WEAK GRAVITATIONAL FIELDS IN GENERALIZED METRIC SPACES

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Abstract

In the first section we consider a weak pseudo-Riemannian metric on a 4-dimensional manifold M and its associated Berwald-type nonlinear connection N on TM, and settle the basic facts related to the study of $(h, v)-$metrics. In the second section we apply a Finslerian perturbation to the weak metric, which yields a pseudo-Riemann - Finslerian $(h, v)-$metric structure on TM; we determine the explicit Einstein equations for this model. In the third section it is shown that the Sasaki N-lift of the conformal deformation of the weak metric provides also a canonic $(h, v)-$ almost Hermitian metric structure on TM, for which the $h-$ and $v-$Einstein equations are also infered. In the last section are determined the equations of the stationary curves and of their deviations for these models, with emphasis on the special cases of $h-$ and $v-$paths.

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1 Introduction

The concept of gravitational waves in a Finsler space was introduced in a recent work of P.C.Stavrinos [18]. On the other hand, in last years was developed the theory of vector bundles endowed with $(h, v)-$metrics, providing relevant models for General Relativity [12], [13]. In this paper we study the geometrical structure of two $(h, v)-$metrics produced by deformations of a weak pseudo-Riemannian metric γ_{ij} defined on a real 4-dimensional differentiable manifold M. The weakness of the

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gravitational field is expressed by the decomposition of the metric γ_{ij} into the flat Minkowski metric and a small perturbation [18]

$$\gamma_{ij}(x) = n_{ij} + \varepsilon^{(1)}_{ij}(x), \tag{1}$$

where $n_{ij} = diag(-1, 1, 1, 1)$, and $\varepsilon^{(1)}_{ij}$ represents a symmetric tensor field with $|\varepsilon^{(1)}_{ij}(x)| << 1$. The indices are raised in a linearized approach via n_{ij}, e.g., $\varepsilon^{rs} = n^{ri}n^{sj}\varepsilon_{ij}$, where we denoted briefly $\varepsilon_{ij} = \varepsilon^{(1)}_{ij}$. This point of view permits us to develop the linearized version of a given generalized model of General Relativity, in which the symmetric tensor field propagates in a weak pseudo-Riemannian gravitational field.

The pseudo-Riemannian metric γ_{ij} implicitly endowes the tangent bundle (TM, π, M) with the *non-linear connection*

$$N^a_i(x, y) = \gamma^a_{jb} y^b, \tag{2}$$

where γ^i_{jk} are the Christoffel symbols of the metric, and where we denoted by (x^i, y^a) the local coordinates in a chart $\tilde{U} \subset TM$. Throughout the paper, the Latin indices $i, j, k, \ldots, a, b, c, \ldots$ will run in the range $\overline{1, 4}$, while the Greek ones $\alpha, \beta, \gamma, \ldots$, in the range $\overline{1, 8}$. The nonlinear connection N produces on $\mathcal{X}(\tilde{U})$ the *local adapted basis*

$$\{\delta_i = \partial_i - N^b_i \partial_b,\ \dot{\partial}_a\}_{i,a=\overline{1,4}} \equiv \{\partial y^\beta\}_{\beta=\overline{1,8}}, \tag{3}$$

with $\partial_i = \frac{\partial}{\partial x^i}$ and $\dot{\partial}_a = \frac{\partial}{\partial y^a}$, as well as *the dual local basis*

$$\{d^i = dx^i,\ \delta^a = \delta y^a = dy^a + N^a_j dx^j\}_{i,a=\overline{1,4}} \equiv \{dy^\beta\}_{\beta=\overline{1,8}}. \tag{4}$$

We shall consider hereafter the *linear approach*, in which the Christoffel symbols of the weak metric γ_{ij} will take the linearized form [18]

$$\varepsilon^i_{jk} = \frac{1}{2} n^{is}(\partial_{\{j}\varepsilon_{sk\}} - \partial_s \varepsilon_{jk}) \approx \gamma^i_{jk}, \tag{5}$$

where we denoted $\tau_{\{ij\}} = \tau_{ij} + \tau_{ji}$. Then the nonlinear connection will be also approximated by the *weak nonlinear connection*

$$\varepsilon^a_{ib} y^b \approx N^a_i. \tag{6}$$

In this framework, the Finslerian and the conformal generalized Lagrange deformations of the weak metric γ_{ij} provide specific $(h, v)-$metrics on TM.

Generally, if the tangent bundle (TM, π, M) is endowed with a $(h, v)-$metric [12],

$$G = g_{ij}(x, y) dx^i \otimes dx^j + h_{ab}(x, y) \delta y^a \otimes \delta y^b, \tag{7}$$

then one can consider the canonic $N-$connection $\mathbf{D}$, of coefficients

$$\{L^i_{jk}, \tilde{L}^a_{bk}, \tilde{C}^i_{ja}, C^a_{bc}\} \equiv \{\Gamma^\alpha_{\beta\gamma}\}$$

which preserves the $h-v$ splitting produced by N, is metrical, $h-$ and $v-$ symmetrical, and depends on N and G only. Its coefficients are [12]

$$\begin{array}{rcl} L^i_{jk} & = & \frac{1}{2}g^{is}(\delta_{\{j}g_{sk\}} - \delta_s g_{jk}) \\ \tilde{L}^a_{bk} & = & \dot{\partial}_b N^a_k + \frac{1}{2}h^{ac}(\delta_k h_{bc} - h_{c\{d}\dot{\partial}_{b\}}N^d_k) \\ \tilde{C}^i_{ja} & = & \frac{1}{2}g^{ih}\dot{\partial}_a g_{jh} \\ C^a_{bc} & = & \frac{1}{2}h^{ad}(\dot{\partial}_{\{b}h_{dc\}} - \dot{\partial}_d h_{bc}) \end{array}$$

Then *the torsion tensor field* $\mathcal{T} \in \mathcal{T}^1_2(TM)$ of the linear $N-$connection $\mathbf{D}$ has the coefficients given by the relation

$$\mathcal{T}(\delta_\alpha, \delta_\beta) = \mathcal{T}_\beta{}^\kappa{}_\alpha \delta_\kappa, \;\; \mathcal{T}_\beta{}^\kappa{}_\alpha = \Gamma^\alpha_{[\beta\kappa]} + B^\alpha_{[\beta\kappa]}, \tag{8}$$

where we denoted $\tau_{[\alpha\beta]} = \tau_{\alpha\beta} - \tau_{\beta\alpha}$ and *the non-holonomy coefficients* $B_\alpha{}^\gamma{}_\beta$ are provided by the relations $[\delta_\alpha, \delta_\beta] = B_\alpha{}^\gamma{}_\beta \delta_\gamma$.

The $h, v-$splitting of $\mathcal{T}$ provides the *torsion $N-$tensor fields* [12]

$$\begin{array}{rcl} T^i_{jk} & = & d^i\mathcal{T}(\delta_k, \delta_j) = L^i_{[jk]}, \;\; S^a_{bc} = \delta^a\mathcal{T}(\dot{\partial}_c, \dot{\partial}_b) = C^a_{[bc]}, \\ R^a_{kl} & = & \delta^a\mathcal{T}(\delta_l, \delta_k) = \delta_{[l}N^a_{k]}, \;\; P^i_{ja} = d^i\mathcal{T}(\dot{\partial}_a, \delta_j) = \tilde{C}^i_{ja}, \\ P^a_{bk} & = & \delta^a\mathcal{T}(\delta_k, \dot{\partial}_b) = \dot{\partial}_b N^a_k - \tilde{L}^a_{bk}. \end{array}$$

Also, the *curvature tensor field* $\mathcal{R} \in \mathcal{T}^1_3(TM)$ of the $N-$connection $\mathbf{D}$ has the coefficients given by

$$\mathcal{R}(\delta_\alpha, \delta_\beta)\delta_\gamma = \mathcal{R}_\gamma{}^\lambda{}_{\beta\alpha}\delta_\lambda, \;\; \mathcal{R}_\beta{}^\alpha{}_{\gamma\theta} = \delta_{[\theta}\Gamma^\alpha_{\beta\gamma]} + \Gamma^\phi_{\beta[\gamma}\Gamma^\alpha_{\phi\theta]} + \Gamma^\alpha_{\beta\phi}B^\phi_{\gamma\theta}, \tag{9}$$

and the $h, v-$splitting of $\mathcal{R}$ provides the *curvature $N-$tensor fields*

$$\begin{array}{l} R_j{}^i{}_{kl} = d^i\mathcal{R}(\delta_l, \delta_k)\delta_j = \delta_{[l}L^i_{jk]} + L^h_{j[k}L^i_{hl]} + \tilde{C}^i_{ja}R^a_{kl} \\ \tilde{R}_b{}^a{}_{kl} = \delta^a\mathcal{R}(\delta_l, \delta_k)\dot{\partial}_b = \delta_{[l}\tilde{L}^a_{bk]} + \tilde{L}^c_{b[k}\tilde{L}^a_{cl]} + C^a_{bc}R^c_{kl} \\ P_j{}^i{}_{kc} = d^i\mathcal{R}(\dot{\partial}_c, \delta_k)\delta_j = \dot{\partial}_c L^i_{jk} - (\delta_k\tilde{C}^i_{jc} + L^i_{hk}\tilde{C}^h_{jc} - L^h_{jk}\tilde{C}^i_{hc} - \tilde{L}^b_{ck}\tilde{C}^i_{jb}) + \tilde{C}^i_{jb}P^b_{kc} \\ \tilde{P}_b{}^a{}_{kc} = \delta^a\mathcal{R}(\dot{\partial}_c, \delta_k)\dot{\partial}_b = \dot{\partial}_c\tilde{L}^a_{bk} - (\delta_k C^a_{bc} + \tilde{L}^a_{dk}C^d_{bc} - \tilde{L}^d_{bk}C^a_{dc} - \tilde{L}^d_{ck}C^a_{db}) + C^a_{bd}P^d_{kc} \\ \tilde{S}_j{}^i{}_{bc} = d^i\mathcal{R}(\dot{\partial}_c, \dot{\partial}_b)\delta_j = \dot{\partial}_{[c}\tilde{C}^i_{jb]} + \tilde{C}^h_{j[b}\tilde{C}^i_{hc]} \\ S_b{}^a{}_{cd} = \delta^a\mathcal{R}(\dot{\partial}_d, \dot{\partial}_c)\dot{\partial}_b = \dot{\partial}_{[d}C^a_{bc]} + C^e_{b[c}C^a_{ed]}. \end{array}$$

These are the basic geometrical objects which will allow us to infer the Einstein equations of the linearized deformed models defined in the following sections.

2 The Finslerian deformed weak model

We shall present two deformations of the weak metric γ_{ij} and study the associated $(h, v)-$metric structures provided on the tangent space.

The first deformation is produced by a weak Finslerian perturbation of the pseudo-Riemannian gravitational field γ_{ij}, which leads to the generalized Finslerian metric [18]

$$f_{ij}(x,y) = \gamma_{ij}(x) + \varepsilon^{(2)}{}_{ij}(x,y), \tag{10}$$

where $\varepsilon^{(2)}{}_{ij}(x,y)$ is the Finslerian perturbation, $|\varepsilon^{(2)}{}_{ij}(x,y)| << 1$. We remark that, in view of (1), the tensor

$$\varepsilon^{*}{}_{ij}(x,y) = \varepsilon^{(1)}_{ij}(x) + \varepsilon^{(2)}{}_{ij}(x,y) \tag{11}$$

provides a weak Finslerian perturbation of the Minkowski metric n_{ij}, and that $\varepsilon^{*}{}_{ij}$ identically vanishes iff γ_{ij} is flat. This point of view permits us to consider the $(h,v)-$metric $v-$Finslerian or $v-$Lagrangian approaches.

From a physical point of view, a weak Finslerian gravitational field appears as a Finslerian perturbation of a pseudo-Riemannian gravitational field (or external field) of the conventional General Relativity. The perturbation can be considered in the geometrical framework developed by R.G.Beil on the Kaluza-Klein theory or in the ansatz of the Randers-type Yang-Mills theory [7], [8]. Namely, the Finslerian perturbation of the pseudo-Riemannian metric can be provided by the electromagnetic field, or by a gauge or spinor extension of the pseudo-Riemannian gravitational field. In each of this models, the original pseudo-Riemannian model appears as a limit case. Therefore, the correspondence principle between Finslerian and pseudo-Riemannian structure depends basically on the type of the generalized Finsler or Lagrange space associated to the deformed metric.

We should note that the metric $f_{ij}(x,y)$ is a Finsler metric itself, and provides on TM a particular case of generalized Lagrange structure $GL^n = (M, f_{ij})$ in the sense of R.Miron [12]. The *almost Hermitian model* of GL^n, given by the $N-$lift of f_{ij} to TM and by the canonic adapted complex structure on TM defined locally by

$$J(\delta_i) = -\dot{\partial}_i, \quad J(\dot{\partial}_i) = \delta_i, \;\; i = \overline{1,4},$$

yields an *almost Kahler structure.* In case that $\varepsilon^{(1)}_{ij} = const.$ and $\varepsilon^{(2)}{}_{ij}(x,y) = \varepsilon^{(2)}{}_{ij}(y)$, then this is a Kahler space.

On the other side, the two components $n + \varepsilon^{(1)}$ and $\varepsilon^{(2)}$ of the weak Finslerian metric (10) provide on TM the $(h,v)-$metric

$$G = (n_{ij} + \varepsilon^{(1)}_{ij}(x))dx^i \otimes dx^j + \varepsilon^{(2)}{}_{ab}(x,y)\delta y^a \otimes \delta y^b. \tag{12}$$

We shall call the structure (TM,G) *the Finslerian deformed weak model* (FDWM). We note, that in the case when $\varepsilon^{(2)}$ depends on y only, we obtain a *pseudo-Riemann - locally Minkowski* $(h,v)-$metric, and the gravitational field of this space is called *weak Riemannian-locally Minkowski gravitational field.* For obtaining the Einstein equations of the deformed model, we set first the following

Lemma 1. *a) The coefficients of the canonic linear $N-$connection of FDWM in linearized approach are*

$$L^i_{jk} \;\; = \;\; \tilde{L}^i_{jk} = \varepsilon^i_{jk} \approx \gamma^i_{jk}$$

$$\tilde{C}^i_{ja} = 0; \quad C^a_{bc} = \frac{1}{2}\varepsilon^{(2)\,ad}C_{dbc},$$

where $C_{abc} = \dot{\partial}_a \varepsilon^{(2)}{}_{bc}$ *is the Cartan tensor field associated to* $\varepsilon^{(2)}$.

b) The N−fields of torsion of the FDWM are

$$T^i{}_{jk} = 0, \quad \tilde{C}^i{}_{ja} = 0, \quad P^a{}_{kb} = 0, \quad R^a{}_{jk} = r_c{}^a{}_{jk} y^c, \quad S^a_{bc} = 0. \tag{13}$$

c) The N−fields of curvature of the FDWM are

$$\begin{aligned} R_j{}^i{}_{kl} &= r_j{}^i{}_{kl}, \quad \tilde{R}_b{}^a{}_{kl} = r_b{}^a{}_{kl}, \quad P_j{}^i{}_{kc} = 0, \\ \tilde{P}_b{}^a{}_{kc} &= -(\delta_k C^a_{bc} + \varepsilon^a_{dk} C^d_{bc} - \varepsilon^d_{\{bk} C^a_{dc\}}) \\ S_j{}^i{}_{bc} &= 0, \quad \tilde{S}_b{}^a{}_{cd} = C^s_{b[d} C^a_{c]s}, \end{aligned}$$

where $r_j{}^i{}_{kl}$ *is the linearized weak curvature,*

$$r_j{}^i{}_{kl} = \partial_{[l}\varepsilon^i_{jk]} = \frac{1}{2}n^{is}(\partial^2_{[lj}\varepsilon_{sk]} + \partial^2_{[ks}\varepsilon_{jl]}). \tag{14}$$

By straightforward computation, one gets

Theorem 1. *a) The Ricci N−tensor fields of the linearized FDWM are*

$$\begin{aligned} R_{ij} &\equiv R_i{}^k{}_{jk} = r_i{}^k{}_{jk} = \frac{1}{2}(\Box\varepsilon_{ij} + \partial^2_{ij}\varepsilon - \partial^2_{\{js}\varepsilon^s_{i\}}) \\ P_{jb} &\equiv P_j{}^k{}_{kb} = 0 \\ \tilde{P}_{bk} &\equiv \tilde{P}_b{}^d{}_{kd} = -(\delta_k C^a_{ba} - \varepsilon^d_{bk} C^a_{da}) \\ S_{ab} &\equiv S_a{}^d{}_{bd} = C^e_{a[d} C^d_{b]e} \end{aligned}$$

where $\varepsilon = n^{ij}\varepsilon_{ij}$, *and "*$\Box$*" denotes the d'Alambertian*

$$\Box = -\partial^2_{00} + \partial^2_{11} + \partial^2_{22} + \partial^2_{33} \equiv -\partial^2_{tt} + \partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz}.$$

b) The Ricci scalars of curvature of the linearized FDWM are

$$R = r = \Box\varepsilon - \partial^2_{ij}\varepsilon^{ij}, \qquad S = C^e_{b[d} C^d_{c]e}\varepsilon^{(2)\,bc}.$$

Corrolary 1. *The Einstein equations of the linearized FDWM are*

$$\begin{aligned} R_{ij} - \frac{1}{2}(R+S)n_{ij} &\equiv \frac{1}{2}(\Box\varepsilon_{ij} + \partial^2_{ij}\varepsilon - \partial^2_{\{js}\varepsilon^s_{i\}}) - n_{ij}(R+S) = \kappa T_{ij} \\ S_{ab} - \frac{1}{2}(R+S)n_{ab} &\equiv C^e_{a[d}C^d_{b]e} - \frac{1}{2}\varepsilon^{(2)}{}_{ab}(R+S) = \kappa T_{ab} \\ \tilde{P}_{jb} &\equiv 0 = \kappa T_{jb}, \\ P_{bk} &\equiv -(\delta_k C^a_{ba} - \varepsilon^d_{bk} C^a_{da}) = \kappa T_{bk}, \end{aligned}$$

where $T_{ij}, T_{ab}, T_{jb}, T_{bk}$ *are the energy-momentum N−tensor fields, and* κ *is a constant.*

3 The conformally deformed weak model

We consider a conformal-type deformation of the weak pseudo-Riemannian metric γ_{ij}, given by

$$f_{ij}(x,y) = e^{2\sigma(x,y)}\gamma_{ij}(x), \tag{15}$$

where $\sigma : \mathbb{R} \to TM$ is a function of class $\mathcal{C}^\infty$ on TM except the null section, and γ_{ij} is the weak metric (1). Then f_{ij} provides a generalized Lagrange metric TM, and its Sasaki $N-$lift defines a canonic $(h,v)-$metric on TM,

$$G = f_{ij}(x,y)dx^i \otimes dx^j + f_{ab}(x,y)\delta y^a \otimes \delta y^b.$$

We shall call the metric structure (TM, G), the *conformally deformed weak model* (CDWM). It should be noticed that this model is used as mathematical model in General Relativity and obeys the Ehlers-Pirani-Shild conditions [1], [14], [15]. For studying the geometry of the CDWM, we shall use the geometrical concepts of the linearized theory of gravitation of the of General Relativity [16], and empower the linear approximations (5) and (6). Then, by straightforward computation, we obtain the following result

Lemma 2. *The canonic connection of the linearized CDWM has the coefficients*

$$\begin{array}{rcl} L^i_{jk} & = & \varepsilon^i_{jk} + \Lambda^i_j k \\ \tilde{L}^a_{bk} & = & \varepsilon^a_{bk} + \dfrac{1}{2}\varepsilon^{(2)ac}(\delta_k \varepsilon^{(2)}{}_{bc} - \varepsilon^{(2)}{}_{c\{d}\varepsilon^d_{b\}k}) \\ \tilde{C}^i_{ja} & = & \dfrac{1}{2}\delta^i_j \dot{\sigma}_a \\ C^a_{bc} & = & \delta^a_{\{b}\dot{\sigma}_{c\}} - \gamma_{bc} n^{ad}\dot{\sigma}_d \end{array}$$

where we denoted $\sigma_k = \delta_k \sigma, \dot{\sigma}_a = \dot{\partial}_a \sigma$ and $\Lambda^i_{jk} = \delta^i_j \sigma_k + \delta^i_k \sigma_j - \gamma_{jk} n^{is}\sigma_s$.

Theorem 2. *a) The $N-$fields of torsion of the linearized CDWM are*

$$\begin{array}{rcl} T^i{}_{jk} & = & 0, \quad \tilde{C}^i{}_{ja} = \dfrac{1}{2}\delta^i_j \dot{\sigma}_a, \quad S^a_{bc} = 0, \\ P^a{}_{kb} & = & \dot{\partial}_b N^a_k - \tilde{L}^a_{bk} = -\dfrac{1}{2} n^{ac}(\delta_k \gamma_{bc} - \gamma_{c\{d}\varepsilon^d_{b\}k}) \\ R^a{}_{jk} & = & r_c{}^a{}_{jk} y^c. \end{array} \tag{16}$$

b) The $h - hhh$ and $v - vvv$ $N-$fields of curvature of the linearized CDWM are

$$\begin{array}{rcl} R_j{}^i{}_{kl} & = & r_j{}^i{}_{kl} + \delta^i_{[k}\sigma_{jl]} - n^{is}\gamma_{j[k}\sigma_{sl]} + \delta^i_j \dot{\sigma}_a r_c{}^a{}_{kl} y^c \\ S_b{}^a{}_{cd} & = & \delta^a_{[c}\dot{\sigma}_{bd]} - n^{as}\gamma_{b[c}\dot{\sigma}_{sd]}, \end{array}$$

where $r_j{}^i{}_{kl}$ are given in (14), and

$$\begin{array}{rcl} \sigma_{sl} & = & \delta_l \sigma_s - L^t_{sl}\sigma_t + \sigma_s \sigma_l - \dfrac{1}{2}\gamma_{sl} n^{ij}\sigma_i \sigma_j \\ \dot{\sigma}_{ab} & = & \dot{\partial}_b \dot{\sigma}_a - C^d_{ab}\dot{\sigma}_d + \dot{\sigma}_a \dot{\sigma}_b - \dfrac{1}{2}\gamma_{ab} n^{cd}\dot{\sigma}_c \dot{\sigma}_d. \end{array}$$

c) The hh− and the vv−Ricci N−tensor fields, and the scalar field curvatures of the linearized CDWM are

$$\begin{aligned} R_{ij} &= r_{i\ jk}^{\ k} - \gamma_{ij} n^{kl} \sigma_{kl} - 2\sigma_{ij} + \dot{\sigma}_a r_{c\ ji}^{\ a} y^c \\ S_{ab} &= -2\dot{\sigma}_{ab} - \gamma_{ab} n^{cd} \dot{\sigma}_{cd} \end{aligned}$$

and respectively,

$$R = e^{-2\sigma}(n^{ij} r_{i\ jk}^{\ k} - 6n^{ij}\sigma_{ij}), \qquad S = -6e^{-2\sigma} n^{cd} \dot{\sigma}_{cd}.$$

Corrolary 2. *The h− and v−Einstein equations of the linearized CDWM are*

$$\begin{aligned} 2R_{ij} - (R+S)\gamma_{ij} &\equiv 2(r_{i\ jk}^{\ k} - 2\sigma_{ij} + \dot{\sigma}_a r_{c\ ji}^{\ a} y^c) - \\ &\quad -(n^{lm} r_{l\ mk}^{\ k} - 6n^{lm}\sigma_{lm} - 8n^{cd}\dot{\sigma}_{cd})\gamma_{ij} \\ &= 2\kappa T_{ij} \\ 2S_{ab} - (R+S)\gamma_{ab} &\equiv -4\dot{\sigma}_{ab} - -(n^{ij} r_{i\ jk}^{\ k} - 6n^{ij}\sigma_{ij} - 8n^{cd}\dot{\sigma}_{cd})\gamma_{ab} \\ &= 2\kappa T_{ab}, \end{aligned}$$

where T_{ij} and T_{ab} are the hh− and vv− energy-momentum N−tensor fields, and κ is a constant.

4 The paths of the deformed weak models

Let $c : I = [a,b] \subset \mathbb{R} \to TM$ be a smooth curve, such that its image lies in a chart $\tilde{U} \subset TM$,

$$c(t) = (x^i(t), y^a(t)) \equiv (y^\alpha(t)), \forall t \in I,$$

and let $\mathbf{D}$ be a linear $N-$connection on TM.

Definitions. a) The fields defined on c by

$$\begin{aligned} \mathcal{V} &= \mathcal{V}^\alpha \delta_\alpha, & \mathcal{V}^\alpha &= \frac{\delta y^\alpha}{dt} \\ \mathcal{F} &= \frac{\mathbf{D}\mathcal{V}}{dt} = \mathcal{F}^\alpha \delta_\alpha, & \mathcal{F}^\alpha &= \frac{\delta \mathcal{V}^\alpha}{dt} + \Gamma^\alpha_{\beta\kappa} \mathcal{V}^\beta \mathcal{V}^\kappa, \ \alpha = \overline{1,8}, \end{aligned} \tag{17}$$

will be called *covariant velocity* field of the curve c, and respectively the *covariant force* on c, the last providing the motion of the test-body along c.

b) We shall say that c is a *stationary curve* with respect to $\mathbf{D}$ iff $\mathcal{F} = 0$ along the curve.

c) The curve c is called

- *h−curve*, if $\pi_v(\mathcal{V}) = 0$;
- *v−curve*, if $\pi_h(\mathcal{V}) = 0$,

where by π_h and π_v we denoted respectively the $h-$ and $v-$projectors of the canonic splitting induced by N. If a $h-/v-$curve satisfies also the extra condition $\mathcal{F} = 0$, then it is called $h-/v-$*path*, respectively.

Remarks. a) Since in the linearized approach, the non-linear connection N is provided by the pseudo-Riemannian metric γ_{ij} on M by means of relation (6), any stationary h-curve (i.e., any $h-$path) of the two models projects onto a geodesic of M.

b) The $v-$paths of the FDWM coincide with the $v-$paths of the Finsler space $(M, F(x,y))$, with $F^2 = \varepsilon^{(2)}{}_{ab}(x,y)y^a y^b$.

c) The paths of CDWM coincide with the paths of the Generalized Lagrange space $(M, e^{2\sigma(x,y)}\gamma_{ij}(x))$.

d) Any $h-$path $c : I \subset \mathbb{R} \to TM$, $c(t) = (x^i(t), y^a(t))$ is a solution of the Volterra-Hamilton-type second-order differential system

$$\begin{cases} \dfrac{dy^a}{dt} + N^a_j(x(t), y(t))\dfrac{dx^j}{dt} = 0 \\ \dfrac{d^2x^i}{dt^2} + L^i_{jk}(x(t), y(t))\dfrac{dx^j}{dt}\dfrac{dx^k}{dt} = 0. \end{cases} \tag{18}$$

It should be noticed, that in the FDWM, the system (18) rewrites

$$\begin{cases} \dfrac{dy^i}{dt} = -\varepsilon^i_{jk}(x(t)) \cdot y^j(t)z^k(t), \quad \dfrac{dx^i}{dt} = z^i(t) \\ \dfrac{dz^i}{dt} = -\varepsilon^i_{jk}(x(t)) \cdot z^j(t)z^k(t), \end{cases}$$

e) Any $v-$path $c : I \subset \mathbb{R} \to TM$, $c(t) = (x^i_0, y^a(t))$ is a solution of the second-order differential system

$$\frac{d^2y^a}{dt^2} + C^a_{bc}(x_0, y(t))\frac{dy^b}{dt}\frac{dy^c}{dt} = 0. \tag{19}$$

f) The system (18) has the unknowns $x^i = x^i(t), y^i = y^i(t), z^i = z^i(t), i = \overline{1,4}$. If we impose initial conditions $x^i(0) = x^i_0$, $y^i(0) = y^i_0$, $z^i(0) = z^i_0$, we obtain a Cauchy problem which is perfectly tractable numerically, e.g., using the Runge-Kutta algorithm [3]. A similar approach can be applied to the system (19).

Let now $c : I_1 \times I_2 \subset \mathbb{R}^2 \to \tilde{U} \subset TM$ be a family of stationary curves, having t as arc-length parameter, and u the deviation parameter [17], [9],

$$c(t,u) = (x^i(t,u), y^a(t,u)) = (y^\alpha(t,u)) \in \tilde{U}, \quad \forall (t,u) \in I_1 \times I_2,$$

where $\tilde{U} \subset TM$ is an open chart-domain. Then let $\mathcal{Z} = \mathcal{Z}^\alpha \delta_\alpha$ be the *deviation vector field*, given by

$$\mathcal{Z}^i = \partial_u x^i, \quad \mathcal{Z}^a = \partial_u y^a + N^a_i \partial_u x^i,$$

and let $\mathcal{V} = \mathcal{V}^\alpha \delta_\alpha$ be the *velocity vector field*, where

$$\mathcal{V}^i = \partial_t x^i, \quad \mathcal{V}^a = \partial_t y^a + N^a_i \partial_t x^i.$$

For any vector field $\mathcal{W} = \mathcal{W}^\alpha \delta_\alpha$, defined on the family $Im(c)$, we can consider the partial covariant derivatives

$$\delta_t \mathcal{W}^\alpha = \partial_t \mathcal{W}^\alpha + \Gamma^\alpha_{\beta\gamma} \mathcal{W}^\beta \mathcal{V}^\gamma, \quad \delta_u \mathcal{W}^\alpha = \partial_u \mathcal{W}^\alpha + \Gamma^\alpha_{\beta\gamma} \mathcal{W}^\beta \mathcal{Z}^\gamma.$$

The equations of deviations of the family with respect to the connection $\mathbf{D}$ characterize the tidal force $\mathcal{Z}$, and have the form [2], [4], [6]

$$\delta_t^2 \mathcal{Z}^\alpha + \delta_t \mathcal{T}^\alpha = \rho^\alpha + \delta_u \mathcal{F}^\alpha,$$

where we denoted $\mathcal{T}^\alpha = \mathcal{T}_{\beta\ \gamma}^{\ \alpha} \mathcal{V}^\beta \mathcal{Z}^\gamma$ and $\rho^{\ \alpha} = \mathcal{R}_{\beta\ \gamma\lambda}^{\ \alpha} \mathcal{V}^\beta \mathcal{Z}^\gamma \mathcal{V}^\lambda$. Actually, these equations split

$$\begin{cases} \delta_t^2 \mathcal{Z}^i + \delta_t \mathcal{T}^i = \rho^i + \delta_u \mathcal{F}^i \\ \\ \delta_t^2 \mathcal{Z}^a + \delta_t \mathcal{T}^a = \rho^a + \delta_u \mathcal{F}^a, \end{cases}$$

and it should be noticed that the equations of deviations of stationary curves are considerably simplified for paths, e.g., if c is an h-path, then these become:

$$\begin{cases} \delta_t^2 \mathcal{Z}^i + \delta_t \mathcal{T}^i = \rho^i, \ i = \overline{1,4} \\ \\ \delta_t^2 \mathcal{Z}^a + \delta_t \mathcal{T}^a = 0, \ a = \overline{1,4}. \end{cases}$$

The equations of deviations of paths for the two models presented above are particular cases of the ones considered in [5], [2], [4], [6], which are extensions of the Finslerian case settled in [17]. In particular, the study of deviation of geodesics for the Finslerian case, which includes the Finsler metric $n + \varepsilon^{(1)} + \varepsilon^{(2)}$ of the linearized FDWM, was performed in [18], [19].

Conclusions. The weak pseudo-Riemannian gravitational model was extended by considering two deformations of the weak pseudo-Riemannian metric γ_{ij} of the 4-dimensional base space M. These gave rise to $(h, v)-$metrics on TM. Thus the two considered deformed models fit in the general theory of $(h, v)-$metric structures on vector bundles developed in [12], [13], [14], [4], [5]. In this framework, the explicit Einstein equations and the equations of stationary curves and of their deviations were determined for the canonic linear $N-$connection, with the Berwald-type nonlinear connection N considered in linearized approach.

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