LEAST SQUARES PROBLEM AND MAGNETIC DYNAMICS

C. UDRIŞTE and M. POSTOLACHE

Abstract

This paper concerns with theoretical, numerical, and graphical aspects for realizing an ATLAS of geometric magnetic dynamics around piecewise rectilinear electric nets, being dedicated both to theorists and practitioners in Electrodynamics. Section 1 proves the link between least squares problem for ODEs and the geometric dynamics induced by a flow in a suitable geometrical framework. Section 2 show the connection between least squares problem for PDEs and the dynamics induced by an integrable distribution in a suitable framework. Section 3 describes the mathematical objects involved dynamics around spatial nets. Section 4 exhibits 100 computer-generated plots for geometric magnetic dynamics, using a specialized MAPLE software designed by our research team.

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1 Least Squares Problem for ODEs and Geometric Dynamics

(T, h) and (M, g) will denote Riemann manifolds of dimension 1 and n, and all the functions are of class C^∞. Local coordinates on T, and M will be written $t = (t^1)$, $x = (x^i)$, $i = 1, 2, \ldots, n$. The components of the metrics h and g, and the associated Christoffel symbols will be denoted respectively by h_{11}, g_{ij}, H^1_{11}, G^i_{jk}.

We use the product bundle $(T \times M, \pi, T)$ whose skorthand is π and the first order jet manifold (bundle) $J^1\pi$ [2].

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Definition 1.1 *A first-order ordinary differential equation (ODE) on π is a closed embedded submanifold Σ of $J^1\pi$. A solution of the ODE Σ is a local section $x \in \Gamma_W(\pi)$, where W is an open interval of T, which satisfies $j_t^1 x \in \Sigma$ for every $t \in W$.*

Of course this definition looks nothing like the usual definition of an ordinary differential equation, but we can see the relationship between the two definitions by using coordinates.

Let $Y = (Y^i)$ be a distinguished vector field on jet bundle of order one $J^1\pi$ and $x_1(t) = \frac{dx}{dt}(t)$. Then the first-order ODE can be written

$$\Sigma\colon Y(t, x, x_1) = 0. \tag{1}$$

Consequently, the use of a submanifold Σ like ODE is a way of separating the description of the equation from a description of its solutions.

If we add to (1) the constraint $x(a) = x_a$ we obtain a Cauchy problem; if we add the constraints $x(a) = x_a$, $x(b) = x_b$, we obtain a boundary value problems. Generally, such problems can have or not solutions.

We modify the preceding boundary value problem. Using the Sasaki-like metric [9], [10]

$$S = h_{11} dt \otimes dt + g_{ij} dx^i \otimes dx^j + h^{11} g_{ij} \delta x_1^i \otimes \delta x_1^j,$$

with respect to the adapted basis

$$\left(dt, \quad dx^i, \quad \delta x_1^i = dx_1^i - H_{11}^1 x_1^i dt + G_{jk}^i x_1^j dx^k\right)$$

on $J^1\pi$, we attach to this problem the Lagrangian density of energy

$$L(x(t)) = \frac{1}{2} h^{11}(t) g_{ij}(x(t)) Y^i(t, x(t), x_1(t)) Y^j(t, x(t), x_1(t))$$

and the total energy

$$E(x; [a, b]) = \int_a^b L(x)(t) \sqrt{h_{11}(t)}\, dt.$$

Let us find a function x (curve) which satisfies the given boundary conditions and requiring that $E(x; [a, b])$ be as close as possible to zero throughout $[a, b]$, for all compactly supported variations. Consequently the initial boundary problem is changed into a least squares problem of variational calculus:

(P_1) minimize $E(x; [a, b])$ subject to $x \in \{C^\infty[a, b] / x(a) = x_a, x(b) = x_b\}$.

A solution x^* of the minimization problem (P_1) must be a critical point of the energy functional E, i.e., an extremal of the Lagrangian

$$\mathcal{L} = L(x)(t) \sqrt{h_{11}(t)},$$

satisfying the boundary conditions $x(a) = x_a$, $x(b) = x_b$. The extremals satisfy the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial x^k} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x_1^k} = 0, \quad k = 1, \ldots, n$$

or

$$\frac{\partial L}{\partial x^k} - \frac{d}{dt}\frac{\partial L}{\partial x_1^k} - H_{11}^1 \frac{\partial L}{\partial x_1^k} = 0.$$

Particularly, if

$$Y^i = x_1^i - X^i(t, x),$$

then the differential system (1) is written $\frac{dx^i}{dt} = X^i(t, x(t))$, $i = 1, \ldots, n$ and the least squares Lagrangian density of energy is reduced to

$$\begin{aligned} L &= \frac{1}{2}h^{11}g_{ij}\left(\frac{dx^i}{dt} - X^i\right)\left(\frac{dx^j}{dt} - X^j\right) \\ &= \frac{1}{2}h^{11}g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} - h^{11}g_{ij}\frac{dx^i}{dt}X^j + f, \end{aligned}$$

where

$$f = \frac{1}{2}h^{11}g_{ij}X^iX^j$$

is a *potential energy* produced by the distinguished vector field $X = (X^i)$. The *helicity* of the distinguished vector field X is the external distinguished tensor field

$$F_j{}^i = \nabla_j X^i - g^{ih}g_{kj}\nabla_h X^k,$$

where ∇ is the connection induced by the Riemannian metric g. Also we shall use the connection D produced by the Riemannian metric h.

In this case the preceeding least squares problem reduces to ODEs of geometric dynamics induced by the distinguished vector field X and the metrics h, g [4]-[10]. Also the extremals are called (harmonic or) potential maps.

Theorem 1.1 *The least squares problem for the boundary value problem*

$$\frac{dx^i}{dt} = X^i(t, x(t)), \ x(a) = x_a, \ x(b) = x_b$$

reduces to the boundary value problem

$$h^{11}\frac{\delta}{dt}\frac{dx^i}{dt} = h^{11}(g^{ij}g_{kj}(\nabla_h X^k)X^j + F_j{}^i\frac{dx^j}{dt} + DX^i), \ x(a) = x_a, \ x(b) = x_b, \quad (2)$$

where

$$\frac{\delta}{dt}\frac{dx^i}{dt} = \frac{d^2x^i}{dt^2} - H_{11}^1\frac{dx^i}{dt} + G_{jk}^i\frac{dx^j}{dt}\frac{dx^k}{dt}.$$

Theorem 1.2 *(Lorentz-Udrişte World-Force Law). Every solution of DEs system (2) is a horizontal potential map on the Riemann-Lagrange manifold*

$$\left(T \times M, \quad h+g, \quad N\binom{j}{1}_j = G^i_{jk}y^k - F_j{}^i, \quad M\binom{i}{1}_1 = -H^1_{11}y^i\right), \quad y^i = \frac{dx^i}{dt}.$$

Remark. The geometric dynamics defined by a vector field X on M can be decomposed into the phase portrait (corresponding to the constant value $\mathcal{H} = 0$) and transversal curves (corresponding to constant values $\mathcal{H} \neq 0$), where

$$\mathcal{H} = \frac{1}{2} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} - f$$

is the Hamiltonian induced by L.

2 Least Squares Problem for PDEs and Geometric Dynamics

Ingredients: all the functions and manifolds are of class C^∞; (T, h), Riemann manifold of dimension p; (M, g), Riemann manifold of dimension n; $t = (t^\alpha)$, $\alpha = 1, \ldots, p$, local coordinates on T; $x = (x^i)$, $i = 1, \ldots, n$, local coordinates on M; $h_{\alpha\beta}$, local components of the Riemannian metric h; g_{ij}, local components of the Riemannian metric g; D, connection induced by h; ∇, connection induced by g; $H^\alpha_{\beta\gamma}$, Christoffel symbols produced by $h_{\alpha\beta}$; G^i_{jk}, Christoffel symbols produced by g_{ij}; π, the product bundle $(T \times M, \pi, T)$; $J^1\pi$, the first order jet bundle [2].

Definition 2.1 *A first-order partial differential equation (PDE) on π is a closed embedded submanifold Σ of $J^1\pi$. A solution of the PDE Σ is a local section $x \in \Gamma_W(\pi)$, where W is an open subset of T, which satisfies $j^1_t x \in \Sigma$ for every $t \in W$.*

Now we use the coordinates. Let $Y = (Y^i_\alpha)$, $\alpha = 1, \ldots, p$; $i = 1, \ldots, n$ be a distinguished tensor field (which can be identified with a set of distinguished vector fields) on jet bundle of order one $J^1\pi$ and $x_\alpha(t) = \dfrac{\partial x}{\partial t^\alpha}$. Then the first-order PDE can be written

$$\Sigma\colon\ Y(t, x, x_\alpha) = 0. \tag{3}$$

Adding to (3) initial conditions or boundary conditions we obtain Cauchy problems, boundary problems or mixed problems, which can have or not solutions. Here we use a boundary value problem modified into a least squares problem. For that we use the Sasakian metric

$$S = h_{\alpha\beta} dt^\alpha \otimes dt^\beta + g_{ij} dx^i \otimes dx^j + h^{\alpha\beta} g_{ij} \delta x^i_\alpha \otimes \delta x^j_\beta$$

with respect to the adapted basis

$$\left(dt^\alpha, \quad dx^i, \quad \delta^i_\alpha = dx^i_\alpha - H^\gamma_{\alpha\beta} x^i_\gamma dt^\beta + G^i_{jk} x^j_\alpha dx^k\right)$$

on $J^1\pi$, and we build the Lagrangian density of energy

$$L(x(t)) = \frac{1}{2}h^{\alpha\beta}(t)g_{ij}(x(t))Y^i_\alpha(t, x(t), x_\gamma(t))Y^j_\beta(t, x(t), x_\gamma(t)).$$

These produce the total energy

$$E(x; W) = \int_W L(x)(t)dv_h,$$

where $dv_h = \sqrt{\det h}dt^1 \wedge \cdots \wedge dt^p$ denotes the volume element induced by the Riemannian metric h, and W is a relatively compact domain in T.

Let us find a function x (parametrized sheet) which satisfies be boundary condition $x|_{\partial W} = \chi$ and requiring that $E(x; W)$ be as close as possible to zero throught W, for all compactly supported variations. Consequently the initial boundary problem is changed into a least squares problem of variational calculus:

$$(P_2) \text{ minimize } E(x; W) \text{ subject to } x \in \{C^\infty W / x|_{\partial W} = \chi\}.$$

A solution x^* of the minimization problem (P_2) must be a critical point of the energy functional E, i.e., an extremal of the Lagrangian

$$\mathcal{L} = L(x)(t)\sqrt{\det h(t)}$$

satisfying the boundary condition $x|_{\partial W} = \chi$.

The extremals satisfy the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial x^k} - \frac{\partial}{\partial t^\alpha}\frac{\partial \mathcal{L}}{\partial x^k_\alpha} = 0, \ k = 1, \dots, n$$

or

$$\frac{\partial L}{\partial x^k} - \frac{\partial}{\partial t^\alpha}\frac{\partial L}{\partial x^k_\alpha} - H^\gamma_{\gamma\alpha}\frac{\partial L}{\partial x^k{}_\alpha} = 0.$$

Particularly, if

$$Y^i_\alpha = x^i_\alpha - X^i_\alpha(t, x),$$

then the PDEs system (2) is written

$$\frac{\partial x^i}{\partial t^\alpha} = X^i_\alpha(t, x), \ \alpha = 1, \dots, p; \ i = 1, \dots, n.$$

The complete integrability conditions

$$\frac{\partial X^i_\alpha}{\partial t^\beta} + \frac{\partial X^i_\alpha}{\partial x^j}X^j_\beta = \frac{\partial X^i_\beta}{\partial t^\alpha} + \frac{\partial X^i_\beta}{\partial x^j}X^j_\alpha$$

ensure the existence of solutions.

The least squares Lagrangian density of energy is reduced to

$$\begin{aligned} L &= \frac{1}{2}h^{\alpha\beta}g_{ij}(x^i_\alpha - X^i_\alpha)(x^j_\beta - X^j_\beta) \\ &= \frac{1}{2}h^{\alpha\beta}g_{ij}x^i_\alpha x^j_\beta - g_{ij}x^i_\alpha X^j_\beta + f, \end{aligned}$$

where

$$f = \frac{-1}{2} h^{\alpha\beta} g_{ij} X^i_\alpha X^j_\beta$$

is the *potential energy* produced by the distinguished tensor $X = (X^i_\alpha)$.

The *helicity* of the distinguished tensor field X is the distinguished tensor field

$$F_j{}^i{}_\alpha = \nabla_j X^i_\alpha - g_{hj} g^{ik} \nabla_k X^h_\alpha.$$

In this case the preceeding least squares problem reduces to PDEs of geometric dynamics induced by the distinguished tensor field X and the metrics h, g [9], [10]. Also the extremals are called harmonic or potential maps.

Theorem 2.1 *The least squares problem for a boundary value problem associated to the PDEs system*

$$\frac{\partial x^i}{\partial t^\alpha} = X^i_\alpha(t, x)$$

reduces to boundary value problem associated to the PDEs system

$$h^{\alpha\beta} \frac{\delta}{\partial t^\beta} x^i_\alpha = h^{\alpha\beta}(g^{ih} g_{kj} (\nabla_h X^k_\alpha) X^j_\beta + F_j{}^i{}_\alpha x^j_\beta + D_\beta X^i_\alpha). \tag{4}$$

where

$$\frac{\delta}{\partial t^\beta} x^i_\alpha = \frac{\partial^2 x^i}{\partial t^\alpha \partial t^\beta} - H^\gamma_{\alpha\beta} x^i_\gamma + G^i_{jk} x^j_\alpha x^k_\beta.$$

Theorem 2.2 *(Lorentz-Udrişte World-Force Law). Every solution of PDEs system (4) is a horizontal potential map on the Riemann-Lagrange manifold*

$$\left(T \times M, \quad h + g, \quad N(^i_\alpha)_j = G^i_{jk} x^k_\alpha - F_j{}^i{}_\alpha, \quad M(^i_\alpha)_\beta = -H^\gamma_{\alpha\beta} x^i_\gamma\right).$$

3 Magnetic Dynamics

One of the main ideas of Sabba Ştefănescu [3], [7], [12] was to study the magnetic field generated by picewise rectilinear electric circuits (of continuous current). That is why we concentrate to the geometric magnetic dynamics.

A piecewise rectilinear electric circuit γ_α is expressed in the form

$$\gamma_\alpha = \bigcup_{\beta=1}^{m} \gamma_{\alpha\beta},$$

where $\gamma_{\alpha\beta}$ is a straight line (when $\beta = 1$), a semiline or a segment in space, disposed under the condition of circuit closedness (either at finite distance, or at infinity). A union of p piecewise rectilinear electric circuits

$$\Gamma = \bigcup_{\alpha=1}^{p} \gamma_\alpha = \bigcup_{\alpha=1}^{p} \bigcup_{\beta=1}^{m} \gamma_{\alpha\beta}$$

is called a configuration (network) in space.

For modelling real phenomena, we agree that a configuration in space has to satisfy the following conditions:

1) Each segment $\gamma_{\alpha\beta}$ has its edges in contact with the extremities of other segments or semilines $\gamma_{\lambda\delta}$. Each semiline $\gamma_{\alpha\beta}$ has its finite edge in contact with the edge of a segment or semiline $\gamma_{\lambda\delta}$.

2) At each knot (contact point), the second Kirchhoff law shall be satisfied, i.e., the algebraic sum of the intensities of currents vanishes. If this condition is not satisfied, then the associated magnetic field does not admit a scalar potential.

The magnetic field produced around the circuit γ_α is given by the Biot-Savart-Laplace formula

$$\overline{H}_\alpha(x) = \int_{\gamma_\alpha} \frac{\overline{J}_\alpha \times \overline{px}}{px^3} d\tau_p, \ \forall x = (x_1, x_2, x_3) \in R^3 \setminus \gamma_\alpha,$$

where $p \in \gamma_\alpha$ is the arbitrary point on the electric circuit γ_α, and $\overline{J}_\alpha$ is the conduction current density (theoretically like versor) on γ_α.

The magnetic field is irrotational (rot $H_\alpha = 0$) and solenoidal (div $H_\alpha = 0$). Also the set $\{H_\alpha, \ \alpha = 1, \ldots, p\}$ determines a Lie algebra of solenoidal vector fields.

The (total) magnetic field produced around the configuration Γ is

$$\overline{H}(x) = \sum_{\alpha=1}^{p} \overline{H}_\alpha(x), \ x \in R^3 \setminus \Gamma.$$

The total magnetic field is also irrotational and solenoidal.

Regarding the magnetic geometric dynamics we have two possibilities:

1) uni-parameter (time) dynamics determined by the total magnetic field H and the canonical Riemannian metric δ_{ij} of R^3 (see §1, and [11], [7], [12]).

2) multi-parameter (multi-time) dynamics determined by the vector fields H_α, $\alpha = 1, \ldots, p$, the metric $\delta_{\alpha\beta}$ on the space of parameters and the metric δ_{ij} of R^3 (see §2, and [9], [10], [13]).

In the sequel, we look for the geometric magnetic dynamics produced by $(T, 1)$, $(R^3 \setminus \Gamma, \delta_{ij})$ and the total magnetic field $H = (H_1, H_2, H_3)$. This dynamics is described by the DEs system

$$\frac{d^2 x_i}{dt^2} = \frac{\partial f}{\partial x_i}, \ i = 1, 2, 3; \ x = (x_1, x_2, x_3), \tag{5}$$

where $f = \frac{1}{2}(H_1^2 + H_2^2 + H_3^2)$.

The DEs system (5) describes the extremals of the Lagrangian

$$L = \frac{1}{2} \sum_{i=1}^{3} \left(\frac{dx_i}{dt} \right)^2 - \sum_{i=1}^{3} H_i \frac{dx_i}{dt} + f$$

as potential maps of the Riemann manifold $(T\times M, 1+g)$. The associated Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{3}\left(\frac{dx_i}{dt}\right)^2 - f.$$

4 Atlas of Magnetic Dynamics

The behaviour of concrete dynamical systems cannot be investigated without modern computer simulation. This involves numerical algorithms, graphical methods, and specialized software able to capture the mathematical information contained in the analytical formulas. However, the computer-ploted results must be analyzed by a specialist in the field, to decide upon their practical or theoretical significance.

For numerical modelling of magnetic dynamics, we use the general explanations given in [1], [11], and a specialized MAPLE software realized in our Laboratory of Mathematical Visualization and Computer Graphics. This important computer program, available on PCs, can be used very efectively for plotting and computing extremals.

Each case is individualized by the following data: the electric configuration, the magnetic field, the parameter values, the boundary condition, the number n of iterations, and the iteration step h. For each set of data, the exhibited output includes numerical computed images of trajectories in geometric magnetic dynamics, graph of the Hamiltonian (energy), and $(x, \dot{x})$, $(y, \dot{y})$, $(z, \dot{z})$ Poincaré projections of the corresponding curves in the phase space R^6. The components of the magnetic field around spatial configurations are found in the papers [12]. Originally they are written in the Cartesian Frame used in Geodynamics. To avoid possible misunderstandings, the present modelling and simulation is made after rotation $x \to -x$, $z \to -z$, who introduce the usual Cartesian Frame.

The sequence of the following figures belongs to an ATLAS OF MAGNETIC DYNAMICS that is working now in our Laboratory. These figures are associated to the next illustrative examples:

4.1 Two parallel wires, currents of the same sense

Magnetic field components (Geodynamics Frame):

$$\begin{aligned} H_x &= \frac{-y}{(x+a)^2+y^2} + \frac{-y}{(x-a)^2+y^2} \\ H_y &= \frac{x+a}{(x+a)^2+y^2} + \frac{x-a}{(x-a)^2+y^2} \\ H_z &= 0. \end{aligned}$$

Parameter value: $a = 1$.

4.2 Two parallel wires, currents of opposite sense

Magnetic field components (Geodynamics Frame):

$$\begin{aligned} H_x &= \frac{-y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} \\ H_y &= \frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2} \\ H_z &= 0. \end{aligned}$$

Parameter value: $a = 1$.

4.3 Two right angle circuits, currents of opposite sense

Magnetic field components (Geodynamics Frame):

$$\begin{aligned} H_x &= \frac{-y}{r_1(r_1-z-b)} + \frac{y}{r_2(r_2+z-b)}, \\ H_y &= \frac{x+a}{r_1(r_1-z-b)} - \frac{z+b}{r_1(r_1+x+a)} - \frac{x-a}{r_2(r_2+z-b)} + \frac{z-b}{r_2(r_2-x+a)}, \\ H_z &= \frac{y}{r_1(r_1+x+a)} - \frac{y}{r_2(r_2-x+a)}, \end{aligned}$$

where $r_1^2 = (x+a)^2 + y^2 + (z+b)^2$, and $r_2^2 = (x-a)^2 + y^2 + (z-b)^2$. The domain of definition of this field is given by $A = \mathbf{R}^3 \setminus \{e_1 \cup f_1 \cup e_2 \cup f_2\}$, where the set

$$\begin{aligned} &f_1: \ x \le -a, \quad y = 0, \quad z = -b; \quad e_1: \ x = -a, \quad y = 0, \quad z \ge -b \\ &f_2: \ x \ge a, \quad y = 0, \quad z = b; \quad e_2: \ x = a, \quad y = 0, \quad z \le -b. \end{aligned}$$

Obviously, H is an irrotational and solenoidal (and thus harmonic) vector field. We assume that a and b does not vanish simultaneously.

Parameter values: $a = 1$ and $b = 2$.

4.4 Two right angle circuits, currents of the same sense

Magnetic field components (Geodynamics Frame):

$$\begin{aligned} H_x &= \frac{-y}{r_1(r_1-z-b)} - \frac{y}{r_2(r_2+z-b)}, \\ H_y &= \frac{x+a}{r_1(r_1-z-b)} - \frac{z+b}{r_1(r_1+x+a)} + \frac{x-a}{r_2(r_2+z-b)} - \frac{z-b}{r_2(r_2-x+a)}, \\ H_z &= \frac{y}{r_1(r_1+x+a)} + \frac{y}{r_2(r_2-x+a)}, \end{aligned}$$

where we put $r_1^2 = (x+a)^2 + y^2 + (z+b)^2$, and $r_2^2 = (x-a)^2 + y^2 + (z-b)^2$. The domain of definition of this field is the same set A as above. Obvoiusly, H is an irrotational and solenoidal (and thus, harmonic) vector field.

We assume that a and b does not vanish simultaneously.
Parameter values: $a = 1$ and $b = 2$.

4.5 Analysis of the figures

We accept the existence of particles sensitive to the magnetic field:

TWO PARALLEL WIRES, CURRENTS OF THE SAME SENSE.

CASE 1: two branches, almost same energy; after jumps, the energy is stabilized; almost identical or parallel Poincaré projections.

CASE 2: two branches, almost same energy; the energy has jumps; almost identical or parallel Poincaré projections.

CASE 3: cusp shape; the energy is increasing; almost continuous Poincaré projections.

CASE 4: two branches, almost same energy; the energy has ajump; almost identical or parallel Poincaré projections.

CASE 5: idem.

TWO PARALLEL WIRES, CURRENTS OF OPPOSITE SENSE.

CASE 1: two branches, almost same energy; after jumps, the energy is stabilized like a constant; almost identical or parallel Poincaré projections.

CASE 2: idem.

CASE 3: cusp shape, almost same energy; after jumps, the energy is stabilized; parallel Poincaré projections.

CASE 4: two branches, almost same energy; after jumps, the energy is stabilized like a constant; almost identical or parallel Poincaré projections.

CASE 5: idem.

TWO RIGHT ANGLE CIRCUITS, CURRENTS OF OPPOSITE SENSE.

CASE 1: one curve; consatnt energy; almost smooth Poincaré projections.

CASE 2: idem.

CASE 3: idem.

CASE 4: idem.

CASE 5: idem.

TWO RIGHT ANGLE CIRCUITS, CURRENTS OF THE SAME SENSE.

CASE 1: one curve; consatnt energy; almost smooth Poincaré projections.

CASE 2: idem.

CASE 3: idem.

CASE 4: idem.

CASE 5: idem.

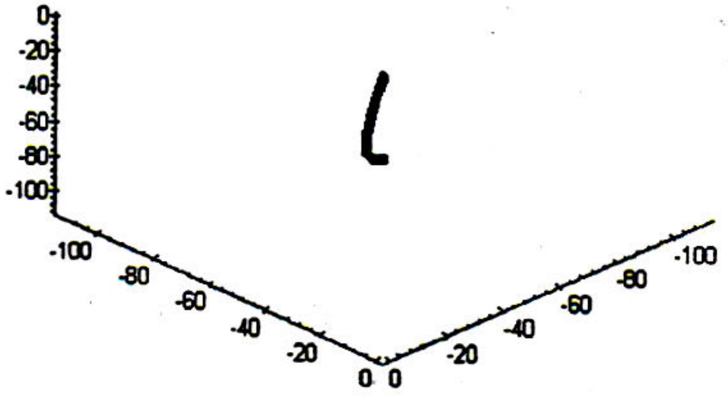
4.6 Illustrative Examples

Two parallel wires, currents of the same sense

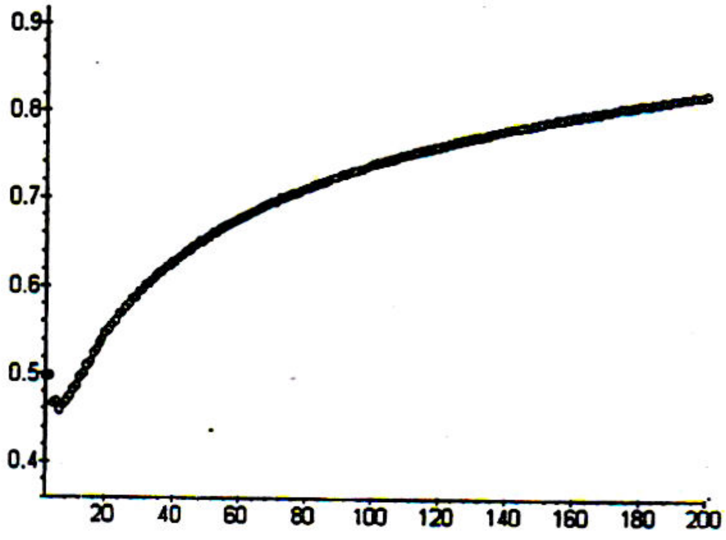
CASE 1. $n = 200$, $h = 0.05$;

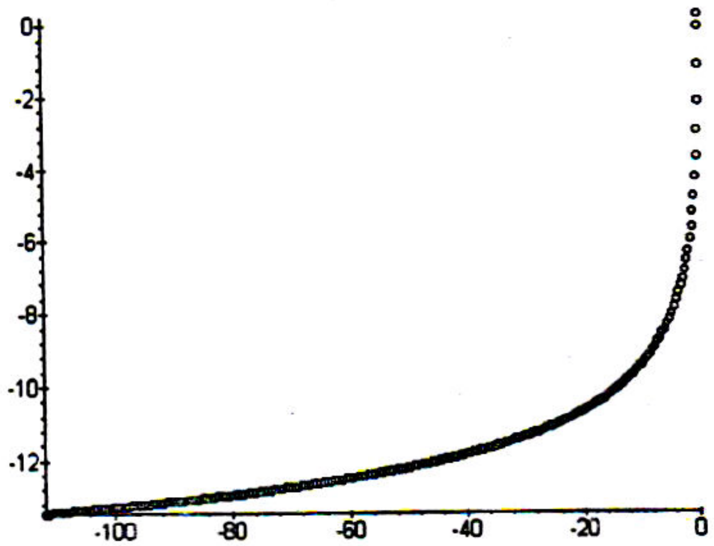
Boundary Conditions: $x(1) = -0.001$; $x(2) = 0.011$; $y(1) = -0.001$; $y(2) = -0.002$; $z(1) = 1$; $z(2) = 0.001$.

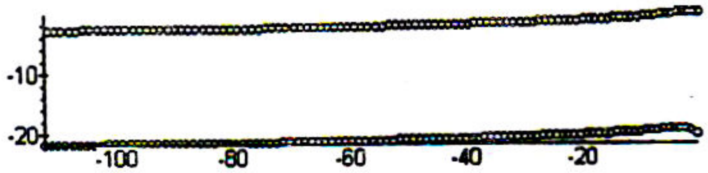
x[k],y[k]],z[k],k=1..n+1

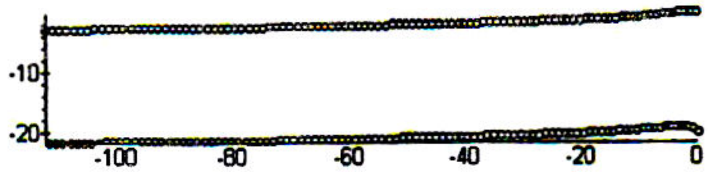


(k,Ed[k]),k=2..n

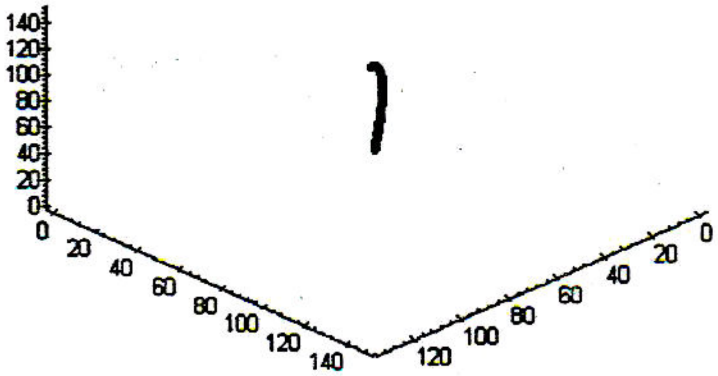


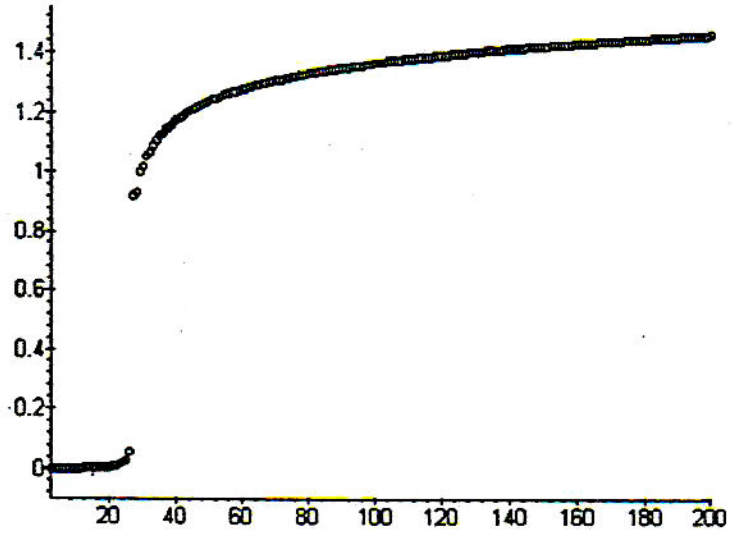

(x[k]+x[k+1])/2,(x[k+1]-x[k])/h

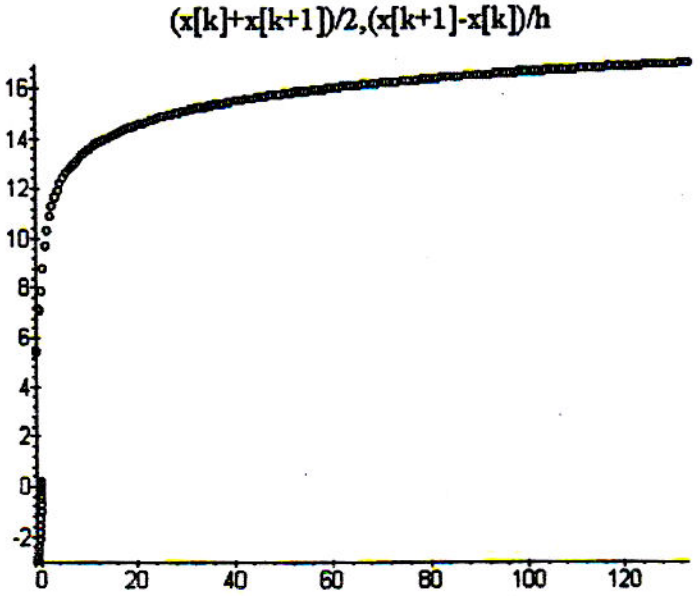

(y[k]+y[k+1])/2,(y[k+1]-y[k])/h


(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

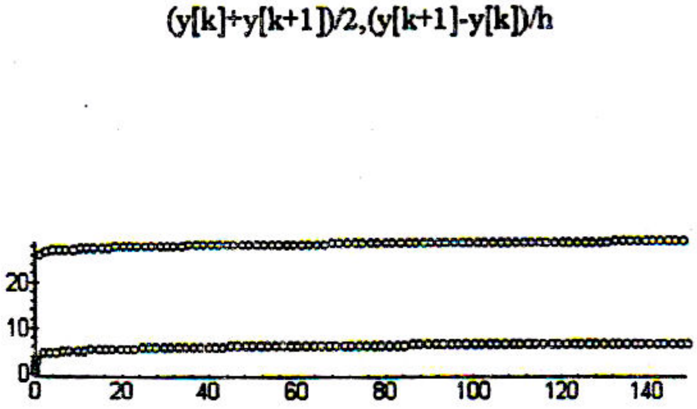
CASE 2. $n = 200$, $h = 0.05$;
Boundary Conditions: $x(1) = -0.0051$; $x(2) = 0.005$; $y(1) = 0$; $y(2) = 0$; $z(1) = 0.0005$; $z(2) = -0.005$.


x[k],y[k]],z[k],k=1..n+1

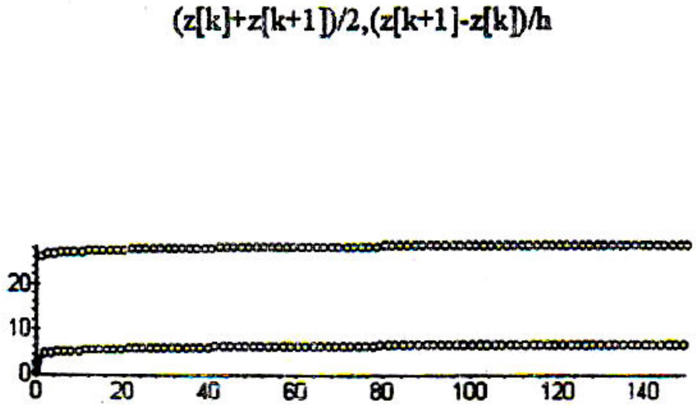

(k,Ed[k]),k=2..n



(x[k]+x[k+1])/2,(x[k+1]-x[k])/h



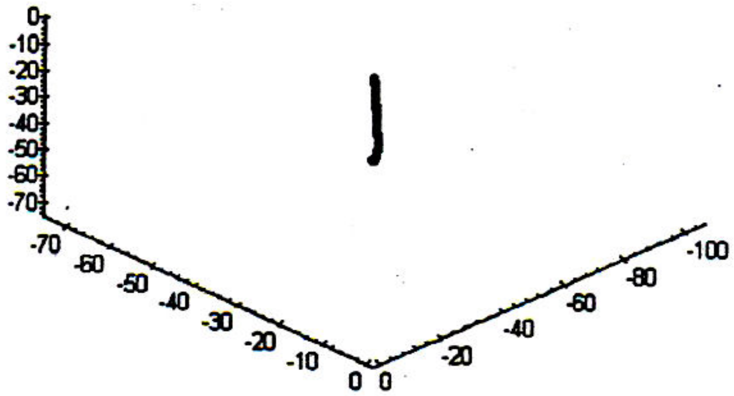
(y[k]+y[k+1])/2,(y[k+1]-y[k])/h



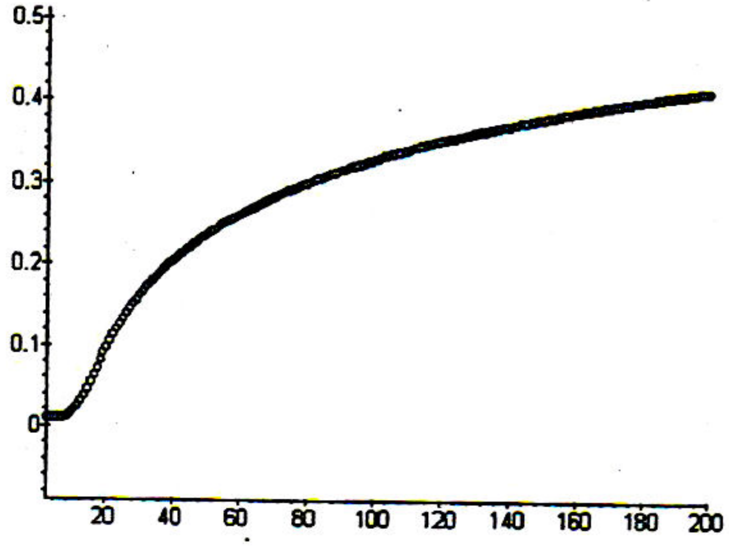
(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

Case 3. $n = 200$, $h = 0.05$;
Boundary Conditions: $x(1) = 0$; $x(2) = 0$; $y(1) = 0.05$; $y(2) = -0.05$; $z(1) = 0.05$; $z(2) = -0.05$.

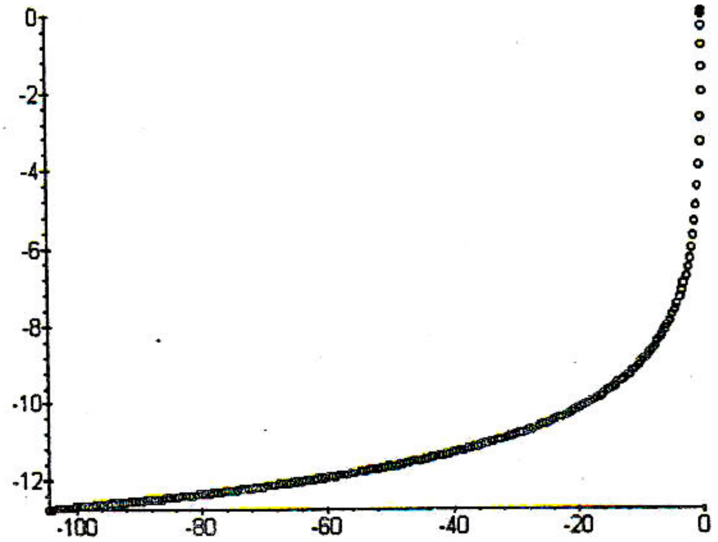
x[k],y[k]],z[k],k=1..n+1



(k,Ed[k]),k=2..n



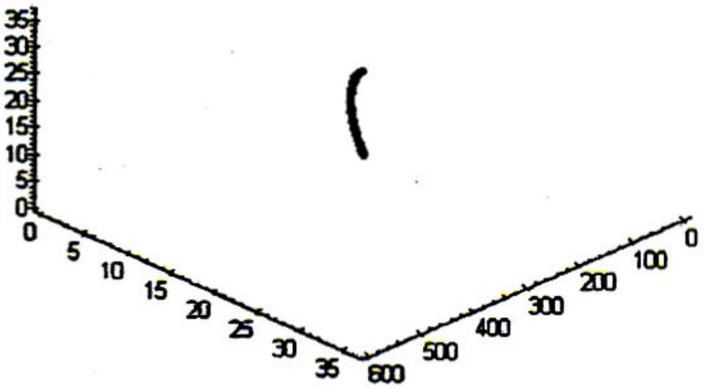
(x[k]+x[k+1])/2,(x[k+1]-x[k])/h

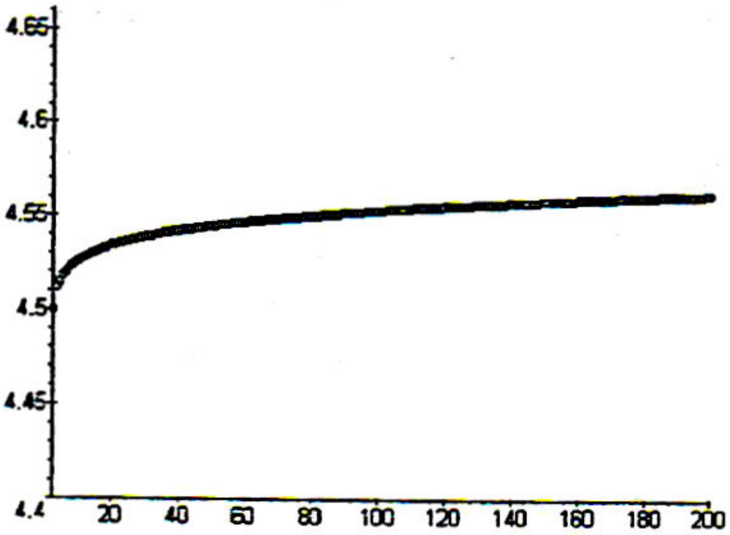


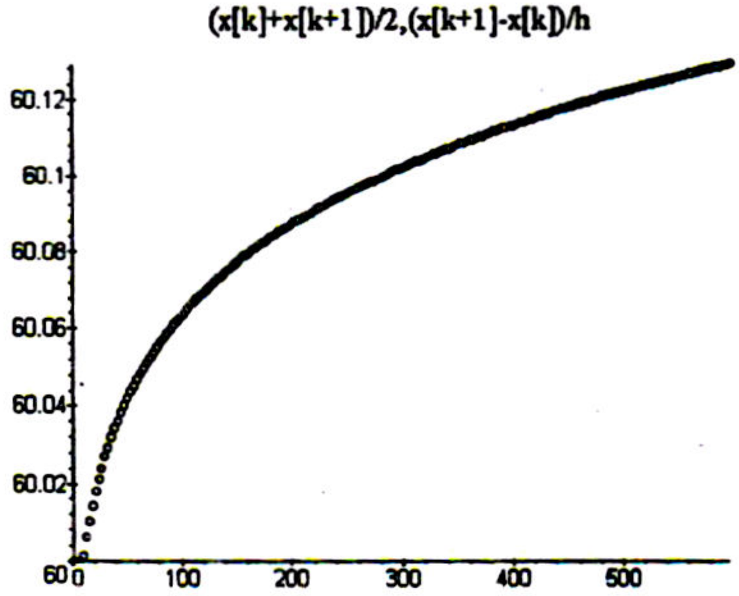
(y[k]+y[k+1])/2,(y[k+1]-y[k])/h

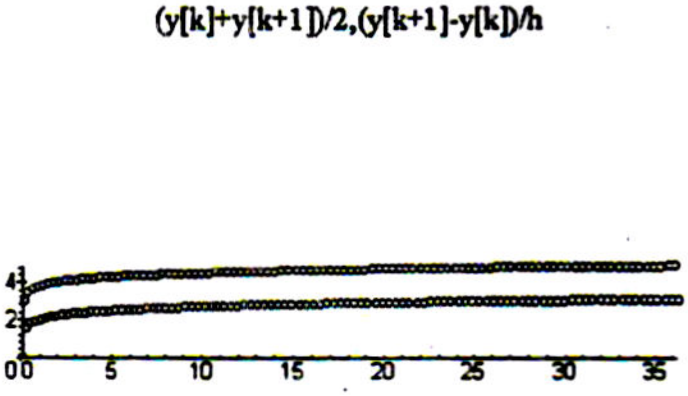
(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

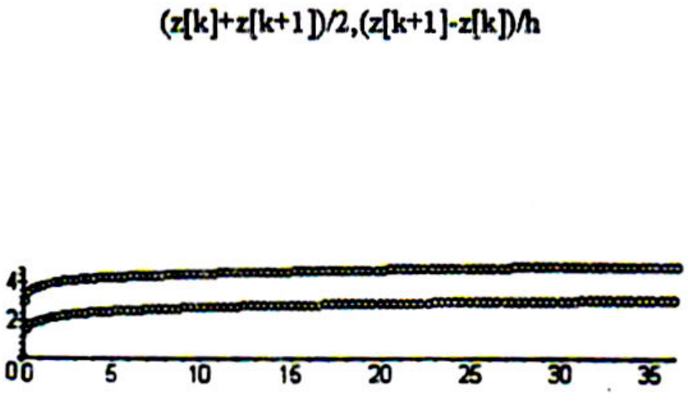
CASE 4. $n = 200$, $h = 0.05$;
Boundary Conditions: $x(1) = -1.5$; $x(2) = 1.5$; $y(1) = 0$; $y(2) = 0$; $z(1) = 0$; $z(2) = 0$.


x[k],y[k]],z[k],k=1..n+1

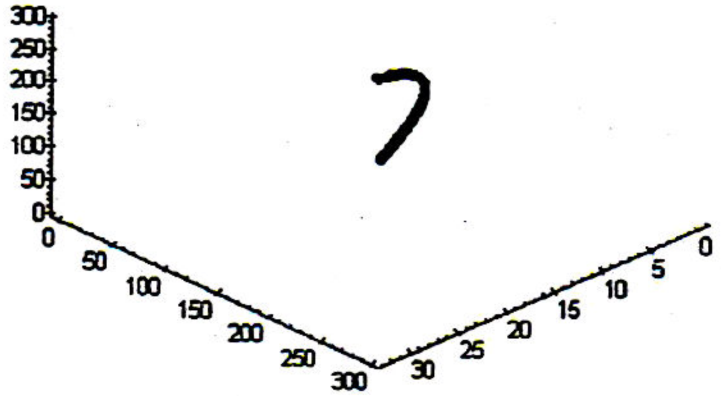

(k,Ed[k]),k=2..n


(x[k]+x[k+1])/2,(x[k+1]-x[k])/h
60.12
60.1
60.08
60.06
60.04
60.02
60
0
100
200
300
400
500

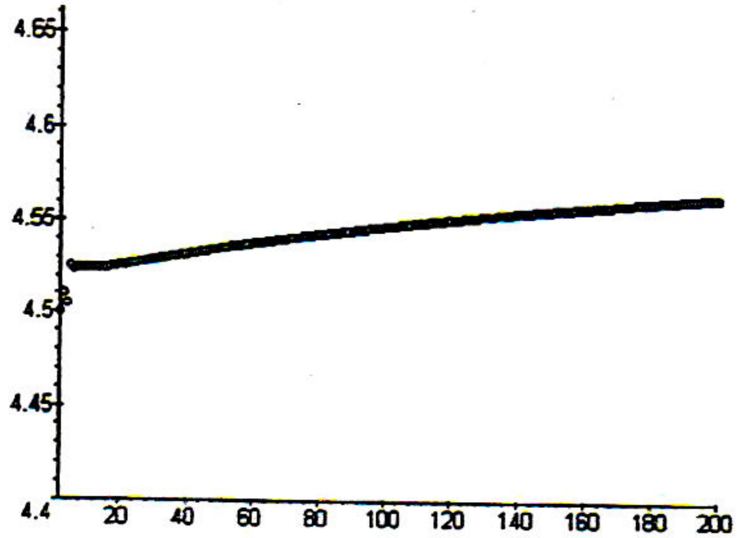

(y[k]+y[k+1])/2,(y[k+1]-y[k])/h
4
2
0
0
5
10
15
20
25
30
35


(z[k]+z[k+1])/2,(z[k+1]-z[k])/h
4
2
0
0
5
10
15
20
25
30
35

CASE 5. $n = 200$, $h = 0.05$;
Boundary Conditions: $x(1) = 0$; $x(2) = 0$; $y(1) = -1.5$; $y(2) = 1.5$; $z(1) = 0$; $z(2) = 0$.

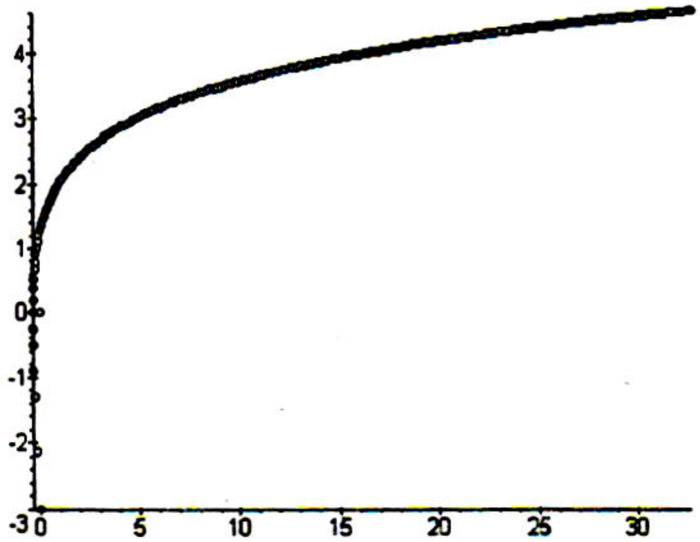


x[k],y[k]],z[k],k=1..n+1



(k,Ed[k]),k=2..n

(x[k]+x[k+1])/2,(x[k+1]-x[k])/h



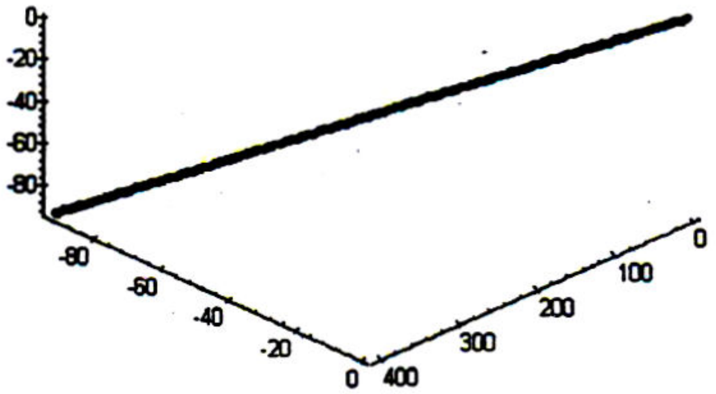
(y[k]+y[k+1])/2,(y[k+1]-y[k])/h

(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

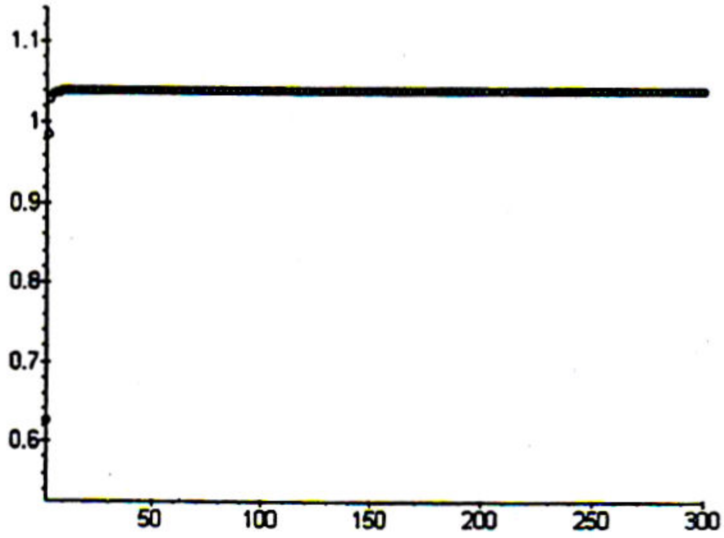
Two parallel wires, currents of opposite sense

CASE 1. $n = 300$, $h = 0.05$;
Boundary Conditions: $x(1) = 0.2$; $x(2) = 1.2$; $y(1) = 0.1$; $y(2) = 0.1$; $z(1) = 0.5$; $z(2) = 0$.

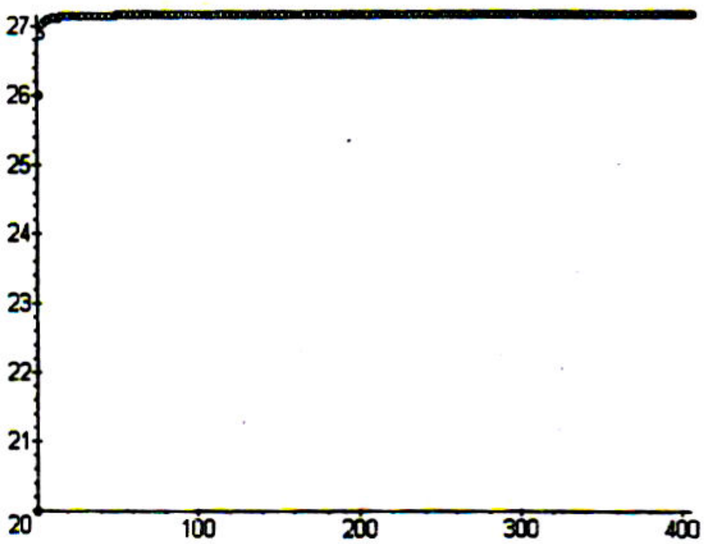
x[k],y[k]],z[k],k=1..n+1



(k,Ed[k]),k=2..n



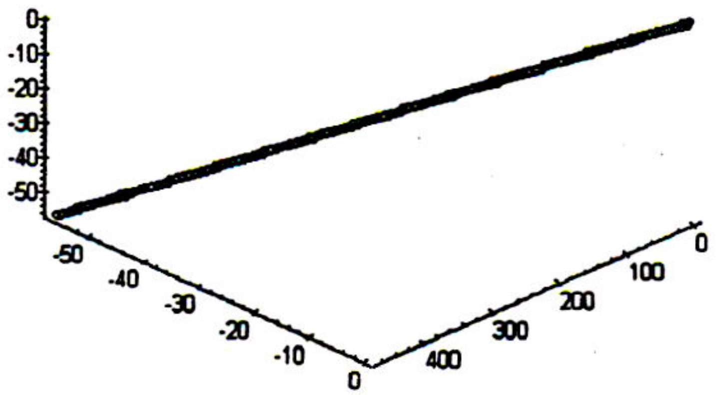
(x[k]+x[k+1])/2,(x[k+1]-x[k])/h



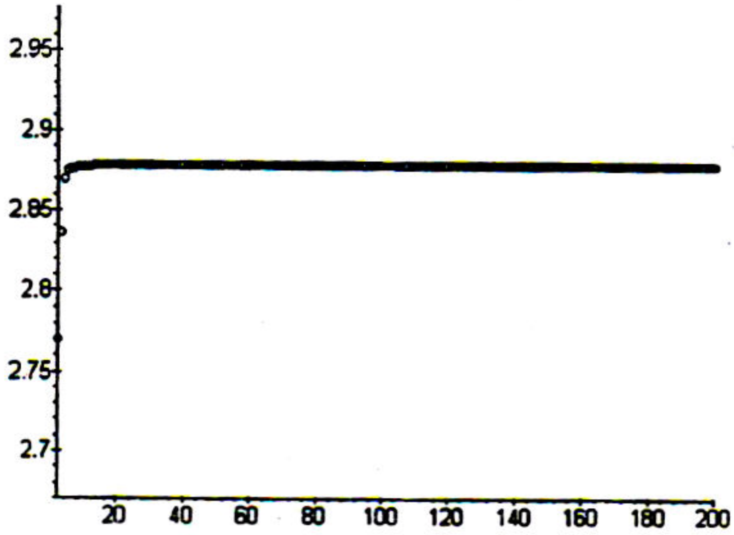
(y[k]+y[k+1])/2,(y[k+1]-y[k])/h

(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

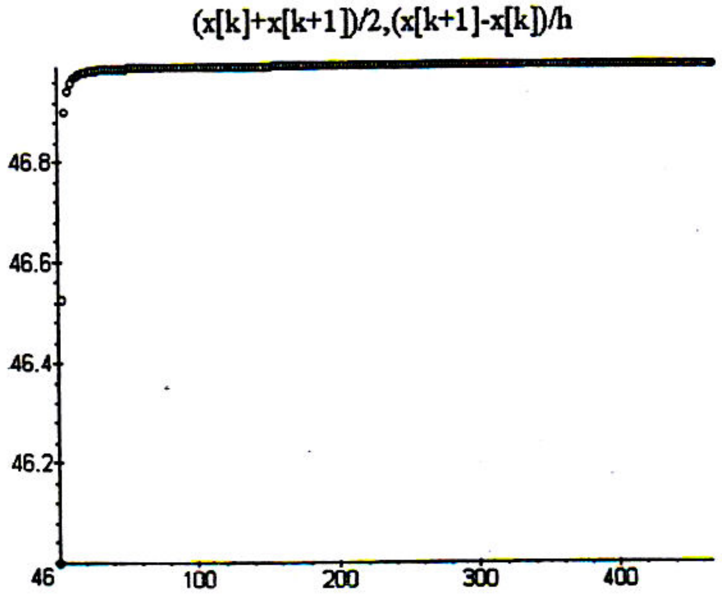
CASE 2. $n = 200$, $h = 0.05$;
Boundary Conditions: $x(1) = -1.1$; $x(2) = 1.2$; $y(1) = 0.1$; $y(2) = 0.1$; $z(1) = 0.5$; $z(2) = 0$.

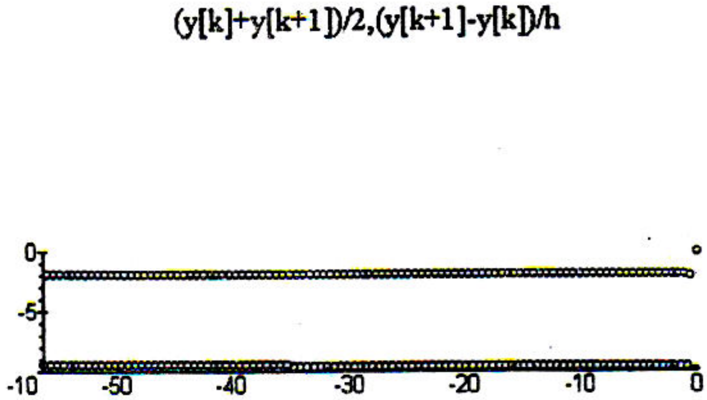


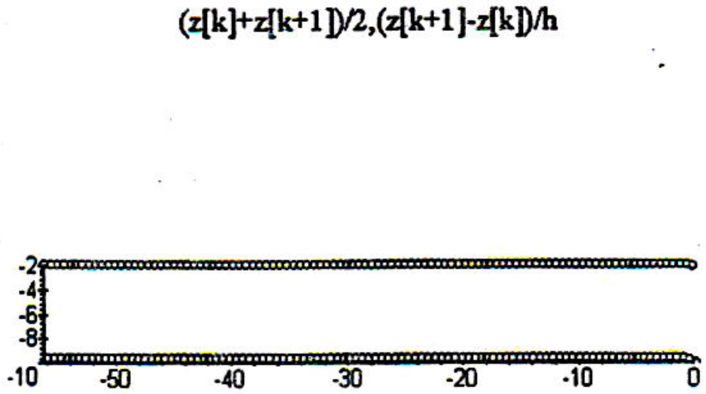
x[k],y[k]],z[k],k=1..n+1



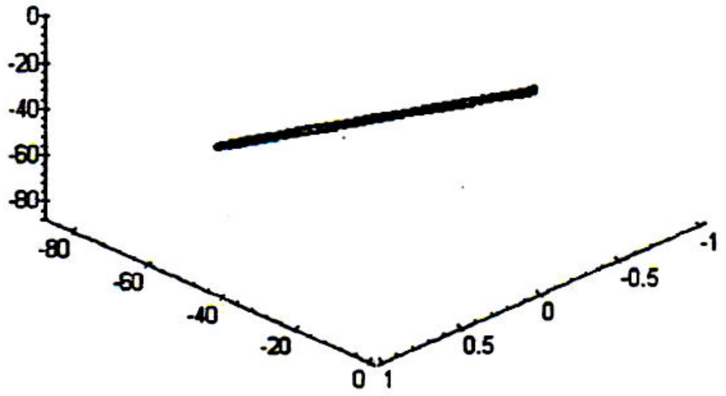
(k,Ed[k]),k=2..n

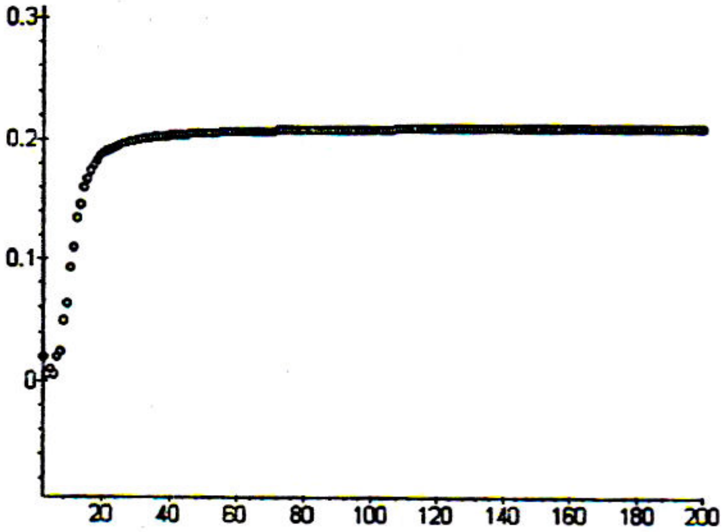

(x[k]+x[k+1])/2,(x[k+1]-x[k])/h


(y[k]+y[k+1])/2,(y[k+1]-y[k])/h


(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

Case 3. $n = 200$, $h = 0.05$;
Boundary Conditions: $x(1) = 0$; $x(2) = 0$; $y(1) = -0.1$; $y(2) = 0.1$; $z(1) = 0$; $z(2) = 0$.


x[k],y[k]],z[k],k=1..n+1


(k,Ed[k]),k=2..n

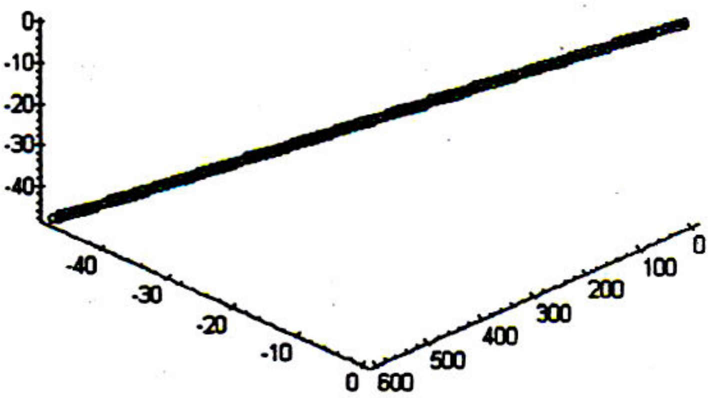
$$(x[k] + x[k+1])/2, (x[k+1] - x[k])/h$$

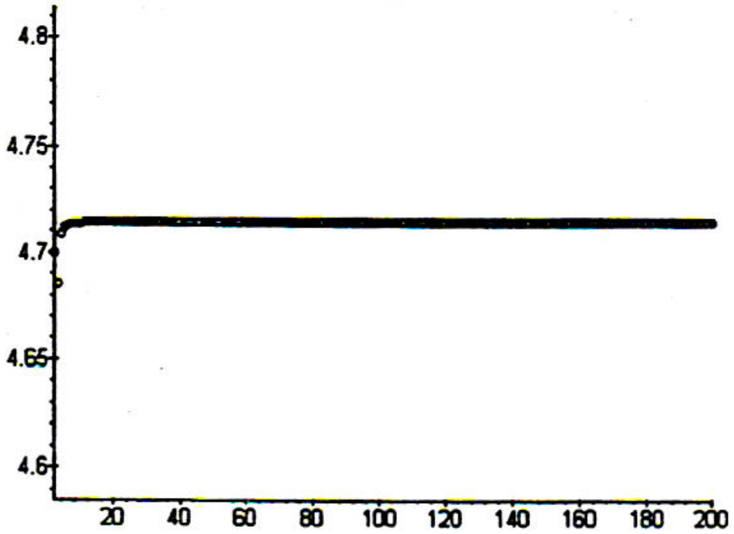
Error, (in plot) Invalid domain specification for plotting points

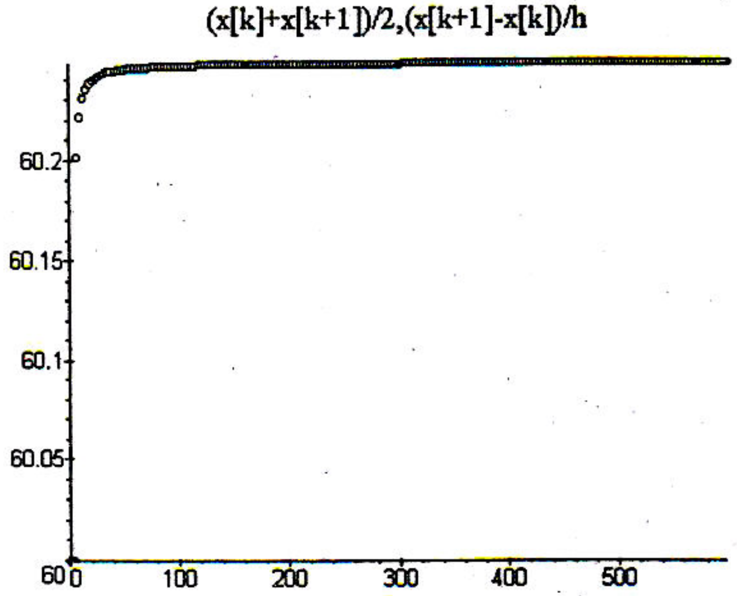
(y[k]+y[k+1])/2,(y[k+1]-y[k])/h

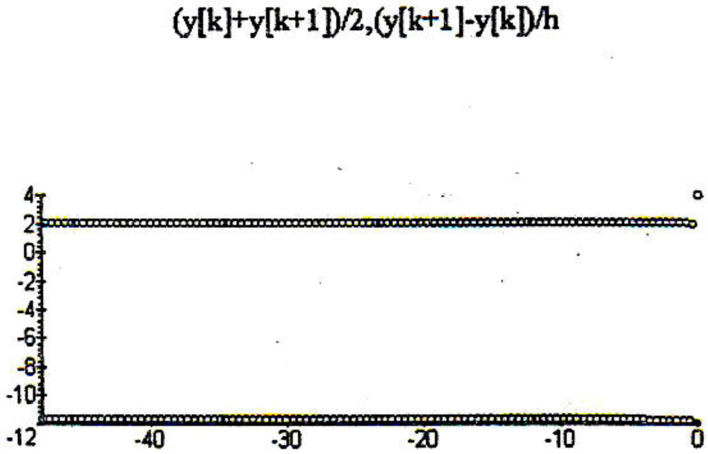
(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

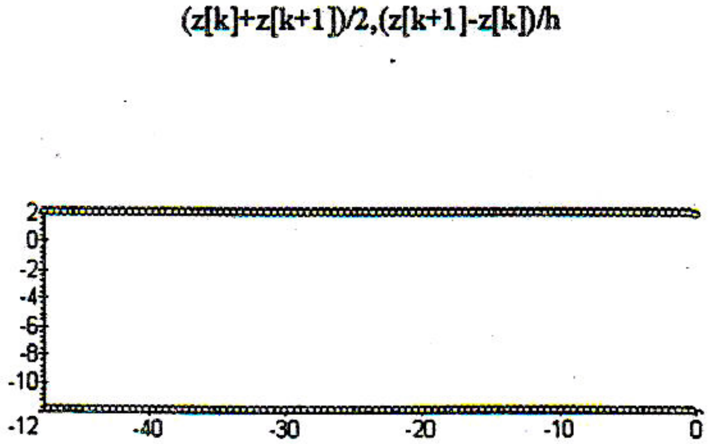
CASE 4. $n = 200$, $h = 0.05$;
Boundary Conditions: $x(1) = -1.5$; $x(2) = 1.5$; $y(1) = -0.1$; $y(2) = 0.1$; $z(1) = 0.5$; $z(2) = -0.1$.


x[k],y[k]],z[k],k=1..n+1


(k,Ed[k]),k=2..n

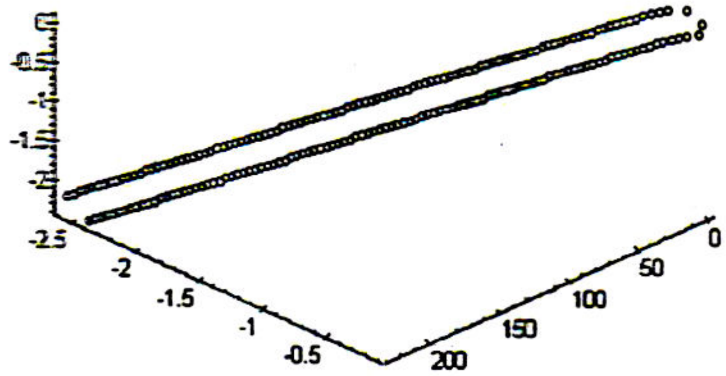

(x[k]+x[k+1])/2,(x[k+1]-x[k])/h


(y[k]+y[k+1])/2,(y[k+1]-y[k])/h

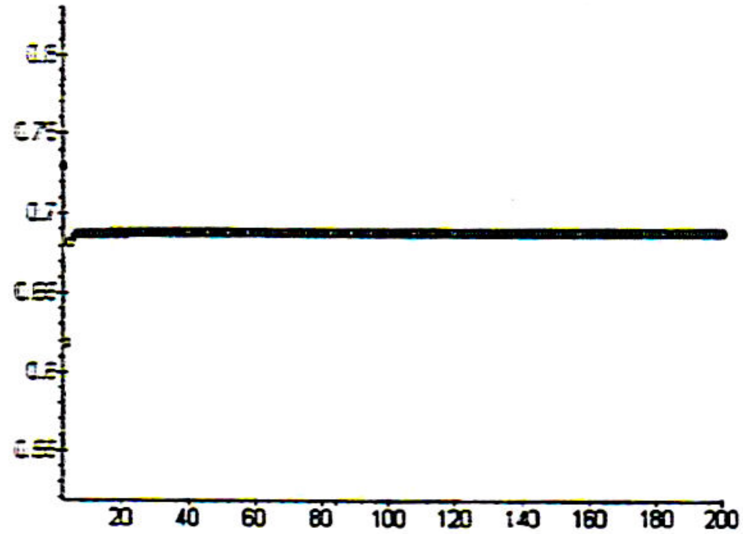

(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

CASE 5. $n = 200$, $h = 0.05$;
Boundary Conditions: $x(1) = -0.1$; $x(2) = 1.1$; $y(1) = -0.1$; $y(2) = -0.2$; $z(1) = 0$; $z(2) = 0.1$.

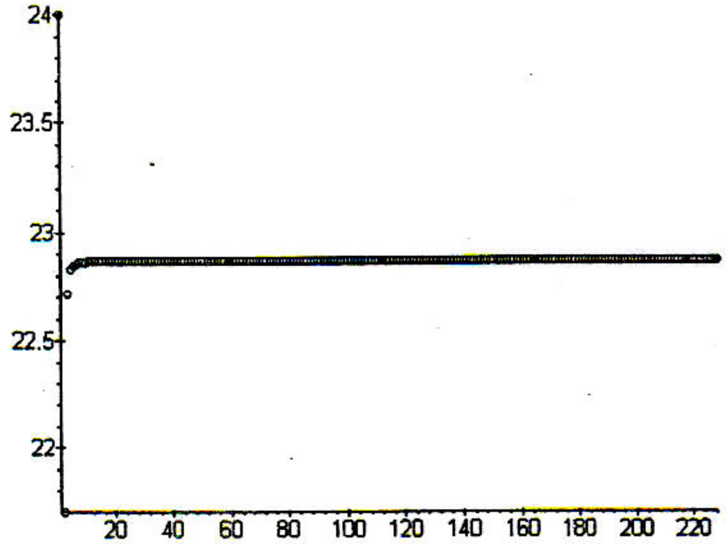
x[k],y[k]],z[k],k=1..n+1



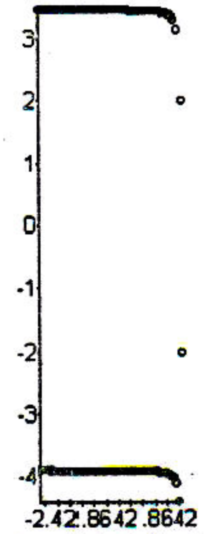
(k,Ed[k]),k=2..n



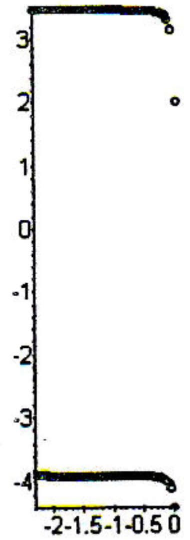
(x[k]+x[k+1])/2,(x[k+1]-x[k])/h



(y[k]+y[k+1])/2,(y[k+1]-y[k])/h

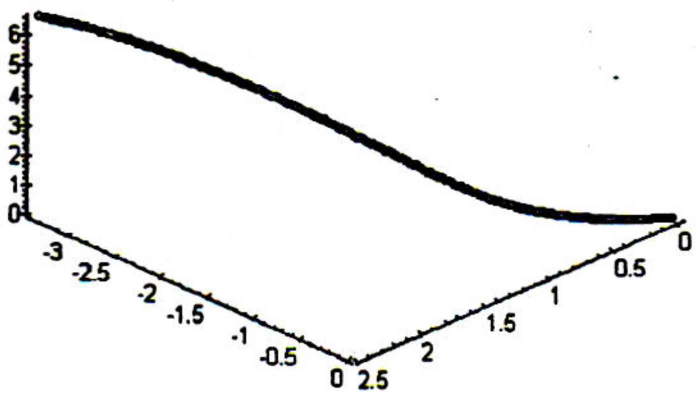


(z[k]+z[k+1])/2,(z[k+1]-z[k])/h



Two right angle circuits, currents of opposite sense

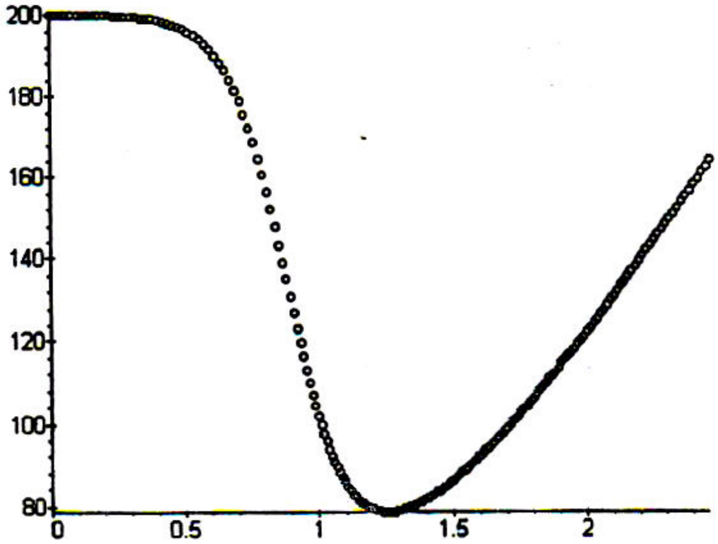
CASE 1. $n = 200$, $h = 0.0001$;
Boundary Conditions: $x(1) = -0.01$; $x(2) = 0.01$; $y(1) = 0$; $y(2) = 0$; $z(1) = -0.01$; $z(2) = 0.02$.



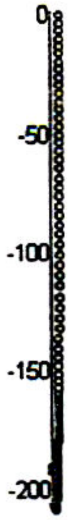
x[k],y[k]],z[k],k=1..n+1

(k,Ed[k]),k=2..n

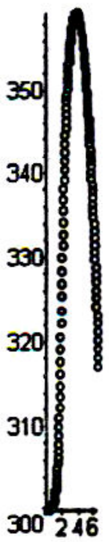
(x[k]+x[k+1])/2,(x[k+1]-x[k])/h



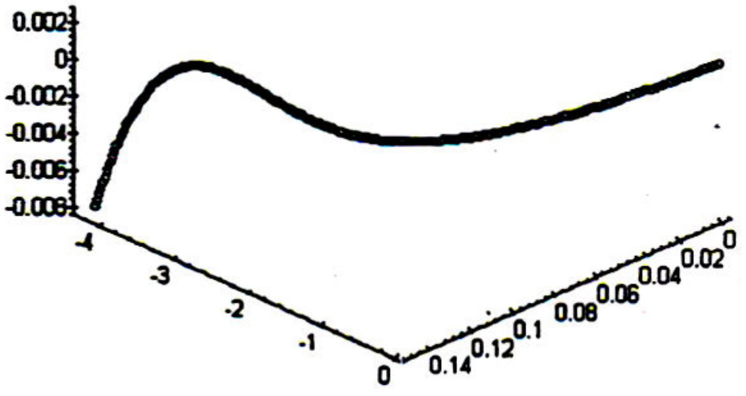
(y[k]+y[k+1])/2,(y[k+1]-y[k])/h

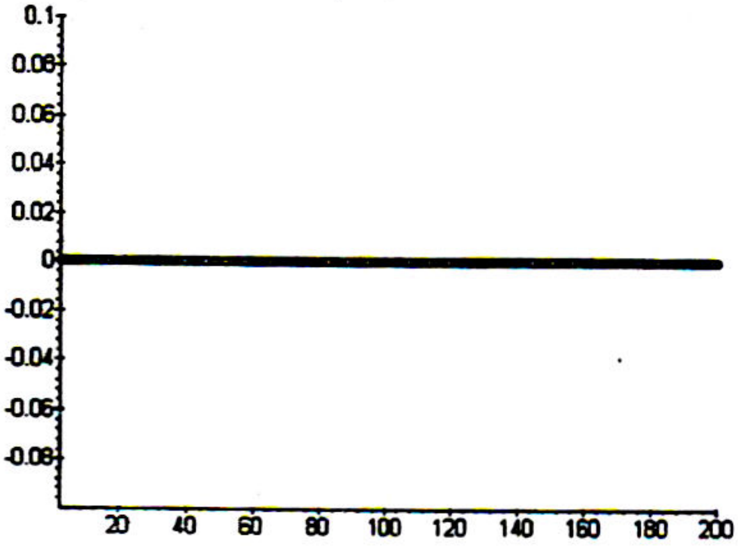


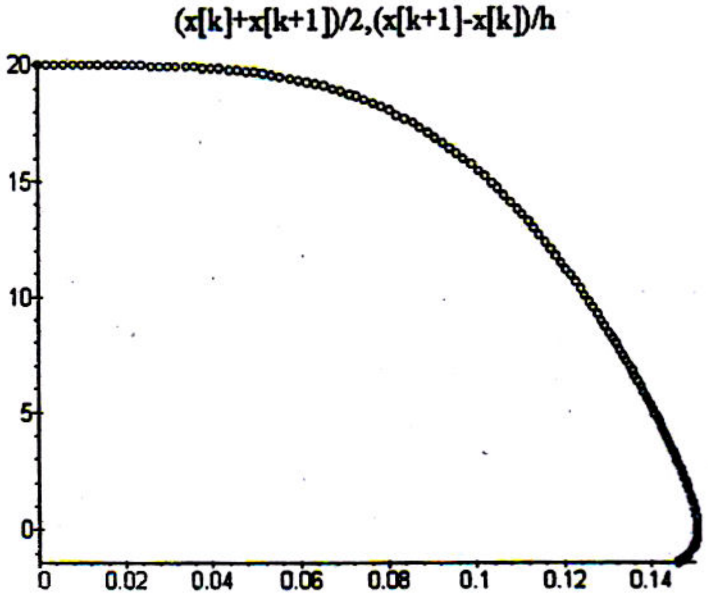
(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

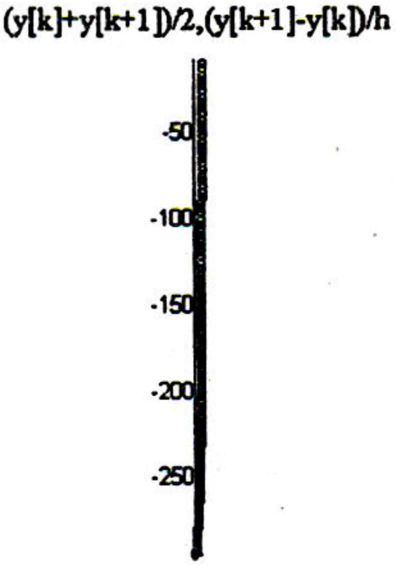


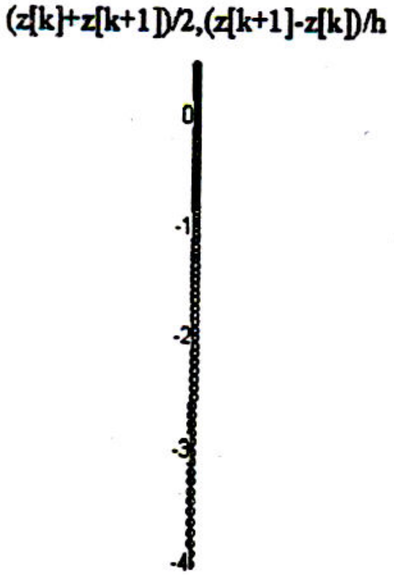
Case 2. $n = 200$, $h = 0.0001$;
Boundary Conditions: $x(1) = -0.001$; $x(2) = 0.001$; $y(1) = 0$; $y(2) = -0.001$; $z(1) = 0$; $z(2) = 0$.


x[k],y[k]],z[k],k=1..n+1


(k,Ed[k]),k=2..n

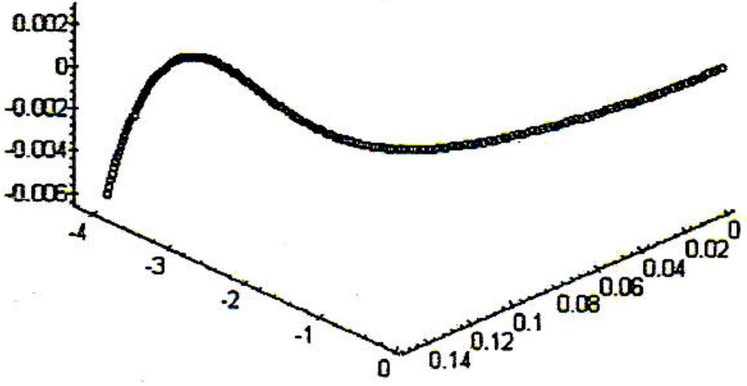

(x[k]+x[k+1])/2,(x[k+1]-x[k])/h
20
15
10
5
0
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14


(y[k]+y[k+1])/2,(y[k+1]-y[k])/h
-50
-100
-150
-200
-250

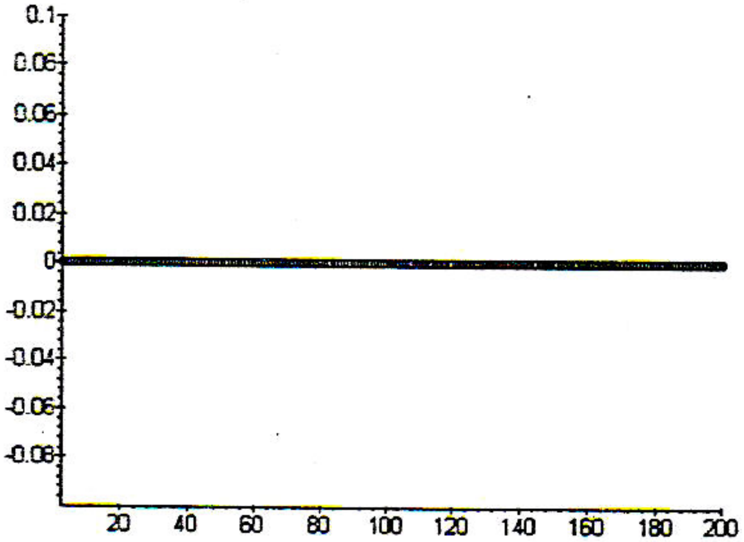

(z[k]+z[k+1])/2,(z[k+1]-z[k])/h
0
-1
-2
-3
-4

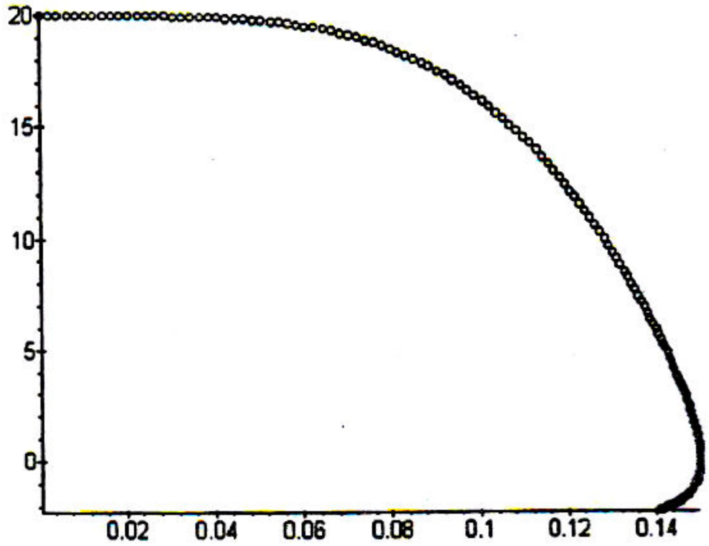
CASE 3. $n = 200$, $h = 0.0001$;
Boundary Conditions: $x(1) = 0$; $x(2) = 0.002$; $y(1) = 0.002$; $y(2) = 0.002$; $z(1) = 0.0001$; $z(2) = 0.0001$.

x[k],y[k]],z[k],k=1..n+1

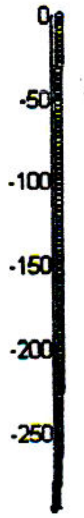


(k,Ed[k]),k=2..n





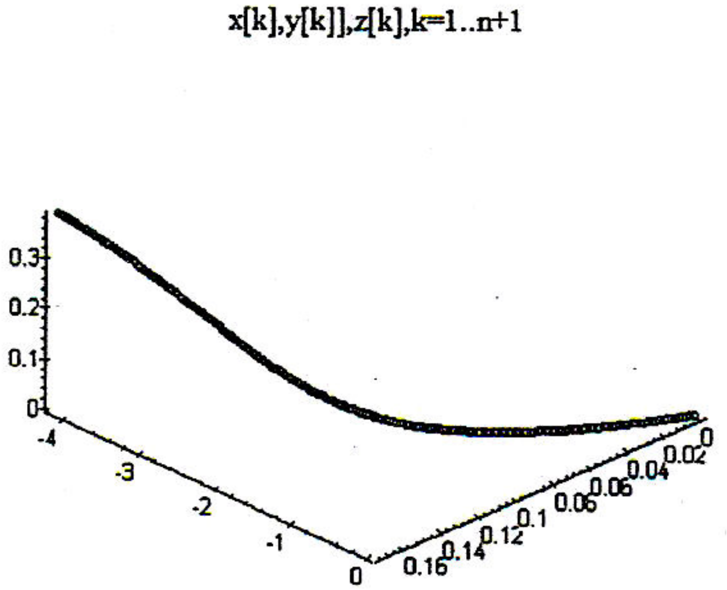
(x[k]+x[k+1])/2,(x[k+1]-x[k])/h

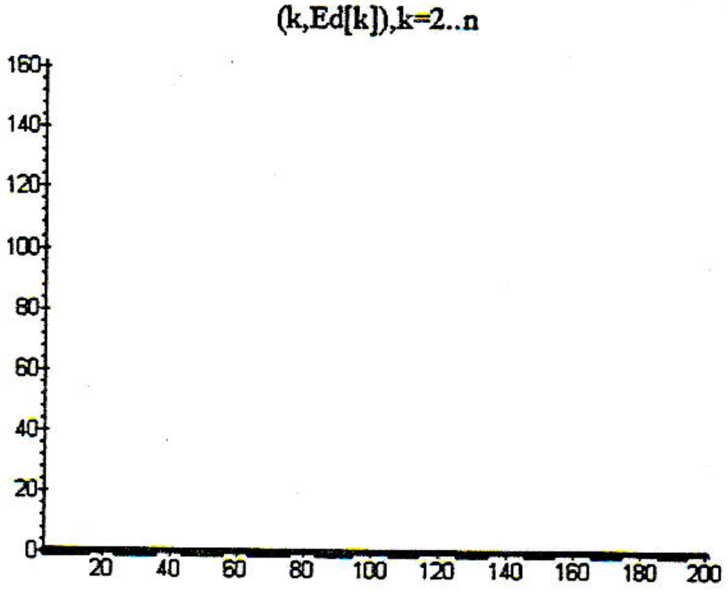


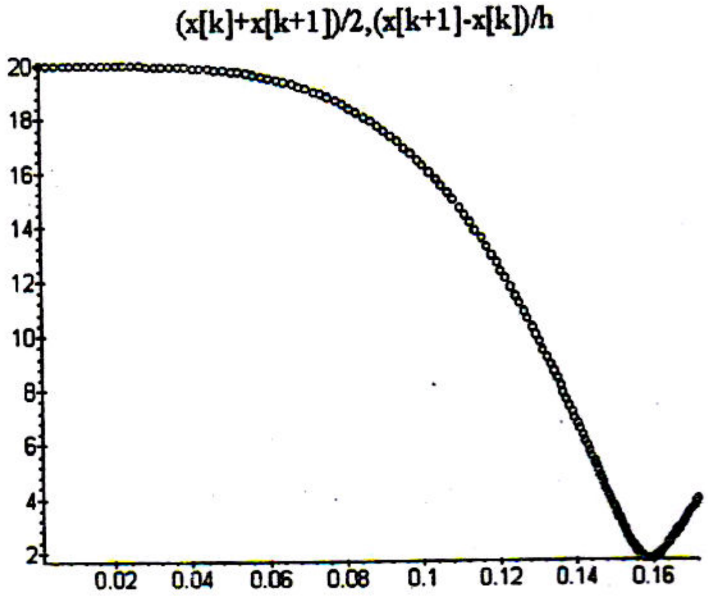
(y[k]+y[k+1])/2,(y[k+1]-y[k])/h

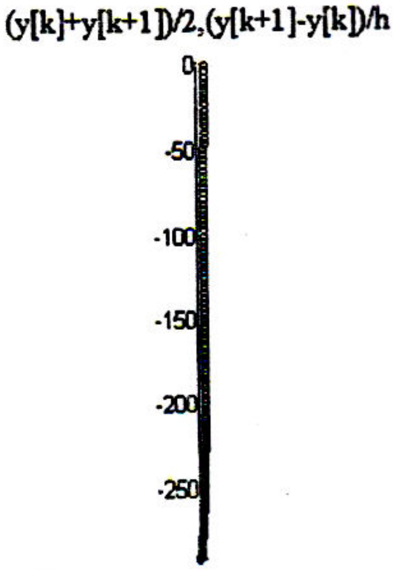
(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

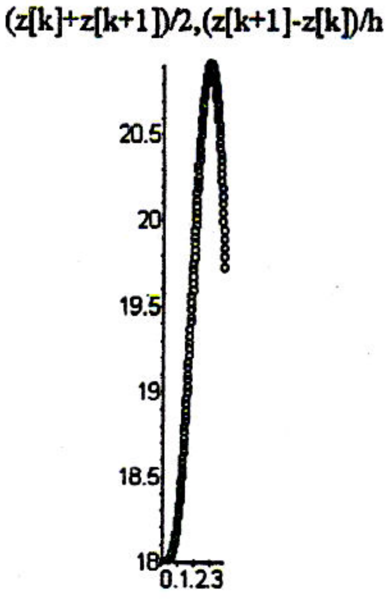
CASE 4. $n = 200$, $h = 0.0001$;
Boundary Conditions: $x(1) = 0$; $x(2) = 0.002$; $y(1) = 0.002$; $y(2) = 0.002$; $z(1) = 0$; $z(2) = 0.0018$.



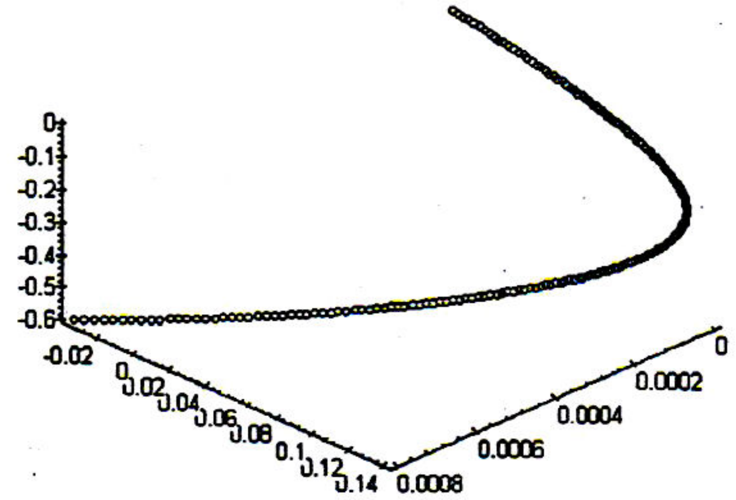



(x[k]+x[k+1])/2,(x[k+1]-x[k])/h
20
18
16
14
12
10
8
6
4
2
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16

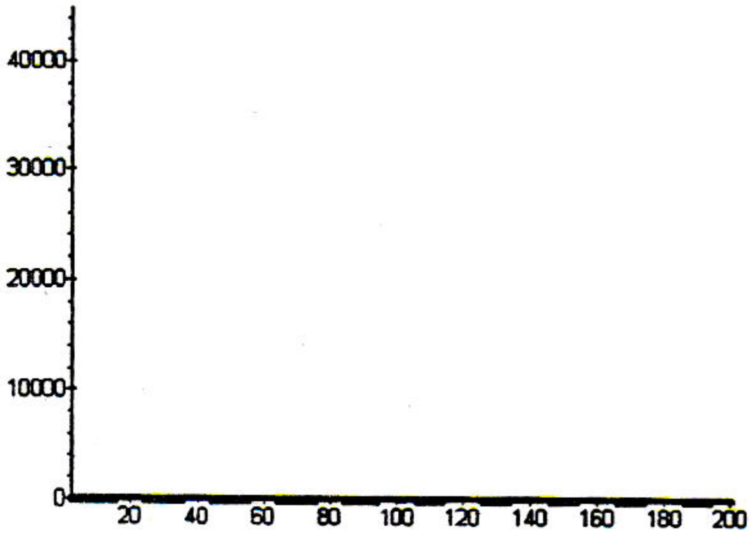

(y[k]+y[k+1])/2,(y[k+1]-y[k])/h
0
-50
-100
-150
-200
-250


(z[k]+z[k+1])/2,(z[k+1]-z[k])/h
20.5
20
19.5
19
18.5
18
0.1.23

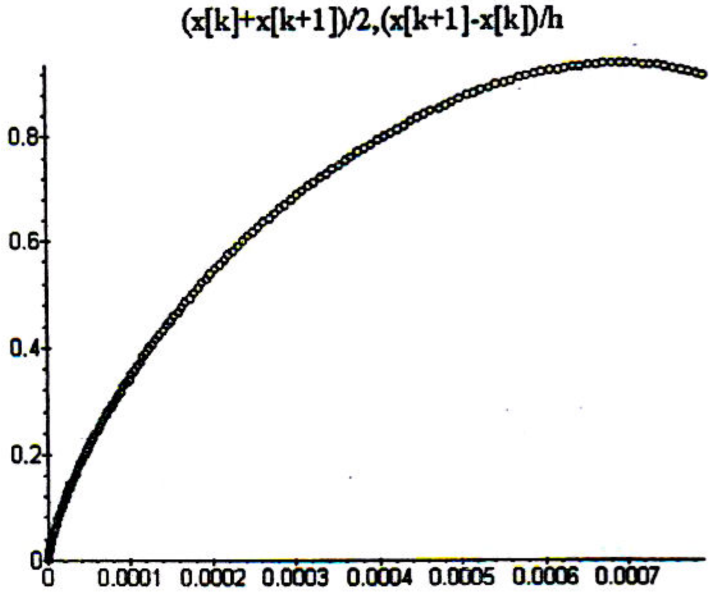
Case 5. $n = 200$, $h = 0.00001$;
Boundary Conditions: $x(1) = 0$; $x(2) = 0$; $y(1) = -0.001$; $y(2) = 0.002$; $z(1) = 0.002$; $z(2) = -0.001$.

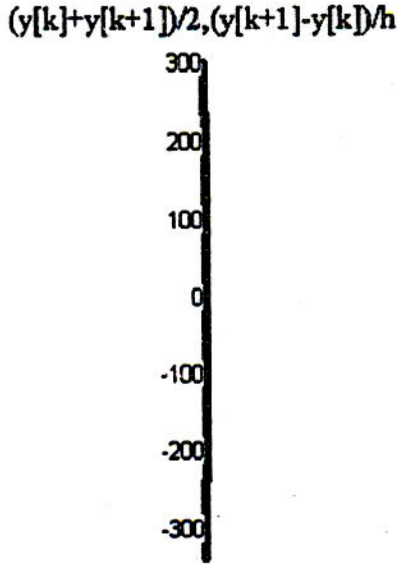


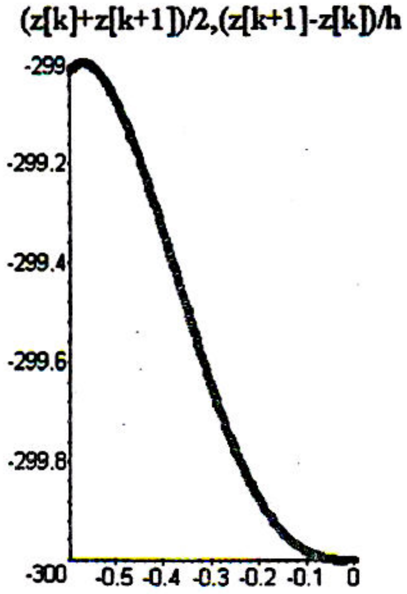
x[k],y[k]],z[k],k=1..n+1



(k,Ed[k]),k=2..n


(x[k]+x[k+1])/2,(x[k+1]-x[k])/h
0.8
0.6
0.4
0.2
0
0
0.0001
0.0002
0.0003
0.0004
0.0005
0.0006
0.0007

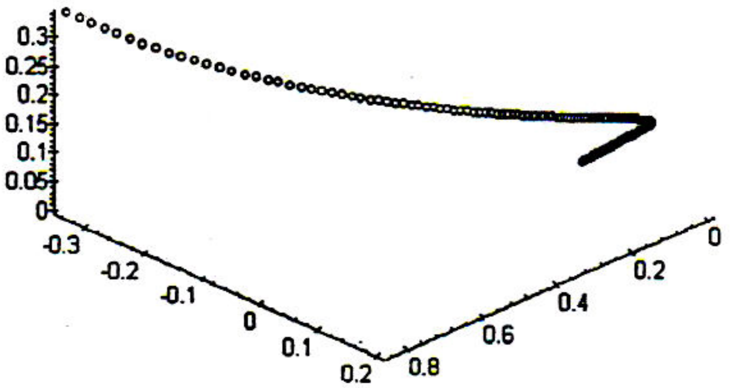

(y[k]+y[k+1])/2,(y[k+1]-y[k])/h
300
200
100
0
-100
-200
-300


(z[k]+z[k+1])/2,(z[k+1]-z[k])/h
-299
-299.2
-299.4
-299.6
-299.8
-300
-0.5
-0.4
-0.3
-0.2
-0.1
0

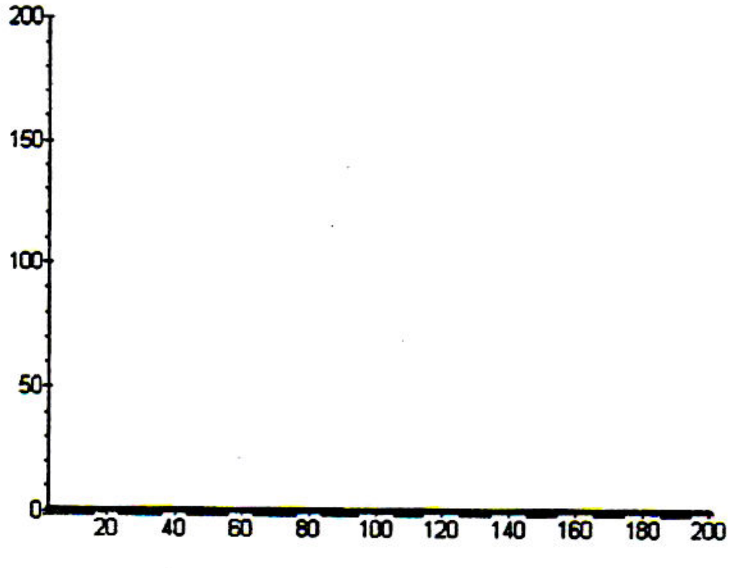
Two right angle circuits, currents of the same sense

CASE 1. $n = 200$, $h = 0.0001$;

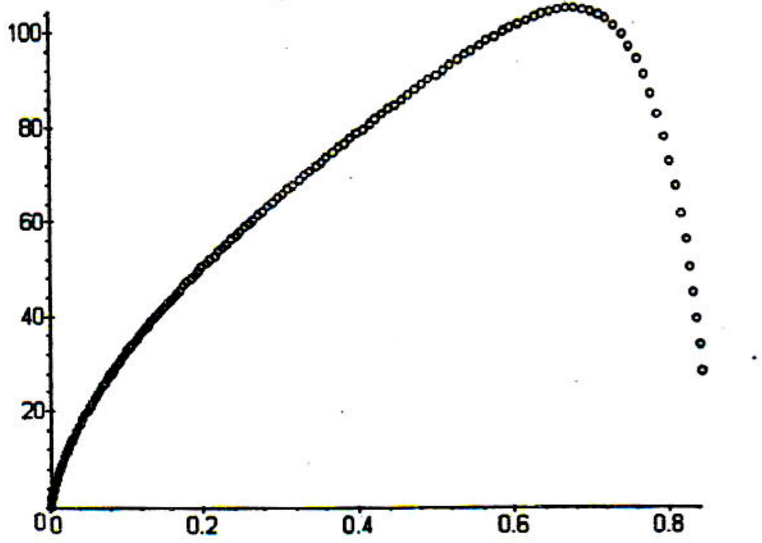
Boundary Conditions: $x(1) = 0.$; $x(2) = 0.$; $y(1) = 0$; $y(2) = 0.002$; $z(1) = -0.001$; $z(2) = 0.001$.



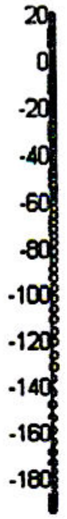
x[k],y[k]],z[k],k=1..n+1



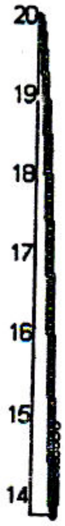
(k,Ed[k]),k=2..n



(x[k]+x[k+1])/2,(x[k+1]-x[k])/h

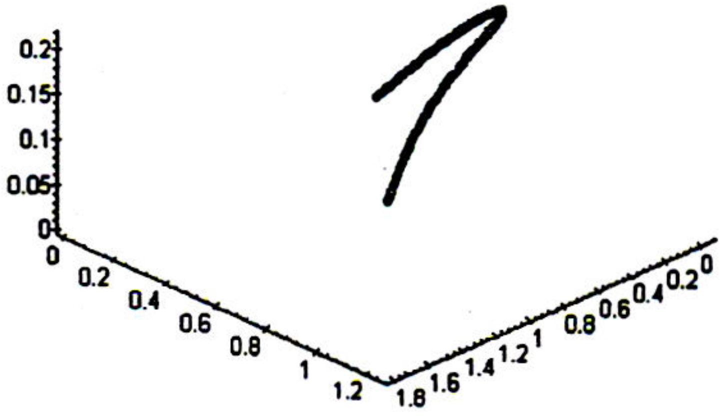


(y[k]+y[k+1])/2,(y[k+1]-y[k])/h

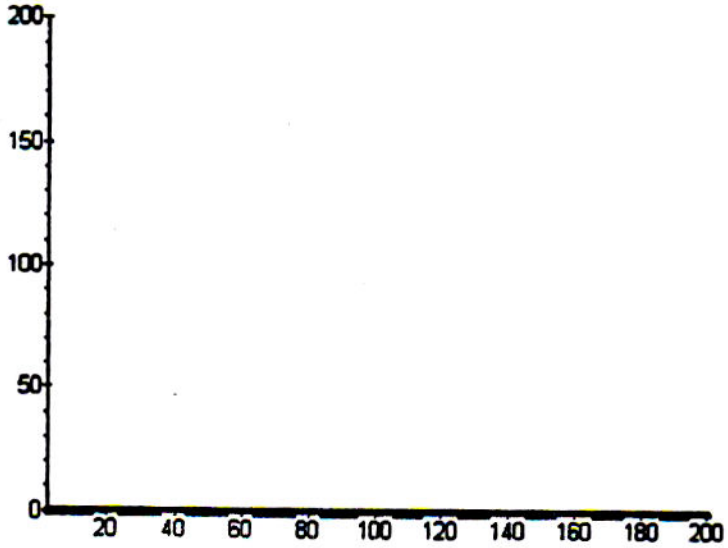


(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

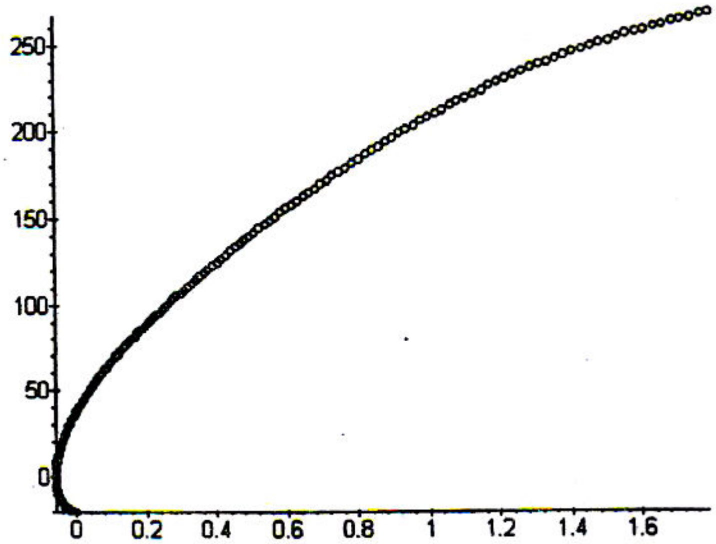
CASE 2. $n = 200$, $h = 0.0001$;
Boundary Conditions: $x(1) = 0.001$; $x(2) = -0.001$; $y(1) = 0$; $y(2) = 0.005$; $z(1) = -0.001$; $z(2) = 0.001$.



x[k],y[k]],z[k],k=1..n+1



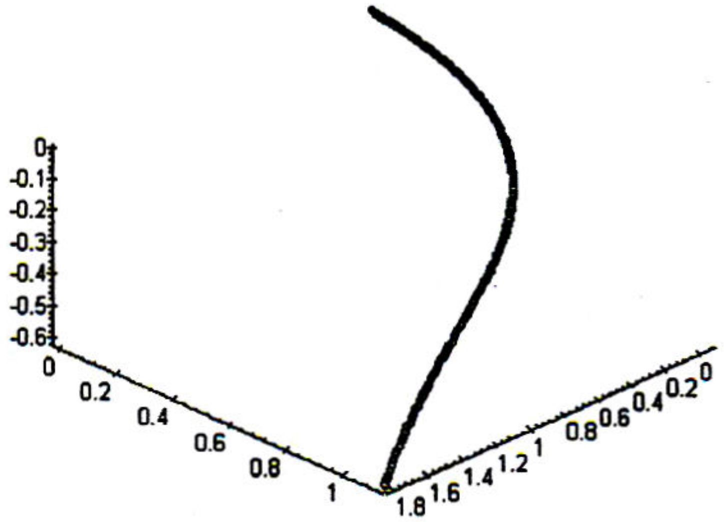
(k,Ed[k]),k=2..n

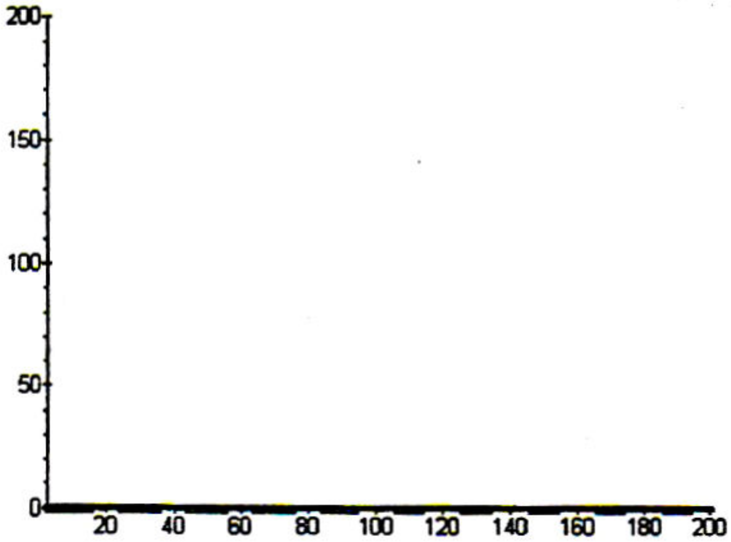

(x[k]+x[k+1])/2,(x[k+1]-x[k])/h

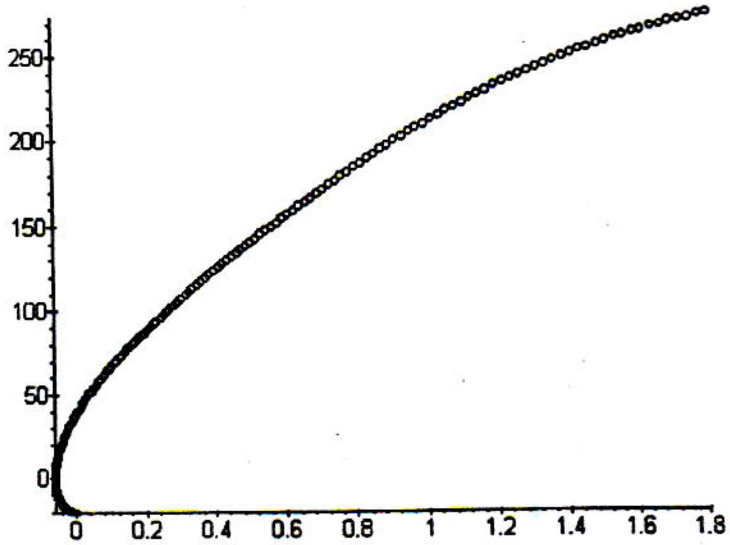

(y[k]+y[k+1])/2,(y[k+1]-y[k])/h

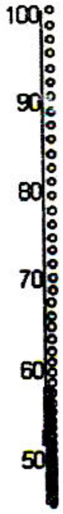

(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

CASE 3. $n = 200$, $h = 0.0001$;
Boundary Conditions: $x(1) = 0.001$; $x(2) = -0.001$; $y(1) = 0$; $y(2)0.005$; $z(1) = 0.001$; $z(2) = -0.001$.


x[k],y[k]],z[k],k=1..n+1

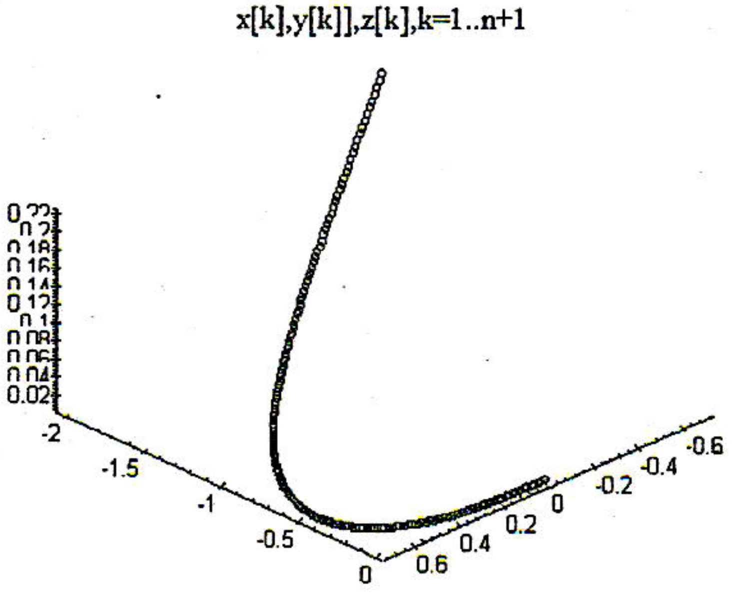

(k,Ed[k]),k=2..n

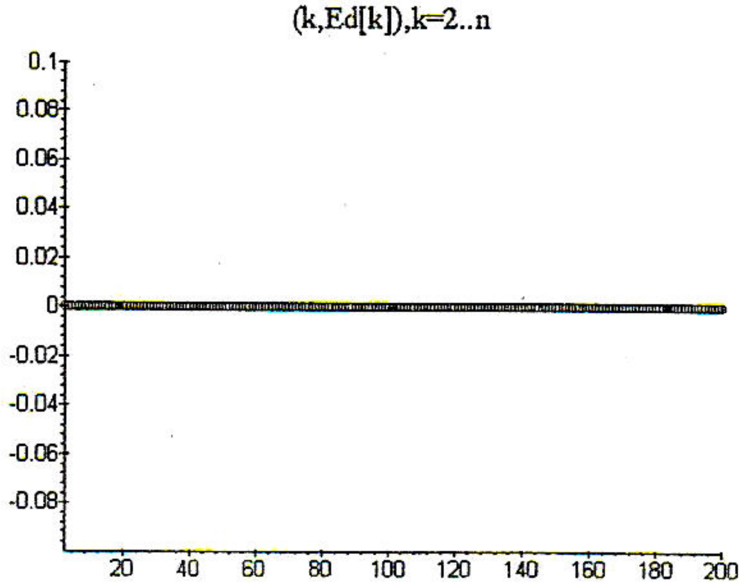

(x[k]+x[k+1])/2,(x[k+1]-x[k])/h

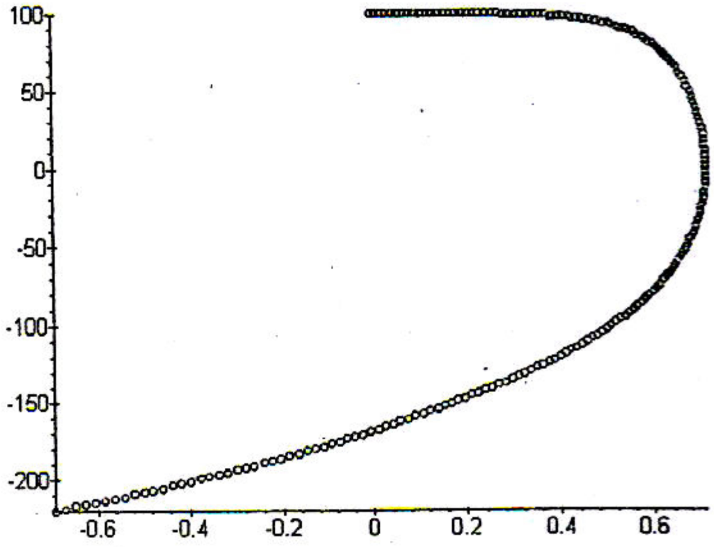

(y[k]+y[k+1])/2,(y[k+1]-y[k])/h


(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

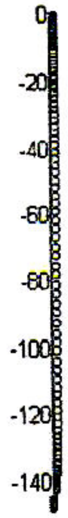
CASE 4. $n = 200$, $h = 0.0001$;
Boundary Conditions: $x(1) = -0.005$; $x(2) = 0.005$; $y(1) = 0$; $y(2) = 0$; $z(1) = 0.005$; $z(2) = 0.005$.







(x[k]+x[k+1])/2,(x[k+1]-x[k])/h

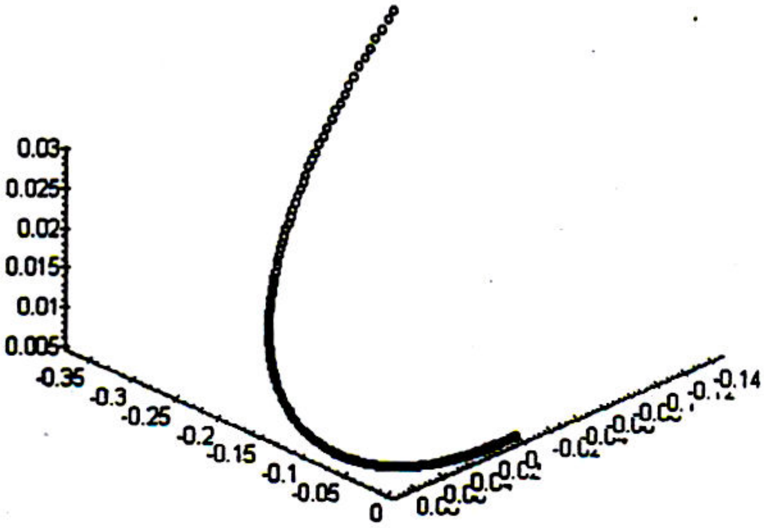


(y[k]+y[k+1])/2,(y[k+1]-y[k])/h

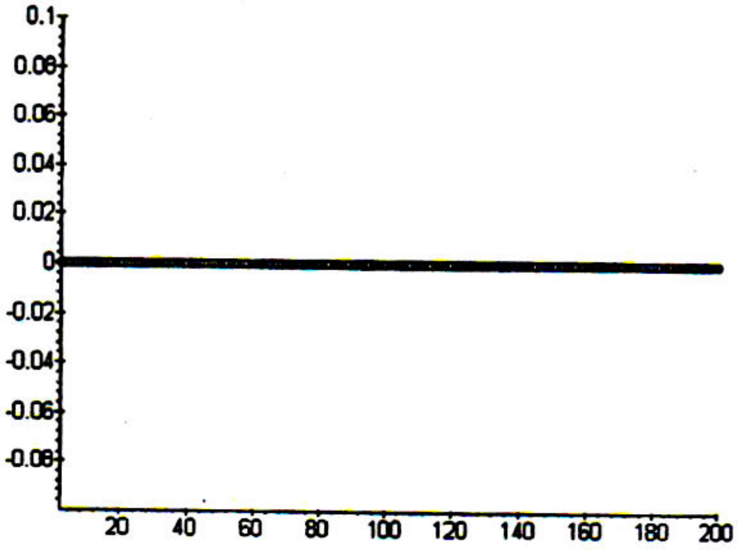
(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

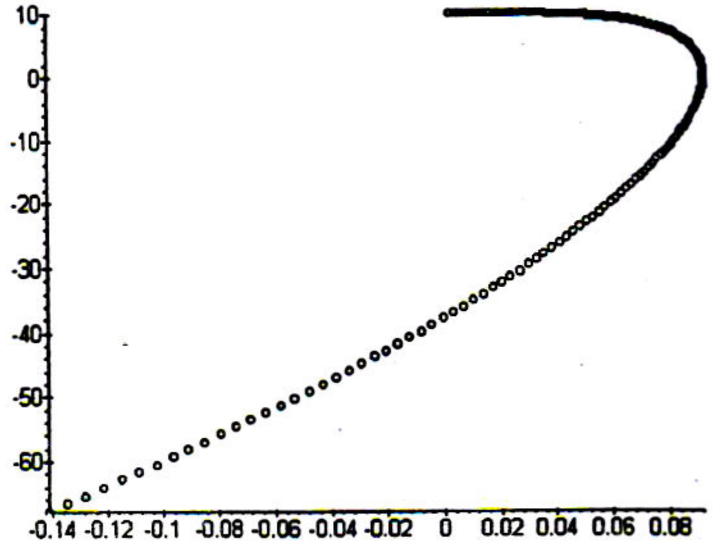
Case 5. $n = 200$, $h = 0.0001$;
Boundary Conditions: $x(1) = 0.001$; $x(2) = 0.002$; $y(1) = 0$; $y(2) = 0$; $z(1) = 0.005$; $z(2) = 0.005$.

x[k],y[k]],z[k],k=1..n+1



(k,Ed[k]),k=2..n




(x[k]+x[k+1])/2,(x[k+1]-x[k])/h


(y[k]+y[k+1])/2,(y[k+1]-y[k])/h


(z[k]+z[k+1])/2,(z[k+1]-z[k])/h

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Author's address:

C. Udrişte and M. Postolache
Univ. POLITEHNICA of Bucharest, Dept. of Mathematics, Splaiul Independenţei 313, 77206
Bucharest (Ro), E-mail: udriste@mathem.pub.ro