BIANCHI IDENTITIES IN THE GEOMETRY OF THE SECOND ORDER

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Abstract

We determine the Bianchi identities for the d-torsions and d-curvatures of a distinguished liniar connection in the geometry of the second order.

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1 Introduction

The Lagrange geometry of the second order has been studied by R.Miron and Gh.Atanasiu [1], [4] and more general in [5-9]. The Lagrange geometry of the order one corresponds to the geometry of the tangent bundle (cf. with R.Miron and M.Anastasiei, [3]).

Here, we determine the Bianchi identities for the d-torsions and d-curvatures of a distinguished liniar connection in the geometry of the second order.

The fundamental notions and notations concerning these geometries are given in [4-9] and in the recent R.Miron's book [2].

At the bigining we give, shortly, the notations we use.

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2 The 2-osculator bundle. Nonlinear connections. Adapted basis. N-linear connections

A transformation of coordinates $(x^i, y^{(1)i}, y^{(2)i}) \to (\tilde{x}^i, \tilde{y}^{(1)i}, \tilde{y}^{(2)i})$ on Osc^2M is given by

$$\begin{cases} \tilde{x}^i &= \tilde{x}^i(x^1, ..., x^n), \qquad rank\| \dfrac{\partial \tilde{x}^i}{\partial x^j} \| = n \\[2ex] \tilde{y}^{(1)i} &= \dfrac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j} \\[2ex] 2\tilde{y}^{(2)i} &= \dfrac{\partial \tilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2 \dfrac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j}. \end{cases} \tag{1}$$

The point $u \in Osc^2M$ of coordinats $(x^i, y^{(1)i}, y^{(2)i})$ will be noted, also, with $u = (x, y^{(1)}, y^{(2)})$.

A *nonlinear connection* N on $E = Osc^2M$ is characterized by the functions $\underset{(\alpha)}{N}{}^i{}_j \ (x, y^{(1)}, y^{(2)}) \quad (\alpha = 1, 2)$ called the coefficients of N which to a transformation of coordinates on E has as effect the rules:

$$\begin{cases} \underset{(1)}{\tilde{N}}{}^i{}_m \dfrac{\partial \tilde{x}^m}{\partial x^j} &= \underset{(1)}{N}{}^m{}_j \dfrac{\partial \tilde{x}^i}{\partial x^m} - \dfrac{\partial \tilde{y}^{(1)i}}{\partial \tilde{x}^j} \\[2ex] \underset{(2)}{\tilde{N}}{}^i{}_m \dfrac{\partial \tilde{x}^m}{\partial x^j} &= \underset{(2)}{N}{}^m{}_j \dfrac{\partial \tilde{x}^i}{\partial x^m} + \underset{(1)}{N}{}^m{}_j \dfrac{\partial \tilde{y}^{(1)i}}{\partial x^j} - \dfrac{\partial \tilde{y}^{(2)i}}{\partial \tilde{x}^j}. \end{cases} \tag{2}$$

We obtain the direct decomposition:

$$T_uE = N_0(u) \oplus N_1(u) \oplus V_2(u) \oplus, \ \forall u \in E \ (N_0 = N) \tag{3}$$

with the local *adapted basis*:

$$\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}} \right\} \quad , \ (i = 1, ..., n) \tag{4}$$

given by:

$$\begin{cases} \dfrac{\delta}{\delta x^i} &= \dfrac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j{}_i \dfrac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N}{}^j{}_i \dfrac{\partial}{\partial y^{(2)j}} \\[2ex] \dfrac{\delta}{\delta y^{(1)i}} &= \dfrac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N}{}^j{}_i \dfrac{\partial}{\partial y^{(2)j}} \end{cases} \tag{5}$$

The fields of geometrical objects wich are important on E are introduced with respect to the direct decomposition (3).

If we consider the projectors h, v_1, v_2, determined by (3) and denote $v_\alpha X = X^{V_\alpha}$, $(\alpha = 1, 2)$ we can uniquely write:

$$X = X^H + X^{V_1} + X^{V_2} \ , \quad \forall X \in \mathcal{X}(E). \tag{6}$$

Thus we have:

$$X^H = X^{(0)i}\frac{\delta}{\delta x^i} \quad , \quad X^{V_1} = X^{(1)i}\frac{\delta}{\delta y^{(1)i}} \quad , \quad X^{V_2} = X^{(2)i}\frac{\delta}{\delta y^{(1)i}}.$$

The coordinates $X^{(\alpha)i}$, $(\alpha = 0, 1, 2)$ change under (1) as follows:

$$\tilde{X}^{(\alpha)i} = \frac{\partial \tilde{x}^i}{\partial x^j} X^{(\alpha)j} \ , \quad (\alpha = 0, 1, 2).$$

Each of them is called a *distinguished vector field*, shortly a *d-vector field.*

Let us consider the dual basis of (4):

$$\{dx^i \ , \ \delta y^{(1)i} \ , \ \delta y^{(2)i}\} \ , \quad (i = 1, ..., n). \tag{7}$$

Then for a field of 1-form ω on E we put:

$$\omega = \omega^H + \omega^{V_1} + \omega^{V_2}, \tag{8}$$

where

$$\omega^H = \omega_i^{(0)} dx^i \ , \quad \omega^{V_1} = \omega_i^{(1)} \delta y^{(1)i} \ , \quad \omega^{V_2} = \omega_i^{(2)} \delta y^{(2)i}$$

and with respect to (1) we have:

$$\omega_i^{(\alpha)} = \frac{\partial \tilde{x}^j}{\partial x^i} \tilde{\omega}_j^{(\alpha)} \ , \quad (\alpha = 0, 1, 2).$$

Each of $\omega_i^{(\alpha)}$ is called a *d-1-form* on E.

Now , we can define a *distinguished tensor field* on E of type (r,s) (shortly a *d-tensor field*) as an element $T \in T_s^r(E)$ with the property:

$$T(\underset{1}{X}\ , ..., \ \underset{s}{X}\ , \overset{1}{\omega}, ..., \overset{r}{\omega}) = T(\underset{1}{X^H}\ , ..., \ \underset{s}{X^{V_2}}\ , \overset{1}{\omega^H}, ..., \overset{r}{\omega^{V_2}}), \tag{9}$$

$$\forall \ \underset{1}{X}\ , ..., \ \underset{s}{X}\ \in \mathcal{X}(E) \ , \quad \forall \ \overset{1}{\omega}, ..., \overset{r}{\omega} \in \overset{*}{\mathcal{X}}(E).$$

Then in adapted basis (4), (7) we obtain:

$$T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s}(x, y^{(1)}, y^{(2)}) \frac{\delta}{\delta x^{i_1}} \otimes .. \otimes \frac{\partial}{\partial y^{(2) i_r}} \otimes dx^{j_1} \otimes .. \otimes \delta y^{(2) j_s}$$

and with respect to (1) we get:

$$\tilde{T}^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \frac{\partial \tilde{x}^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial \tilde{x}^{i_r}}{\partial x^{m_r}} \frac{\partial x^{q_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{q_s}}{\partial \tilde{x}^{j_s}} T^{m_1 \ldots m_r}_{q_1 \ldots q_s}.$$

Consequently we can give a d-tensor field T, of type (r,s) by its local components $T^{i_1 \ldots i_r}_{j_1 \ldots j_s}(x, y^{(1)}, y^{(2)})$.

Let us consider the $F(E)$-linear map $J : \mathcal{X}(E) \to \mathcal{X}(E)$ given on the natural basis of $\mathcal{X}(E)$ by:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}}, \quad J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, \quad J\left(\frac{\partial}{\partial y^{(2)i}}\right) = 0, \qquad (i = 1, .., n). \tag{10}$$

We define an *N-linear connection* on E as a connection D on E wich preserves by parallelism the horizontal distribution N and wich is compatible with the structure J (i.e. $D_X J = 0, \forall X \in \mathcal{X}(E)$).

In the adapted basis (4) it is sufficient to give

$$D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta y^{(\alpha)i}} = L^{m}_{\ ij} \frac{\delta}{\delta y^{(\alpha)m}}, \quad D_{\frac{\delta}{\delta y^{(\beta)j}}} \frac{\delta}{\delta y^{(\alpha)i}} = \underset{(\beta)}{C}{}^{m}_{\ ij} \frac{\delta}{\delta y^{(\beta)m}}, \tag{11}$$

$$(\alpha = 0,\ 1,\ 2, \quad \beta = 1,\ 2, \quad y^{(0)i} = x^i)$$

in order to obtain all the coefficients $D\Gamma(N) = (L^{i}_{\ jm}, \underset{(1)}{C}{}^{i}_{\ jm}, \underset{(2)}{C}{}^{i}_{\ jm})$ of a N-linear connection D.

The h-covariant derivative noted with $|$ and the v_α-covariant derivative noted with $\overset{(\alpha)}{|}$ $(\alpha = 1, 2)$ in the algebra of the d-tensors act, for exemple, for a d-tensor field $K^i_j(x, y^{(1)}, y^{(2)})$ of the type (1,1) as:

$$\begin{cases} K^i_{j|m} = \dfrac{\delta K^i_j}{\delta x^m} + L^{i}_{\ rm} K^r_j - L^{s}_{\ jm} K^i_s \\ K^i_j \overset{(\alpha)}{|}_m = \dfrac{\delta K^i_j}{\delta y^{(\alpha)m}} + \underset{(\alpha)}{C}{}^{i}_{\ rm} K^r_j - \underset{(\alpha)}{C}{}^{s}_{\ jm} K^i_s, \ (\alpha = 1, 2). \end{cases} \tag{12}$$

3 d-torsions of N-linear connections

Let $E = Osc^2M$ be a 2-osculator bundle endowed with a nonlinear connection N. Since an N-linear connection D is a particular connection on E, its torsion is given by:

$$T(X,Y) = D_X Y - D_Y X - [X,Y] \quad , \quad X,Y \in \mathcal{X}(E). \tag{13}$$

By using the projectors h, v_1 and v_2 associated to N the following decomposition of T is derived:

$$\begin{array}{rcl} T(X,Y) & = & T(X^H,Y^H) + T(X^H,Y^{V_1}) + T(X^H,Y^{V_2}) + \\ & & T(X^{V_1},Y^H) + T(X^{V_1},Y^{V_1}) + T(X^{V_1},Y^{V_2}) + \\ & & T(X^{V_2},Y^H) + T(X^{V_2},Y^{V_1}) + T(X^{V_2},Y^{V_2}). \end{array} \tag{14}$$

Every vector field from the right side of (14) has an horizontal and v_1-vertical and v_2-vertical components.

In order to get the local form of the d-tensor fields which determine the d-tensors of torsion of D, called simple the *d-torsions* of D, it is necessary to study the Lie brackets of the vector fields of the adapted basis (4).

Theorem 3.1 *The Lie brackets of the vector fields* $\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\partial}{\partial y^{(2)i}}\}$ *are given by*

$$\begin{array}{rcl} [\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k}] & = & \underset{(01)}{R^i}{}_{jk} \frac{\delta}{\delta y^{(1)i}} + \underset{(02)}{R^i}{}_{jk} \frac{\partial}{\partial y^{(2)i}}, \\ [\frac{\delta}{\delta x^j}, \frac{\delta}{\delta y^{(1)k}}] & = & \underset{(11)}{B^i}{}_{jk} \frac{\delta}{\delta y^{(1)i}} + \underset{(12)}{B^i}{}_{jk} \frac{\partial}{\partial y^{(2)i}}, \\ [\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^{(2)k}}] & = & \underset{(21)}{B^i}{}_{jk} \frac{\delta}{\delta y^{(1)i}} + \underset{(22)}{B^i}{}_{jk} \frac{\partial}{\partial y^{(2)i}}, \\ [\frac{\delta}{\delta y^{(1)j}}, \frac{\delta}{\delta y^{(1)k}}] & = & \underset{(12)}{R^i}{}_{jk} \frac{\partial}{\partial y^{(2)i}}, \\ [\frac{\delta}{\delta y^{(1)j}}, \frac{\partial}{\partial y^{(2)k}}] & = & \underset{(21)}{B^i}{}_{jk} \frac{\partial}{\partial y^{(2)i}}, \end{array}$$

where:

$$\begin{aligned}
\underset{(01)}{R}{}^i{}_{jk} &= \frac{\delta \underset{(1)}{N}{}^i{}_j}{\delta x^k} - \frac{\delta \underset{(1)}{N}{}^i{}_k}{\delta x^j}, \quad \underset{(02)}{R}{}^i{}_{jk} = \frac{\delta \underset{(2)}{N}{}^i{}_j}{\delta x^k} - \frac{\delta \underset{(2)}{N}{}^i{}_k}{\delta x^j} + \underset{(1)}{N}{}^i{}_m \underset{(01)}{R}{}^m{}_{jk}, \\
\underset{(11)}{B}{}^i{}_{jk} &= \frac{\delta \underset{(1)}{N}{}^i{}_j}{\delta y^{(1)k}}, \quad \underset{(12)}{B}{}^i{}_{jk} = \frac{\delta \underset{(2)}{N}{}^i{}_j}{\delta y^{(1)k}} - \frac{\delta \underset{(1)}{N}{}^i{}_k}{\delta x^j} + \underset{(1)}{N}{}^i{}_m \underset{(11)}{B}{}^m{}_{jk}, \\
\underset{(21)}{B}{}^i{}_{jk} &= \frac{\partial \underset{(1)}{N}{}^i{}_j}{\partial y^{(2)k}}, \quad \underset{(22)}{B}{}^i{}_{jk} = \frac{\partial \underset{(2)}{N}{}^i{}_j}{\partial y^{(2)k}} + \underset{(1)}{N}{}^i{}_m \underset{(21)}{B}{}^m{}_{jk}, \\
\underset{(12)}{R}{}^i{}_{jk} &= \frac{\delta \underset{(1)}{N}{}^i{}_j}{\delta y^{(1)k}} - \frac{\delta \underset{(1)}{N}{}^i{}_k}{\delta y^{(1)j}}.
\end{aligned}$$

Now, taking into account the skew-symmetry of the mapping T, the equations $hT(X^{V_\alpha}, X^{V_\beta}) = 0 \quad (\alpha, \beta = 1, 2)$, and (13), (14), we deduce:

Theorem 3.2 *The torsion of an N-linear connection D is completely determined by the following components, which are d-tensor fields of the type (1,2):*

$$\begin{aligned}
T(X^H, Y^H) &= hT(X^H, Y^H) + v_1 T(X^H, Y^H) + v_2 T(X^H, Y^H), \\
T(X^H, Y^{V_\beta}) &= hT(X^H, Y^{V_\beta}) + v_1 T(X^H, Y^{V_\beta}) + v_2 T(X^H, Y^{V_\beta}), \\
T(X^{V_\alpha}, X^{V_\beta}) &= v_1 T(X^{V_\alpha}, Y^{V_\beta}) + v_2 T(X^{V_\alpha}, Y^{V_\beta}), \\
(\alpha, \beta = 1, 2).
\end{aligned}$$

By the Theorem 3.1 and the Theorem 3.2 easily follows:

Theorem 3.3 *The d-torsions of an N-linear connection with the coefficients* $D\Gamma(N) =$

$(L^i{}_{jk}, \underset{(1)}{C}{}^i{}_{jk}, \underset{(2)}{C}{}^i{}_{jk})$ in the adapted basis (4) have the following expressions:

$$\begin{cases} hT(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}) = \underset{(0)}{T}{}^i{}_{jk} \frac{\delta}{\delta x^i}, v_\beta T(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}) = \underset{(0\beta)}{R}{}^i{}_{jk} \frac{\delta}{\delta y^{(\beta)i}}, \\ hT(\frac{\delta}{\delta y^{(\alpha)k}}, \frac{\delta}{\delta x^j}) = \underset{(\alpha)}{C}{}^i{}_{jk} \frac{\delta}{\delta x^i}, v_\beta T(\frac{\delta}{\delta y^{(\alpha)k}}, \frac{\delta}{\delta x^j}) = \underset{(\alpha\beta)}{P}{}^i{}_{jk} \frac{\delta}{\delta y^{(\beta)i}}, \\ v_\beta T(\frac{\delta}{\delta y^{(1)k}}, \frac{\delta}{\delta y^{(1)j}}) = \underset{(1\beta)}{Q}{}^i{}_{jk} \frac{\delta}{\delta y^{(\beta)i}}, v_\beta T(\frac{\partial}{\partial y^{(2)k}}, \frac{\delta}{\delta y^{(1)j}}) = \underset{(2\beta)}{Q}{}^i{}_{jk} \frac{\delta}{\delta y^{(\beta)i}}, \\ v_1 T(\frac{\partial}{\partial y^{(2)k}}, \frac{\partial}{\partial y^{(2)j}}) = 0, v_2 T(\frac{\partial}{\partial y^{(2)k}}, \frac{\partial}{\partial y^{(2)j}}) = \underset{(2)}{S}{}^i{}_{jk} \frac{\partial}{\partial y^{(2)i}}, \\ \qquad (\alpha, \beta = 1, 2) \quad (\frac{\delta}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}), \end{cases} \tag{15}$$

where:

$$\begin{array}{lllllll} \underset{(0)}{T}{}^i{}_{jk} & = & L^i{}_{jk} - L^i{}_{kj}, & \underset{(11)}{P}{}^i{}_{jk} = \underset{(11)}{B}{}^i{}_{jk} - L^i{}_{kj}, & \underset{(12)}{P}{}^i{}_{jk} = \underset{(12)}{B}{}^i{}_{jk}, \\ \underset{(21)}{P}{}^i{}_{jk} & = & \underset{(21)}{B}{}^i{}_{jk}, & \underset{(22)}{P}{}^i{}_{jk} = \underset{(22)}{B}{}^i{}_{jk} - L^i{}_{kj}, & \underset{(12)}{Q}{}^i{}_{jk} = \underset{(12)}{R}{}^i{}_{jk}, \\ \underset{(11)}{Q}{}^i{}_{jk} & = & \underset{(1)}{C}{}^i{}_{jk} - \underset{(1)}{C}{}^i{}_{kj} = \underset{(1)}{S}{}^i{}_{jk}, & \underset{(21)}{Q}{}^i{}_{jk} = \underset{(2)}{C}{}^i{}_{jk}, \\ \underset{(22)}{Q}{}^i{}_{jk} & = & \underset{(21)}{B}{}^i{}_{jk} - \underset{(1)}{C}{}^i{}_{kj}, & \underset{(2)}{S}{}^i{}_{jk} = \underset{(2)}{C}{}^i{}_{jk} - \underset{(2)}{C}{}^i{}_{kj}. \end{array}$$

4 d-curvatures of N-linear connections

In the following we shall deal with the curvature of an N-linear connection D. This is expressed by:

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, \qquad \forall X, Y, Z \in \mathcal{X}(E).$$

Since D preserves by parallelism the horizontal distribution and $D_X(JY) = J(D_X Y)$, $\forall X, Y \in \mathcal{X}(E)$ we have:

$$\begin{array}{rcl} v_\beta(D_X Y^H) & = & 0, h(D_X Y^{V_\alpha}) = 0, v_\beta(D_X Y^{V_\alpha}) = 0 \quad (\beta \neq \alpha; \alpha, \beta = 1, 2), \\ D_X(JY^H) & = & J(D_X Y^H), D_X(JY^{V_\beta}) = J(D_X Y^{V_\beta}) \quad (\beta = 1, 2). \end{array}$$

Consequently, we can formulate:

Theorem 4.1 *a) The essential components of the curvature tensor field* R *are* $R(X,Y)Z^H$ *for any vector field* $X, Y, Z \in \mathcal{X}(E)$.

b) $R(X,Y)Z^H$ *is a horizontal vector field.*

c) the essential d-tensors of curvatures (or simple d-curvatures*) of an* N*-linear connection* D *are:*

$$R(X^H, Y^H)Z^H, R(X^{V_\beta}, Y^H)Z^H, R(X^{V_\beta}, Y^{V_\alpha})Z^H, (\alpha, \beta = 1, 2; \beta \leq \alpha).$$

By the Theorem 4.1 and the equation $R(X,Y) = -R(Y,X) \quad , \quad \forall X, Y \in \mathcal{X}(E)$ we get:

Theorem 4.2 *The d-curvatures of an* N*-linear connection* D*, with the coefficients* $D\Gamma(N) = ({L^i}_{jk}, \underset{(1)}{C^i}_{jk} \ , \ \underset{(2)}{C^i}_{jk} \ )$ *in the adapted basis (4) have the following expressions:*

$$\begin{cases} R(\dfrac{\delta}{\delta x^k}, \dfrac{\delta}{\delta x^j})\dfrac{\delta}{\delta x^h} = {R_h}^i{}_{jk}\dfrac{\delta}{\delta x^i} \quad , \\ R(\dfrac{\delta}{\delta y^{(1)\alpha}}, \dfrac{\delta}{\delta x^j})\dfrac{\delta}{\delta x^h} = \ \underset{(\alpha)}{{P_h}^i{}_{jk}} \ \dfrac{\delta}{\delta x^i}, \\ R(\dfrac{\partial}{\partial y^{(2)k}}, \dfrac{\delta}{\delta y^{(1)j}})\dfrac{\delta}{\delta x^h} = \ \underset{(12)}{{S_h}^i{}_{jk}} \ \dfrac{\delta}{\delta x^i} \quad , \\ R(\dfrac{\delta}{\delta y^{(\alpha)k}}, \dfrac{\delta}{\delta y^{(\alpha)j}})\dfrac{\delta}{\delta x^i} = \ \underset{(\alpha\alpha)}{{S_h}^i{}_{jk}} \ \dfrac{\delta}{\delta x^i}, \end{cases} \tag{16}$$

$(\alpha = 1, 2)$*, where:*

$${R_h}^i{}_{jk} = \frac{\delta {L^i}_{hj}}{\delta x^k} - \frac{\delta {L^i}_{hk}}{\delta x^j} + L^m{}_{hj}{L^i}_{mk} - L^m{}_{hk}{L^i}_{mj} + \sum_{\beta=1}^{2} \underset{(\beta)}{C^i}_{hm} \underset{(0\beta)}{R^m}_{jk},$$

$$\underset{(\alpha)}{{P_h}^i{}_{jk}} = \frac{\delta {L^i}_{hj}}{\delta y^{(\alpha)k}} - \frac{\delta \ \underset{(\alpha)}{C^i}_{hk}}{\delta x^j} + L^m{}_{hj}\underset{(\alpha)}{C^i}_{mk} - \underset{(\alpha)}{C^m}_{hk}{L^i}_{mj} + \sum_{\beta=1}^{2} \underset{(\beta)}{C^i}_{hm} \underset{(\alpha\beta)}{B^m}_{jk},$$

$$\underset{(12)}{S_h{}^i{}_{jk}} = \frac{\partial \underset{(1)}{C^i{}_{hj}}}{\partial y^{(2)k}} - \frac{\delta \underset{(2)}{C^i{}_{hk}}}{\delta y^{(1)j}} + \underset{(1)}{C^m{}_{hj}}\underset{(2)}{C^i{}_{mk}} - \underset{(2)}{C^m{}_{hk}}\underset{(1)}{C^i{}_{mj}} + \underset{(2)}{C^i{}_{hm}}\underset{(21)}{B^m{}_{jk}},$$

$$\underset{(\alpha\alpha)}{S_h{}^i{}_{jk}} = \frac{\delta \underset{(\alpha)}{C^i{}_{hj}}}{\delta y^{(\alpha)k}} - \frac{\delta \underset{(\alpha)}{C^i{}_{hk}}}{\delta y^{(\alpha)j}} + \underset{(\alpha)}{C^m{}_{hj}}\underset{(\alpha)}{C^i{}_{mk}} - \underset{(\alpha)}{C^m{}_{hk}}\underset{(\alpha)}{C^i{}_{mj}} + \underset{(2)}{C^i{}_{hm}}\underset{(\alpha 2)}{R^m{}_{jk}},$$

$$\left(\frac{\delta}{\delta y^{(2)}} = \frac{\partial}{\partial y^{(2)}} \quad and \quad \underset{(22)}{R^m{}_{jk}} = 0\right).$$

5 The Bianchi identities

The d-torsions and d-curvatures of an N-linear connection D are not independent. As it is well known the torsion T and curvature R of every linear connection D on E satisfy the following Bianchi identities:

$$\sum[(D_X T)(Y,Z) - R(X,Y)Z + T(T(X,Y),Z)] = 0, \tag{17}$$

$$\sum[(D_X R)(U,Y,Z) + R(T(X,Y),Z)U] = 0, \tag{18}$$

where $\sum$ means cyclic sum over X,Y,Z.

If D is an N-linear connection on $E = Osc^2 M$, then by the Theorem 4.1 and

$$\begin{aligned} v_\beta(D_X R)(hU,Y,Z) = 0 \quad &, \quad h(D_X R)(v_\beta U,Y,Z) = 0, \\ v_\beta(D_X R)(v_\alpha U,Y,Z) = 0 \quad &, \quad \alpha,\beta = 1,2 \quad , \quad \alpha \neq \beta \end{aligned}$$

the identities (17) and (18) become:

$$\left\{ \begin{array}{l} \sum[h(D_X T)(Y,Z) - hR(X,Y)Z + hT(hT(X,Y),Z) + \\ \sum\limits_{\alpha=1}^{2} hT(v_\alpha T(X,Y),Z)] = 0, \\ \sum[v_\beta(D_X T)(Y,Z) - v_\beta R(X,Y)Z + v_\beta T(hT(X,Y),Z) + \\ \sum\limits_{\alpha=1}^{2} v_\beta T(v_\alpha T(X,Y),Z)] = 0. \end{array} \right. \tag{19}$$

$$\begin{cases} \sum[h(D_X R)(U,Y,Z) + hR(hT(X,Y),Z)U + \sum\limits_{\alpha=1}^{2} hR(v_\alpha T(X,Y),Z)U] = 0, \\ \sum[v_\beta(D_X R)(U,Y,Z) + v_\beta R(hT(X,Y),Z)U + \sum\limits_{\alpha=1}^{2} v_\beta R(v_\alpha T(X,Y),Z)U] = 0, \\ (\beta = 1,2). \end{cases} \tag{20}$$

In order to get the form of these identities in the adapted basis (4), we shall insert in (19) and (20) the vector fields X,Y,Z as in the following table:

X	$\frac{\delta}{\delta x}$	$\frac{\delta}{\delta x}$	$\frac{\delta}{\delta x}$	$\frac{\delta}{\delta x}$	$\frac{\delta}{\delta x}$	$\frac{\delta}{\delta x}$	$\frac{\delta}{\delta y^{(1)}}$	$\frac{\delta}{\delta y^{(1)}}$	$\frac{\delta}{\delta y^{(1)}}$	$\frac{\partial}{\partial y^{(2)}}$
Y	$\frac{\delta}{\delta x}$	$\frac{\delta}{\delta x}$	$\frac{\delta}{\delta y^{(1)}}$	$\frac{\delta}{\delta y^{(1)}}$	$\frac{\delta}{\delta x}$	$\frac{\partial}{\partial y^{(2)}}$	$\frac{\delta}{\delta y^{(1)}}$	$\frac{\delta}{\delta y^{(1)}}$	$\frac{\partial}{\partial y^{(2)}}$	$\frac{\partial}{\partial y^{(2)}}$
Z	$\frac{\delta}{\delta x}$	$\frac{\delta}{\delta y^{(1)}}$	$\frac{\delta}{\delta y^{(1)}}$	$\frac{\partial}{\partial y^{(2)}}$	$\frac{\partial}{\partial y^{(2)}}$	$\frac{\partial}{\partial y^{(2)}}$	$\frac{\delta}{\delta y^{(1)}}$	$\frac{\partial}{\partial y^{(2)}}$	$\frac{\partial}{\partial y^{(2)}}$	$\frac{\partial}{\partial y^{(2)}}$

and U will be taken succesively equal to $\frac{\delta}{\delta x}$, $\frac{\delta}{\delta y^{(1)}}$ and $\frac{\partial}{\partial y^{(2)}}$.

Theorem 5.1 *The local form of the Bianchi identities in the adapted basis (4) are:*

$$\begin{cases} \overset{\circ}{\sum}\Big[\ \underset{(0)}{T}{}^i{}_{jk|l} + \underset{(0)}{T}{}^m{}_{jk}\underset{(0)}{T}{}^i{}_{lm} + \sum\limits_{\alpha=1}^{2}\underset{(0\alpha)}{R}{}^m{}_{jk}\underset{(\alpha)}{C}{}^i{}_{lm} - R_j{}^i{}_{kl}\ \Big] = 0, \\ \overset{\circ}{\sum}\Big[\ \underset{(0\beta)}{R}{}^i{}_{jk|l} + \underset{(0)}{T}{}^m{}_{jk}\underset{(0\beta)}{R}{}^i{}_{lm} + \sum\limits_{\alpha=1}^{2}\underset{(0\alpha)}{R}{}^m{}_{jk}\underset{(\alpha\beta)}{P}{}^i{}_{lm}\ \Big] = 0 \quad ,(\beta = 1,2)\ ; \end{cases} \tag{21}$$

$$\begin{cases} \underset{(0)}{T}{}^i{}_{jk}\overset{(1)}{|}_l - \underset{(1)}{C}{}^i{}_{jl|k} + \underset{(1)}{C}{}^i{}_{kl|j} - \underset{(0)}{T}{}^m{}_{jk}\underset{(1)}{C}{}^i{}_{ml} - \underset{(1)}{C}{}^m{}_{jl}\underset{(0)}{T}{}^i{}_{km} + \underset{(1)}{C}{}^m{}_{kl}\underset{(0)}{T}{}^i{}_{jm} - \\ - \sum\limits_{\alpha=1}^{2}[\underset{(1\alpha)}{P}{}^m{}_{jl}\underset{(\alpha)}{C}{}^i{}_{km} - \underset{(1\alpha)}{P}{}^m{}_{kl}\underset{(\alpha)}{C}{}^i{}_{jm}] - \underset{(1)}{P}{}_j{}^i{}_{kl} + \underset{(1)}{P}{}_k{}^i{}_{jl} = 0, \\ \underset{(0\beta)}{R}{}^i{}_{jk}\overset{(1)}{|}_l - \underset{(1\beta)}{P}{}^i{}_{jl|k} + \underset{(1\beta)}{P}{}^i{}_{kl|j} - \underset{(0)}{T}{}^m{}_{jk}\underset{(1\beta)}{P}{}^i{}_{ml} - \underset{(1)}{C}{}^m{}_{jl}\underset{(0\beta)}{R}{}^i{}_{km} + \underset{(1)}{C}{}^m{}_{kl}\underset{(0\beta)}{R}{}^i{}_{jm} + \\ + \sum\limits_{\alpha=1}^{2}[\underset{(0\alpha)}{R}{}^m{}_{jk}\underset{(\alpha\beta)}{Q}{}^i{}_{lm} - \underset{(1\alpha)}{P}{}^m{}_{jl}\underset{(\alpha\beta)}{P}{}^i{}_{km} + \underset{(1\alpha)}{P}{}^m{}_{kl}\underset{(\alpha\beta)}{P}{}^i{}_{jm}] - \underset{(\beta)}{A}{}_l{}^i{}_{jk} = 0, (\beta = 1,2), \end{cases} \tag{22}$$

where $\underset{(1)}{A}{}_l{}^i{}_{jk} = R_l{}^i{}_{jk}$ *and* $\underset{(2)}{A}{}_l{}^i{}_{jk} = 0$;

$$\left\{\begin{aligned}
&\underset{(1)}{C}{}^{i}{}_{lj}\overset{(1)}{|}_{k} - \underset{(1)}{C}{}^{i}{}_{lk}\overset{(1)}{|}_{j} - \underset{(1)}{C}{}^{m}{}_{lj}\underset{(1)}{C}{}^{i}{}_{mk} + \underset{(1)}{C}{}^{m}{}_{lk}\underset{(1)}{C}{}^{i}{}_{mj} + \\
&+ \sum_{\alpha=1}^{2}\underset{(1\alpha)}{Q}{}^{m}{}_{jk}\underset{(\alpha)}{C}{}^{i}{}_{lm} - \underset{(11)}{S}{}_{l}{}^{i}{}_{jk} = 0, \\
&\underset{(1\beta)}{P}{}^{i}{}_{lj}\overset{(1)}{|}_{k} - \underset{(1\beta)}{P}{}^{i}{}_{lk}\overset{(1)}{|}_{j} + \underset{(1\beta)}{Q}{}^{i}{}_{jk|l} - \underset{(1)}{C}{}^{m}{}_{lj}\underset{(1\beta)}{P}{}^{i}{}_{mk} + \underset{(1)}{C}{}^{m}{}_{lk}\underset{(1\beta)}{P}{}^{i}{}_{mj} + \\
&+ \sum_{\alpha=1}^{2}\left[\underset{(1\alpha)}{Q}{}^{m}{}_{jk}\underset{(\alpha\beta)}{P}{}^{i}{}_{lm} + \underset{(1\alpha)}{P}{}^{m}{}_{lj}\underset{(\alpha\beta)}{Q}{}^{i}{}_{km} - \underset{(1\alpha)}{P}{}^{m}{}_{lk}\underset{(\alpha\beta)}{Q}{}^{i}{}_{jm}\right] - \underset{(1\beta)}{P}{}_{j}{}^{i}{}_{kl} + \underset{(1\beta)}{P}{}_{j}{}^{i}{}_{kl} = 0, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\beta = 1, 2),
\end{aligned}\right. \tag{23}$$

where $\underset{(11)}{P}{}_{j}{}^{i}{}_{kl} = \underset{(1)}{P}{}_{j}{}^{i}{}_{kl}$ *and* $\underset{(12)}{P}{}_{j}{}^{i}{}_{kl} = 0;$

$$\left\{\begin{aligned}
&\underset{(1)}{C}{}^{i}{}_{lj}\overset{(2)}{|}_{k} - \underset{(2)}{C}{}^{i}{}_{lk}\overset{(1)}{|}_{j} - \underset{(1)}{C}{}^{m}{}_{lj}\underset{(2)}{C}{}^{i}{}_{mk} + \underset{(2)}{C}{}^{m}{}_{lk}\underset{(1)}{C}{}^{i}{}_{mj} + \\
&+ \sum_{\alpha=1}^{2}\underset{(2\alpha)}{Q}{}^{m}{}_{jk}\underset{(\alpha)}{C}{}^{i}{}_{lm} - \underset{(12)}{S}{}_{h}{}^{i}{}_{jk} = 0, \\
&\underset{(11)}{P}{}^{i}{}_{lj}\overset{(2)}{|}_{k} - \underset{(21)}{P}{}^{i}{}_{lk}\overset{(1)}{|}_{j} + \underset{(2)}{C}{}^{i}{}_{jk|l} - \underset{(1)}{C}{}^{m}{}_{lj}\underset{(21)}{P}{}^{i}{}_{mk} + \underset{(2)}{C}{}^{m}{}_{lk}\underset{(11)}{P}{}^{i}{}_{mj} - \underset{(11)}{P}{}^{m}{}_{lj}\underset{(2)}{C}{}^{i}{}_{mk} - \\
&- \underset{(21)}{P}{}^{m}{}_{lk}\underset{(1)}{S}{}^{i}{}_{jm} - \underset{(22)}{P}{}^{m}{}_{lk}\underset{(2)}{C}{}^{i}{}_{jm} + \sum_{\alpha=1}^{2}\underset{(2\alpha)}{Q}{}^{m}{}_{jk}\underset{(\alpha 1)}{P}{}^{i}{}_{lm} - \underset{(2)}{P}{}_{j}{}^{i}{}_{kl} = 0, \\
&\underset{(12)}{P}{}^{i}{}_{lj}\overset{(2)}{|}_{k} - \underset{(22)}{P}{}^{i}{}_{lk}\overset{(1)}{|}_{j} + \underset{(22)}{Q}{}^{i}{}_{jk|l} - \underset{(1)}{C}{}^{m}{}_{lj}\underset{(22)}{P}{}^{i}{}_{mk} + \underset{(2)}{C}{}^{m}{}_{lk}\underset{(12)}{P}{}^{i}{}_{mj} - \underset{(11)}{P}{}^{m}{}_{lj}\underset{(22)}{Q}{}^{i}{}_{mk} - \\
&- \underset{(21)}{P}{}^{m}{}_{lk}\underset{(12)}{Q}{}^{i}{}_{jm} - \underset{(12)}{P}{}^{m}{}_{lj}\underset{(2)}{S}{}^{i}{}_{mk} - \underset{(22)}{P}{}^{m}{}_{lk}\underset{(22)}{Q}{}^{i}{}_{jm} + \sum_{\alpha=1}^{2}\underset{(2\alpha)}{Q}{}^{m}{}_{jk}\underset{(\alpha 2)}{P}{}^{i}{}_{lm} + \underset{(1)}{P}{}_{k}{}^{i}{}_{jl} = 0;
\end{aligned}\right. \tag{24}$$

$$\left\{\begin{array}{l}
\underset{(0)}{T}{}^i{}_{jk}\overset{(2)}{|}_l - \underset{(2)}{C}{}^i{}_{jl|k} + \underset{(2)}{C}{}^i{}_{kl|j} - \underset{(0)}{T}{}^m{}_{jk}\underset{(2)}{C}{}^i{}_{ml} - \underset{(2)}{C}{}^m{}_{jl}\underset{(0)}{T}{}^i{}_{km} + \underset{(2)}{C}{}^m{}_{kl}\underset{(0)}{T}{}^i{}_{jm} - \\
-\sum\limits_{\alpha=1}^{2}[\underset{(2\alpha)}{P}{}^m{}_{jl}\underset{(\alpha)}{C}{}^i{}_{km} - \underset{(2\alpha)}{P}{}^m{}_{kl}\underset{(\alpha)}{C}{}^i{}_{jm}] - \underset{(2)}{P_j}{}^i{}_{kl} + \underset{(2)}{P_k}{}^i{}_{jl} = 0, \\
\underset{(0\beta)}{R}{}^i{}_{jk}\overset{(2)}{|}_l - \underset{(2\beta)}{P}{}^i{}_{jl|k} + \underset{(2\beta)}{P}{}^i{}_{kl|j} - \underset{(0)}{T}{}^m{}_{jk}\underset{(2\beta)}{P}{}^i{}_{ml} - \underset{(2)}{C}{}^m{}_{jl}\underset{(0\beta)}{R}{}^i{}_{km} + \underset{(2)}{C}{}^m{}_{kl}\underset{(0\beta)}{R}{}^i{}_{jm} - \\
-\underset{(01)}{R}{}^m{}_{jk}\underset{(2\beta)}{Q}{}^i{}_{ml} - \sum\limits_{\alpha=1}^{2}[\underset{(2\alpha)}{P}{}^m{}_{jl}\underset{(\alpha\beta)}{P}{}^i{}_{km} - \underset{(2\alpha)}{P}{}^m{}_{kl}\underset{(\alpha\beta)}{P}{}^i{}_{jm}] - \underset{(\beta)}{A_j}{}^i{}_{kl} = 0, (\beta = 1, 2),
\end{array}\right. \tag{25}$$

where $\underset{(1)}{A_j}{}^i{}_{kl} = 0$ *and* $\underset{(2)}{A_j}{}^i{}_{kl} = \underset{(02)}{R}{}^m{}_{jk}\underset{(2)}{S}{}^i{}_{ml} + R_l{}^i{}_{jk}$;

$$\left\{\begin{array}{l}
\underset{(2)}{C}{}^i{}_{lj}\overset{(2)}{|}_k - \underset{(2)}{C}{}^i{}_{lk}\overset{(2)}{|}_j - \underset{(2)}{C}{}^m{}_{lj}\underset{(2)}{C}{}^i{}_{mk} + \underset{(2)}{C}{}^m{}_{lk}\underset{(2)}{C}{}^i{}_{mj} + \underset{(2)}{S}{}^m{}_{jk}\underset{(2)}{C}{}^i{}_{lm} - \underset{(22)}{S_l}{}^i{}_{jk} = 0, \\
\underset{(2\beta)}{P}{}^i{}_{lj}\overset{(2)}{|}_k - \underset{(2\beta)}{P}{}^i{}_{lk}\overset{(2)}{|}_j - \underset{(2)}{C}{}^m{}_{lj}\underset{(2\beta)}{P}{}^i{}_{mk} + \underset{(2)}{C}{}^m{}_{lk}\underset{(2\beta)}{P}{}^i{}_{mj} - \\
-\underset{(2\beta)}{P}{}^m{}_{lj}\underset{(2\beta)}{Q}{}^i{}_{mk} + \underset{(2\beta)}{P}{}^m{}_{lk}\underset{(2\beta)}{Q}{}^i{}_{mj} + \underset{(2)}{S}{}^m{}_{jk}\underset{(2\beta)}{P}{}^i{}_{lm} + \underset{(\beta)}{A_j}{}^i{}_{kl} = 0, (\beta = 1, 2),
\end{array}\right. \tag{26}$$

where $\underset{(1)}{A_j}{}^i{}_{kl} = 0$ *and*

$$\underset{(2)}{A_j}{}^i{}_{kl} = \underset{(2)}{S}{}^i{}_{jk|l} - \underset{(22)}{P}{}^m{}_{lj}\underset{(2)}{S}{}^i{}_{mk} + \underset{(22)}{P}{}^m{}_{lk}\underset{(2)}{S}{}^i{}_{mj} - \underset{(2)}{P_j}{}^i{}_{kl} + \underset{(2)}{P_k}{}^i{}_{jl} = 0;$$

$$\overset{\circ}{\sum}\ [\underset{(1\beta)}{Q}{}^i{}_{jk}\overset{(1)}{|}_h + \sum_{\alpha=1}^{2}\underset{(1\alpha)}{Q}{}^m{}_{jk}\underset{(\alpha\beta)}{Q}{}^i{}_{lm} - \underset{(\beta)}{A_j}{}^i{}_{kl}] = 0, (\beta = 1, 2), \tag{27}$$

where $\underset{(1)}{A}{}_j{}^i{}_{kl} = \underset{(11)}{S}{}_j{}^i{}_{kl}$ *and* $\underset{(2)}{A}{}_j{}^i{}_{kl} = 0;$

$$\underset{(1\beta)}{Q^i}{}_{jk} \overset{(2)}{|}_l - \underset{(2\beta)}{Q^i}{}_{jl} \overset{(1)}{|}_k + \underset{(2\beta)}{Q^i}{}_{kl}\overset{(1)}{|}_j - \underset{(1)}{S^m_{jk}} \underset{(2\beta)}{Q^i}{}_{ml} - \tag{28}$$

$$-\sum_{\alpha=1}^{2} [\underset{(2\alpha)}{Q^m_{jl}} \underset{(\alpha\beta)}{Q^i}{}_{km} - \underset{(2\alpha)}{Q^m_{kl}}\underset{(\alpha\beta)}{Q^i}{}_{jm}] - \underset{(\beta)}{A}{}_j{}^i{}_{kl} = 0, (\beta=1,2),$$

where $\underset{(1)}{A}{}_j{}^i{}_{kl} = \underset{(12)}{S}{}_j{}^i{}_{kl} - \underset{(12)}{S}{}_k{}^i{}_{jl}$ *and* $\underset{(2)}{A}{}_j{}^i{}_{kl} = \underset{(12)}{Q^m_{jk}}\underset{(2)}{S^i}{}_{ml} + \underset{(11)}{S}{}_l{}^i{}_{jk};$

$$\underset{(2\beta)}{S^i}{}_{jk}\overset{(1)}{|}_l + \underset{(2\beta)}{Q^i}{}_{lj}\overset{(2)}{|}_k - \underset{(2\beta)}{Q^i}{}_{lk}\overset{(2)}{|}_j - \underset{(2)}{C^m_{lj}}\underset{(2\beta)}{Q^i}{}_{mk} + \underset{(2)}{C^m_{lk}}\underset{(2\beta)}{Q^i}{}_{mj} +$$

$$+\underset{(2)}{S^m_{jk}}\underset{(2\beta)}{Q^i}{}_{lm} + \underset{(22)}{Q^m_{lj}}\underset{(2\beta)}{S^i}{}_{km} - \underset{(22)}{Q^m_{lk}}\underset{(2\beta)}{S^i}{}_{jm} - \underset{(\beta)}{A}{}_j{}^i{}_{kl} = 0, (\beta=1,2), \tag{29}$$

where $\underset{(21)}{S^i}{}_{jk}=0$, $\underset{(22)}{S^i}{}_{jk} = \underset{(2)}{S^i}{}_{jk}$, $\underset{(1)}{A}{}_j{}^i{}_{kl} = \underset{(22)}{S}{}_l{}^i{}_{jk}$ *and* $\underset{(2)}{A}{}_j{}^i{}_{kl} = \underset{(12)}{S}{}_j{}^i{}_{kl} - \underset{(12)}{S}{}_k{}^i{}_{jl};$

$$\overset{\circ}{\sum}[\underset{(2)}{S^i}{}_{jk}\overset{(2)}{|}_l + \underset{(2)}{S^m_{jk}}\underset{(2)}{S^i}{}_{lm} - \underset{(22)}{S}{}_j{}^i{}_{kl}] = 0; \tag{30}$$

$$\overset{\circ}{\sum}[R_h{}^i{}_{jk|l} + \underset{(0)}{T^m_{jk}} R_h{}^i{}_{ml} + \sum_{\alpha=1}^{2}\underset{(0\alpha)}{R^m_{jk}}\underset{(\alpha)}{P}{}_h{}^i{}_{ml}] = 0; \tag{31}$$

$$R_h{}^i{}_{jk}\overset{(1)}{|}_l - \underset{(1)}{P}{}_h{}^i{}_{jl|k} + \underset{(1)}{P}{}_h{}^i{}_{kl|j} + \underset{(0)}{T^m_{jk}}\underset{(1)}{P}{}_h{}^i{}_{ml} - \underset{(1)}{C^m_{jl}} R_h{}^i{}_{mk} + \underset{(1)}{C^m_{kl}}R_h{}^i{}_{mj} +$$

$$+\sum_{\alpha=1}^2[\underset{(0\alpha)}{R^m_{jk}}\underset{(1\alpha)}{S}{}_h{}^i{}_{ml} - \underset{(1\alpha)}{P^m_{jl}}\underset{(\alpha)}{P}{}_h{}^i{}_{mk} + \underset{(1\alpha)}{P^m_{kl}}\underset{(\alpha)}{P}{}_h{}^i{}_{mj}] = 0; \tag{32}$$

$$\underset{(1)}{P}{}_{h}{}^{i}{}_{lj}\overset{(1)}{|}{}_{k} - \underset{(1)}{P}{}_{h}{}^{i}{}_{lk}\overset{(1)}{|}{}_{j} + \underset{(11)}{S}{}_{h}{}^{i}{}_{jk|l} + \underset{(1)}{C}{}^{m}_{lj}\underset{(1)}{P}{}_{h}{}^{i}{}_{mk} - \underset{(1)}{C}{}^{m}_{lk}\underset{(1)}{P}{}_{h}{}^{i}{}_{mj} + \sum_{\alpha=1}^{2}[\underset{(1\alpha)}{P}{}^{m}_{lj}\underset{(1\alpha)}{S}{}_{h}{}^{i}{}_{mk} - \underset{(1\alpha)}{P}{}^{m}_{lk}\underset{(1\alpha)}{S}{}_{h}{}^{i}{}_{mj} + \underset{(1\alpha)}{Q}{}^{m}_{jk}\underset{(\alpha)}{P}{}_{h}{}^{i}{}_{ml}] = 0; \tag{33}$$

$$\underset{(1)}{P}{}_{h}{}^{i}{}_{lj}\overset{(2)}{|}{}_{k} - \underset{(2)}{P}{}_{h}{}^{i}{}_{lk}\overset{(1)}{|}{}_{j} + \underset{(12)}{S}{}_{h}{}^{i}{}_{jk|l} + \underset{(1)}{C}{}^{m}_{lj}\underset{(2)}{P}{}_{h}{}^{i}{}_{mk} - \underset{(2)}{C}{}^{m}_{lk}\underset{(1)}{P}{}_{h}{}^{i}{}_{mj} + \sum_{\alpha=1}^{2}[\underset{(1\alpha)}{P}{}^{m}_{lj}\underset{(\alpha 2)}{S}{}_{h}{}^{i}{}_{mk} - \underset{(2\alpha)}{P}{}^{m}_{lk}\underset{(1\alpha)}{S}{}_{h}{}^{i}{}_{mj} + \underset{(2\alpha)}{Q}{}^{m}_{jk}\underset{(\alpha)}{P}{}_{h}{}^{i}{}_{ml}] = 0; \tag{34}$$

$$R_{h}{}^{i}{}_{jk}\overset{(2)}{|}{}_{l} - \underset{(2)}{P}{}_{h}{}^{i}{}_{jl|k} + \underset{(2)}{P}{}_{h}{}^{i}{}_{kl|j} + \underset{(0)}{T}{}^{m}_{jk}\underset{(2)}{P}{}_{h}{}^{i}{}_{ml} - \underset{(2)}{C}{}^{m}_{jl}R_{h}{}^{i}{}_{mk} + \underset{(2)}{C}{}^{m}_{kl}R_{h}{}^{i}{}_{mj} + \sum_{\alpha=1}^{2}[\underset{(0\alpha)}{R}{}^{m}_{jk}\underset{(\alpha 2)}{S}{}_{h}{}^{i}{}_{ml} - \underset{(2\alpha)}{P}{}^{m}_{jl}\underset{(\alpha)}{P}{}_{h}{}^{i}{}_{mk} + \underset{(2\alpha)}{P}{}^{m}_{kl}\underset{(\alpha)}{P}{}_{h}{}^{i}{}_{mj}] = 0; \tag{35}$$

$$\underset{(2)}{P}{}_{h}{}^{i}{}_{lj}\overset{(2)}{|}{}_{k} - \underset{(2)}{P}{}_{h}{}^{i}{}_{lk}\overset{(2)}{|}{}_{j} + \underset{(22)}{S}{}_{h}{}^{i}{}_{jk|l} + \underset{(2)}{C}{}^{m}_{lj}\underset{(2)}{P}{}_{h}{}^{i}{}_{mk} - \underset{(2)}{C}{}^{m}_{lk}\underset{(2)}{P}{}_{h}{}^{i}{}_{mj} + \sum_{\alpha=1}^{2}[\underset{(2\alpha)}{P}{}^{m}_{lj}\underset{(\alpha 2)}{S}{}_{h}{}^{i}{}_{mk} - \underset{(2\alpha)}{P}{}^{m}_{lk}\underset{(\alpha 2)}{S}{}_{h}{}^{i}{}_{mj} + \underset{(2\alpha)}{S}{}^{m}_{jk}\underset{(\alpha)}{P}{}_{h}{}^{i}{}_{ml}] = 0, \tag{36}$$

where $\underset{(21)}{S}{}^{i}{}_{jk} = 0;$

$$\overset{\circ}{\sum}[\underset{(11)}{S}{}_{h}{}^{i}{}_{jk}\overset{(1)}{|}{}_{l} + \sum_{\alpha=1}^{2}\underset{(1\alpha)}{Q}{}^{m}_{jk}\underset{(1\alpha)}{S}{}_{h}{}^{i}{}_{ml}] = 0; \tag{37}$$

$$\underset{(11)}{S}{}_h{}^i{}_{jk}\overset{(2)}{|}_l - \underset{(12)}{S}{}_h{}^i{}_{jl}\overset{(1)}{|}_k + \underset{(12)}{S}{}_h{}^i{}_{kl}\overset{(1)}{|}_j +$$

$$+\sum_{\alpha=1}^{2}[\underset{(1\alpha)}{Q}{}^m{}_{jk}\underset{(\alpha 2)}{S}{}_h{}^i{}_{ml} - \underset{(2\alpha)}{Q}{}^m{}_{jl}\underset{(1\alpha)}{S}{}_h{}^i{}_{mk} + \underset{(2\alpha)}{Q}{}^m{}_{kl}\underset{(1\alpha)}{S}{}_h{}^i{}_{mj}] = 0; \tag{38}$$

$$\underset{(12)}{S}{}_h{}^i{}_{lj}\overset{(2)}{|}_k - \underset{(12)}{S}{}_h{}^i{}_{lk}\overset{(2)}{|}_j + \underset{(22)}{S}{}_h{}^i{}_{jk}\overset{(1)}{|}_l +$$

$$+\sum_{\alpha=1}^{2}[\underset{(2\alpha)}{Q}{}^m{}_{lj}\underset{(\alpha 2)}{S}{}_h{}^i{}_{mk} - \underset{(2\alpha)}{Q}{}^m{}_{lk}\underset{(\alpha 2)}{S}{}_h{}^i{}_{mj} + \underset{(2\alpha)}{S}{}^m{}_{jk}\underset{(1\alpha)}{S}{}_h{}^i{}_{ml}] = 0, \tag{39}$$

where $\underset{(21)}{S}{}^m{}_{jk} = 0;$ *and* $\underset{(22)}{S}{}^m{}_{jk} = \underset{(2)}{S}{}^m{}_{jk};$

$$\overset{\circ}{\sum}[\underset{(22)}{S}{}_h{}^i{}_{jk}\overset{(2)}{|}_l + \underset{(2)}{S}{}^m{}_{jk}\underset{(22)}{S}{}_h{}^i{}_{ml}] = 0. \tag{40}$$

Remark. $\overset{\circ}{\sum}$ means cyclic sum over (l,k,j) in (21), (27), (30) and over (j,k,l) in (31), (37)and (40).

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