta x^j_\beta + h^{\alpha\gamma}h^{\beta\lambda} g_{ij}\delta x^i_{\alpha\beta} \otimes \delta x^j_{\gamma\lambda}$$

on $J^2\pi$.

3 (First order and second-order) Lagrangians induced by 1-forms and metrics

First we remark that the derivative along a local section,

$$\frac{\delta}{\partial t^\beta} x^i_\alpha = x^i_{\alpha\beta} - H^\gamma_{\alpha\beta} x^i_\gamma + G^i_{jk} x^j_\alpha x^k_\beta,$$

is a distinguished tensor on $J^1\pi$.

Suppose $\omega = (\omega_i)$ is an 1-form on M representing n *potentials*. Its covariant derivative $\nabla_j \omega_i$ can be decomposed into skew-symmetric part (*Maxwell or helicity tensor field*) and symmetric part (*deformation rate tensor field*),

$$\nabla_j \omega_i = m_{ij} + n_{ij}, \quad m_{ij} = \frac{1}{2}(\nabla_j \omega_i - \nabla_i \omega_j), \quad n_{ij} = \frac{1}{2}(\nabla_j \omega_i + \nabla_i \omega_j).$$

The deformation rate tensor field $n = (n_{ij})$ represents $\frac{n(n+1)}{2}$ potentials.

The semi-Riemann metric $g = (g_{ij})$ and the deformation rate tensor field $n = (n_{ij})$ produce a new tensor field a of componets

$$a_{ij} = g_{ij} + n_{ij}.$$

The preceding mathematical objects define the following Lagrangian densities of energy:

1) *second-order general deformation density of energy*,

$$L_a = h^{\alpha\beta}\omega_i \frac{\delta}{\partial t^\beta} x^i_\alpha + h^{\alpha\beta} a_{ij} x^i_\alpha x^j_\beta;$$

2) *second-order deformation density of energy*

$$L_n = h^{\alpha\beta}\omega_i \frac{\delta}{\partial t^\beta} x^i_\alpha + h^{\alpha\beta} n_{ij} x^i_\alpha x^j_\beta;$$

3) *first-order gravitational density of energy* (used in the theory of classical harmonic maps)

$$L_g = h^{\alpha\beta} g_{ij} x^i_\alpha x^j_\beta.$$

These verify

$$L_a = L_n + L_g.$$

Also, we remark that L_a and L_n are linear functions with respect to the second-order derivatives (*partial accelerations*), as is sometimes required in Mechanics [3].

The second-order deformation density of energy is zero along a local section ϕ which is an integral manifold of the distribution generated by the given 1-form $\omega = (\omega_i)$. Indeed,

$$\omega_i x^i_\alpha = 0$$

implies

$$n_{ij} x^i_\alpha x^j_\beta + \omega_i \frac{\delta}{\partial t^\beta} x^i_\alpha = 0,$$

and hence

$$L_n(j^2_t \phi) = 0.$$

In this case the dimension p depends on dimension n and on the rank of Mawwell tensor.

The second-order general deformation density of energy L_a determines the energy functional

$$E(\phi; W) = \int_W L_a(j^2_t\phi) dv_h, \tag{1}$$

where W is a relatively compact domain in T.

Theorem. *The extremals of the energy functional (1) are described by the PDEs*

$$g_{ki} h^{\alpha\beta} x^i_{\alpha\beta} + \left[\frac{1}{2}\left(\frac{\partial a_{kl}}{\partial x^j} + \frac{\partial a_{kj}}{\partial x^l} - \frac{\partial a_{jl}}{\partial x^k}\right) - \frac{1}{2}\left(\frac{\partial^2 \omega_k}{\partial x^l \partial x^j} + \frac{\partial^2 \omega_j}{\partial x^l \partial x^k} - \frac{\partial^2 \omega_l}{\partial x^j \partial x^k}\right) + \right.$$

$$\left. + \frac{1}{2}\left(\frac{\partial}{\partial x^j}(\omega_i G^i_{kl}) + \frac{\partial}{\partial x^l}(\omega_i G^i_{kj}) - \frac{\partial}{\partial x^k}(\omega_i G^i_{jl})\right)\right] h^{\alpha\beta} x^j_\alpha x^l_\beta - g_{ki} h^{\alpha\gamma} H^\beta_{\alpha\gamma} x^i_\beta = 0.$$

Proof. We use the second-order Lagrangian

$$\mathcal{L} = (h^{\alpha\beta}\omega_i \frac{\delta}{\partial t^\beta} x^i_\alpha + h^{\alpha\beta} a_{ij} x^i_\alpha x^j_\beta)\sqrt{|h|}.$$

Since $\mathcal{L} = L\sqrt{|h|}$, the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial x^k} - \frac{\partial}{\partial t^\alpha}\frac{\partial \mathcal{L}}{\partial x^k_\alpha} + \frac{\partial^2}{\partial t^\alpha \partial t^\beta}\frac{\partial \mathcal{L}}{\partial x^k_{\alpha\beta}} = 0$$

can be written

$$\begin{aligned} \frac{\partial L}{\partial x^k} &- \frac{\partial}{\partial t^\alpha}\frac{\partial L}{\partial x^k_\alpha} + \frac{\partial^2}{\partial t^\alpha \partial t^\beta}\frac{\partial L}{\partial x^k_{\alpha\beta}} - \\ &- H^\gamma_{\gamma\alpha}\frac{\partial L}{\partial x^k_\alpha} + \frac{1}{\sqrt{|h|}}\frac{\partial^2\sqrt{|h|}}{\partial t^\alpha \partial t^\beta}\frac{\partial L}{\partial x^k_{\alpha\beta}} + \frac{2}{\sqrt{|h|}}\frac{\partial\sqrt{|h|}}{\partial t^\alpha}\frac{\partial}{\partial t^\beta}\frac{\partial L}{\partial x^k_{\alpha\beta}} = 0, \end{aligned} \tag{2}$$

where

$$H^{\gamma}_{\gamma\alpha} = \frac{1}{\sqrt{|h|}}\frac{\partial\sqrt{|h|}}{\partial t^{\alpha}}, \quad \frac{1}{\sqrt{|h|}}\frac{\partial^2\sqrt{|h|}}{\partial t^{\alpha}\partial t^{\beta}} = \frac{\partial H^{\gamma}_{\gamma\alpha}}{\partial t^{\beta}} + H^{\gamma}_{\gamma\alpha}H^{\delta}_{\delta\beta}.$$

Explicitly,

$$L = h^{\alpha\beta}\omega_i(x^i_{\alpha\beta} - H^{\gamma}_{\alpha\beta}x^i_{\gamma} + G^i_{jl}x^j_{\alpha}x^l_{\beta}) + h^{\alpha\beta}a_{jl}x^j_{\alpha}x^l_{\beta},$$

and consequently

$$\begin{aligned}\frac{\partial L}{\partial x^k} &= h^{\alpha\beta}\frac{\partial\omega_i}{\partial x^k}(x^i_{\alpha\beta} - H^{\gamma}_{\alpha\beta}x^i_{\gamma} + G^i_{jl}x^j_{\alpha}x^l_{\beta}) + \\ &+ h^{\alpha\beta}\omega_i\frac{\partial G^i_{jl}}{\partial x^k}x^j_{\alpha}x^l_{\beta} + h^{\alpha\beta}\frac{\partial a_{jl}}{\partial x^k}x^j_{\alpha}x^l_{\beta},\end{aligned}$$

$$\frac{\partial L}{\partial x^k_{\alpha}} = -h^{\beta\gamma}\omega_k H^{\alpha}_{\beta\gamma} + 2h^{\alpha\beta}\omega_i G^i_{kl}x^l_{\beta} + 2h^{\alpha\beta}a_{kl}x^l_{\beta},$$

$$\frac{\partial L}{\partial x^k_{\alpha\beta}} = h^{\alpha\beta}\omega_k,$$

$$\begin{aligned}-\frac{\partial}{\partial t^{\alpha}}\frac{\partial L}{\partial x^k_{\alpha}} &= \frac{\partial h^{\beta\gamma}}{\partial t^{\alpha}}\omega_k H^{\alpha}_{\beta\gamma} + h^{\beta\gamma}\omega_k\frac{\partial H^{\alpha}_{\beta\gamma}}{\partial t^{\alpha}} + h^{\beta\gamma}H^{\alpha}_{\beta\gamma}\frac{\partial\omega_k}{\partial x^l}x^l_{\alpha} - \\ &- 2\frac{\partial h^{\alpha\beta}}{\partial t^{\alpha}}\omega_i G^i_{kl}x^l_{\beta} - 2h^{\alpha\beta}\frac{\partial\omega_i}{\partial x^j}G^i_{kl}x^l_{\beta}x^j_{\alpha} - 2h^{\alpha\beta}\omega_i\frac{\partial G^i_{kl}}{\partial x^j}x^l_{\beta}x^j_{\alpha} - \\ &- 2h^{\alpha\beta}\omega_i G^i_{kl}x^l_{\alpha\beta} - 2\frac{\partial h^{\alpha\beta}}{\partial t^{\alpha}}a_{kl}x^l_{\beta} - 2h^{\alpha\beta}\frac{\partial a_{kl}}{\partial x^j}x^l_{\beta}x^j_{\alpha} - \\ &- 2h^{\alpha\beta}a_{kl}x^l_{\alpha\beta},\end{aligned}$$

$$\frac{\partial}{\partial t^{\beta}}\frac{\partial L}{\partial x^k_{\alpha\beta}} = \frac{\partial h^{\alpha\beta}}{\partial t^{\beta}}\omega_k + h^{\alpha\beta}\frac{\partial\omega_k}{\partial x^i}x^i_{\beta},$$

$$\begin{aligned}\frac{\partial^2}{\partial t^{\alpha}\partial t^{\beta}}\frac{\partial L}{\partial x^k_{\alpha\beta}} &= \frac{\partial^2 h^{\alpha\beta}}{\partial t^{\alpha}\partial t^{\beta}}\omega_k + 2\frac{\partial h^{\alpha\beta}}{\partial t^{\beta}}\frac{\partial\omega_k}{\partial x^j}x^j_{\alpha} + \\ &+ h^{\alpha\beta}\frac{\partial^2\omega_k}{\partial x^i\partial x^j}x^i_{\beta}x^j_{\alpha} + h^{\alpha\beta}\frac{\partial\omega_k}{\partial x^i}x^i_{\alpha\beta}.\end{aligned}$$

Replacing in (2), after a long computation, we find the PDEs

$$\begin{aligned}g_{ki}h^{\alpha\beta}x^i_{\alpha\beta} + \Bigg[\frac{1}{2}\left(\frac{\partial a_{kj}}{\partial x^i} + \frac{\partial a_{ki}}{\partial x^j} - \frac{\partial a_{ij}}{\partial x^k}\right) - \frac{1}{2}\Bigg(\frac{\partial}{\partial x^i}(\nabla_j\omega_k) + \frac{\partial}{\partial x^j}(\nabla_k\omega_i) - \\ - \frac{\partial}{\partial x^k}(\nabla_i\omega_j)\Bigg)\Bigg] h^{\alpha\beta}x^i_{\alpha}x^j_{\beta} - g_{ki}h^{\alpha\beta}H^{\gamma}_{\alpha\beta}x^i_{\gamma} = 0,\end{aligned}$$

which coincide to those in Theorem.

The pure gravitational potentials are given by

$$a_{ij} = g_{ij} + \frac{1}{2}(\nabla_j\omega_i + \nabla_i\omega_j).$$

Denoting $b_{ij} = \nabla_i \omega_j$, we define $\Gamma_{kji} = a_{kji} - b_{kji}$, where

$$a_{kji} = \frac{1}{2}\left(\frac{\partial a_{kj}}{\partial x^i} + \frac{\partial a_{ki}}{\partial x^j} - \frac{\partial a_{ij}}{\partial x^k}\right),\ b_{kji} = \frac{1}{2}\left(\frac{\partial b_{kj}}{\partial x^i} + \frac{\partial b_{ki}}{\partial x^j} - \frac{\partial b_{ij}}{\partial x^k}\right)$$

are respectively the Christoffel symbols of a_{ij} and b_{ij}. It is verified that

$$\Gamma_{kji} = g_{kji} + m_{kji},$$

where g_{kji} are the Christoffel symbols of the metric g_{ij} and m_{kji} are the Christoffel symbols of the Maxwell tensor (helicity)

$$m_{ij} = \frac{1}{2}(\nabla_j \omega_i - \nabla_i \omega_j).$$

Consequently, we obtain the following

Corollary. *The extremals of the energy functional (1) are described by the PDEs*

$$h^{\alpha\beta}\frac{\delta}{\delta t^\beta} x^k_\alpha + M^k{}_{ji} h^{\alpha\beta} x^i_\alpha x^j_\beta = 0,$$

where $(g^{ki}m_{ij}, M^k{}_{ji})$ is the Otsuki connection [2].

The preceding extremals may be interpreted like p-dimensional sheets in a multi-time dynamics generated by the 1-form ω and the semi-Riemann metrics h and g.

4 Electromagnetic multi-parameter dynamics

Let $U \subset R^3 = M$ be a domain of linear homogeneous isotropic media. Maxwell's equations on $U \times R$ reflect the relations between the characteristic objects of electromagnetic fields. The objects are:

E	$[V/m]$	*electric field strength*
H	$[A/m]$	*magnetic field strength*
J	$[A/m^2]$	*electric curent density*
ε	$[As/Vm]$	*permitivity*
μ	$[Vs/Am]$	*permeability*
$D = \varepsilon * E$	$[C/m^2] = [As/m^2]$	*electric displacement (flux)*
$B = \mu * H$	$[T] = [Vs/m^2]$	*magnetic induction (flux)*

In terms of differential forms, E, H are differential 1-forms, J, D, B are differential 2-forms, ρ is a differential 3-form, and the star operator $*$ is the Hodge operator. If d is the exterior derivative operator, and ∂_t is the time derivative operator, then the Maxwell's equations for static media are

$$dE = -\partial_t B, \quad dH = J + \partial_t D, \quad dD = \rho, \quad dB = 0$$

(coupled PDEs of first order).

The local components E_i, $i = 1, 2, 3$, of E are called *electric potentials*, and the local components H_i, $i = 1, 2, 3$, of H are called *magnetic potentials*. Since the electric field E, and the magnetic field H are 1-forms, each generate a multi-parameter dynamics (in the sense of Section 3) if we add the semi-Riemann manifold (T, h), and the semi-Riemann manifold $(U \times R, g)$, where g is a Lorentz metric (for example, the Minkowski metric).

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