SOME REMARKS ON 3-DIMENSIONAL CONTRACT RIEMANNIAN MANIFOLDS

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Abstract

In this paper we prove that a conformally flat 3-τ-manifold is either flat or a Sasakian manifold.

AMS Subject Classification: 53C15, 53C25.
Key words: Contact manifold, Riemannian manifold, Sasakian manifold, $3-\tau$-manifold.

1 Introduction.

D.E.Blair in [3] proved that there are no flat Riemannian metrics associated to contact structure on a contact manifold of dimension > 3.

In [7] M.Okumura proved that every Sasakian conformally flat manifold of dimension >3 has a constant curvature 1. In [9] S.Tanno proved that a 3-dimensional conformally flat K-contact Riemannian manifold has a constant curvature 1. In [4] D. E. Blair and T.Koufogiorgos proved that a conformally flat contact metric manifold M^{2n+1} with $Q\varphi = \varphi Q$ has a constant curvature 1 if $n > 1$, and has a constant curvature 0 or 1 if $n = 1$. In [1] K.Bang proved that in dimension greater than 3, there are no conformally flat contact metric manifolds with $R(X, \xi)\xi = 0$. In [8] Z.Olszak proved that on a conformally flat contact metric manifold of dimension $2n + 1 \geq 5$, the scalar curvature S satisfies $S \leq 2n(2n + 1)$.

In the present paper we prove that a conformally flat 3-τ-manifold is either flat or a Sasakian manifold. This is an extension of Bang's result proved in [1]. Further results will be given in [6].

2 Preliminaries

By a contact manifold we mean a C^∞ manifold M^{2n+1}, endowed with a global 1-from η such that $\eta\wedge (d\eta)^n \neq 0$ everywhere on M^{2n+1}. Then, it has an underlying contact

Editor Gr.Tsagas *Proceedings of the Workshop on Global Analysis, Differential Geometry and Lie Algebras, 1997,* 138-141

metric structure (η, g, φ, ξ), where g is a Riemannian metric (called associated metric), φ a global tensor of type $(1,1)$ and ξ a global vector field (called characteristic vector field). These structure tensors satisfy:

$$\varphi^2 = -I + \eta \oplus \xi,\ \eta = g(\xi, \cdot),\ \eta(\xi) = 1,$$

$$d\eta(Y, Z) = g(Y, \varphi Z),\ g(\varphi Y, \varphi Z) = g(Y, Z) - \eta(Y)\eta(Z).$$

The associated metrics can be constructed by the polarization of $d\eta$ evaluated on a local orthonormal basis of an arbitrary metric on the contact distribution B defined by $ker\,\eta$.

Denoting by L and R the Lie differentiation and the curvature tensor respectively, we can define the tensors τ, h and l by:

$$\tau = L_\xi g,\ h = \frac{1}{2} L_\xi \varphi,\ l = R(\cdot, \xi)\xi.$$

Denoting by ∇ the Riemannian connection of g, we have the following formulas (see [2]):

$$\begin{array}{ll}
\varphi\xi = h\xi = l\xi = 0, & \eta \circ \varphi = \eta \circ h = 0, \\
d\eta(\xi, Z) = 0, & Trh = Trh\varphi = 0, \\
h\varphi = -\varphi h, & \nabla_Z \xi = -\varphi Z - \varphi h Z, \\
\nabla_\xi h = \varphi - \varphi l - \varphi h^2, & \varphi l \varphi - l = 2(\varphi^2 + h^2) \\
\nabla_\xi \varphi = 0, & Tr\,l = g(Q\xi, \xi) = 2 - Trh^2.
\end{array}$$

A contact metric structure is K-contact if ξ is a Killing field, case fulfilled if and only if $h = 0$. If the structure is normal, then it is Sasakian. A Sasakian structure is K-contact only for dimension 3.

The sectional curvature $K(X, \xi)\,(= g(R(X, \xi)\xi, X))$ of a plane section spanned by ξ and a vector X orthogonal to ξ, is called ξ-sectional curvature. The sectional curvature $K(X, \varphi X)\,(= g(R(X, \varphi X)\varphi X, X))$ of a plane section spanned by vectors X and φX with X orthogonal to ξ, is called φ-sectional curvature.

On a 3-dimensional Riemannian manifold M, we denote by Q the Ricci operator, by $S = TrQ$ the scalar curvature and by P the tensor field $-Q + \frac{S}{4} I$. The curvature tensor $R(Y, Z)W$ is given by:

$$R(Y, Z)W = g(Z, W)QY - g(Y, W)QZ + g(QZ, W)Y - g(QY, W)Z - \frac{S}{2}[g(Z, W)Y - g(Y, W)Z]. \tag{2.1}$$

A Riemannian manifold is said to be conformally flat if it is conformally equivalent to a Euclidean space. A 3-dimensional Riemannian manifold M is conformally flat if and only if:

$$(\nabla_Y P)Z = (\nabla_Z P)Y,\ (\forall) Y, Z \in \mathcal{X}(M). \tag{2.2}$$

A 3-dimensional contact metric manifold satisfying $\nabla_\xi \tau = 0$ is called a 3-τ-*manifold* (see e.g. [5]). In [5] we prove some results which we use in this paper.

Proposition 2.1 *Let M^3 be a non-Sasakian 3-τ-manifold. If X is a unit-eigenvector of h corresponding to the eigenvalue λ and orthogonal to ξ, then:*
$\nabla_\xi X = \nabla_\xi(\varphi X) = 0$,
$\nabla_X \xi = -(\lambda+1)\varphi X$, $\nabla_{\varphi X}\xi = (1-\lambda)X$,
$\nabla_X X = \frac{1}{2\lambda}[\varphi X \cdot \lambda + \eta(QX)]\varphi X$,
$\nabla_{\varphi X}\varphi X = \frac{1}{2\lambda}[X \cdot \lambda + \eta(Q\varphi X)]X$,
$\nabla_X(\varphi X) = -\frac{1}{2\lambda}[\varphi X \cdot \lambda + \eta(QX)]X + (\lambda+1)\xi$,
$\nabla_{\varphi X} X = -\frac{1}{2\lambda}[X \cdot \lambda + \eta(Q\varphi X)]\varphi X + (\lambda-1)\xi$.

Corollary 2.1 *On a 3-τ-manifold holds $\xi \cdot Tr\, l = 0$.*

Proposition 2.2 *Let M^3 be a 3-dimensional contact metric manifold. If for every Z of the contact distribution B, holds $QZ \in B$, then the conditions $Q\varphi = \varphi Q$ and $\nabla_\xi \tau = 0$ are equivalent.*

If $\lambda(\neq 0)$ is the eigenvalue of h with unit-eigenvector X orthogonal to ξ, then:

$$Tr\, l = 2(1-\lambda^2) \leq 2. \tag{2.3}$$

For every vector field Z on a 3-τ-manifold M^3 we have:

$$QZ = \alpha Z + b\eta(Z)\xi + \eta(Z)Q\xi + \eta(QZ)\xi, \tag{2.4}$$

where $\alpha = \dfrac{1}{2}(S - Tr\, l)$ and $b = -\dfrac{1}{2}(S + Tr\, l)$.

3 Conformally flat 3-τ-manifold

In [6] we proved the following result.

Proposition 3.1 *A conformally flat 3-τ-manifold with $Tr\, l$ =constant is either flat or Sasakian with constant curvature l.*

The proof of the above proposition uses the relations (2.1), (2.2), (2.3) and (2.4) and the Propositions 2.1, 2.2 and Corollary 2.1.

Using the Propositions 3.1 and 2.2, [4] we obtain the main result:

Theorem 3.1 *A conformally flat 3-τ-manifold is either flat or Sasakian manifold.*

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