# THE CHARACTERISTIC ELEMENTS OF SPECIAL NILPOTENT LIE ALGEBRAS OF DIMENSION TEN

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**Abstract**

The aim of the present paper is to give the characteristic elements of a 10-dimensional Nilpotent Lie algebra with maximal abelian ideal of dimensional nine.

**AMS Subject Classification:** 17B30.

**Key words:** Nilpotent Lie Algebra, Abelian Ideal , Invariant Functions.

## 1 Introduction.

The determination of characteristic elements of a given nilpotent Lie algebra is an open problem. In this paper we estimate the characteristic elements of the nilpotent Lie algebras of dimension ten whose the maximal abelian ideal is of dimension nine.

The whole paper contains four paragraphs. The first paragraph contains the introduction and the second the preliminaries. In the third paragraph are described the nilpotent Lie algebras of dimension ten whose the dimension of maximal abelian ideal is nine .The characteristic elements of these nilpotent Lie algebras are estimated in the forth paragraph.

## 2 Preliminaries

We denote a base of a nilpotent Lie algebra $g$ of dimension $n$ by $\{X_1, ...X_n\}$.For this Lie algebra $g$ we have the relation:

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k \quad i, j = 1, ...n \tag{1}$$

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where $c_{ij}^k$ are the structure constants of the Lie algebra $g$.

The invariants of $g$ are the functions $F(X)$ which satisfy the relations

$$[X_i, F(X)] = 0, \quad i = 1, 2, ...n \tag{2}$$

It has been discussed [4] to derive, which we adopt of replacing the $X$' s by c-number differential operators:

$$X_i \to x_i = \sum x_k c_{ij}^k \partial x_i \tag{3}$$

which have the same commutation rule and act on a space of continuously differentiable functions of n real variables. The commutation relations (2) are replaced by the set of partial differential equations:

$$x_i F(x) = 0 \tag{4}$$

which can be usually solved by the standard methods.Therefore, the solutions of (2) are obtained from those of (4) by the replacements $x_i \to X_i$ provided that the factors $X_i$ in $F(X)$ can be ordered so that (4) implies (2).

If the solution is a rational function, then we symmetrize the dominator and numerator separately and after that we replace x$_i$ by X$_i$ and obtain rational invariants.

It is possible the invariants to be no polynomials neither rational functions but to have a general form. This depends on the solutions of the first order partial differential equation (4).

The first order partial differential equation (4), by means (3) takes the form

$$\sum_{j,k=1}^{n} c_{ij}^k x_k \frac{\partial f}{\partial x_j} = 0$$

which leads to the following system of ordinary differential equations.

$$\frac{dx_1}{\sum_{i=1}^n c_{i1}^k dx_k} = \frac{dx_2}{\sum_{i=1}^n c_{i2}^k x_k} = ... = \frac{dx_n}{\sum_{i=1}^n c_{in}^k x_k} = \frac{dt}{s}, \tag{5}$$

where

$$t = x_1 + ... + x_n, \qquad s = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij}^k x_k.$$

The system of differential equations (5) can be replaced by a set of equivalent differential equations

$$\frac{dt}{s} = \frac{d(\beta_1 x_1 + ... + \beta_n x_n)}{\sum_{j=1}^n \sum_{i=1}^n \beta_k c_{ij}^k x_k},$$

where $\beta_{1,...,}\beta_n$ are arbitrary constants. When we change the constants, we obtain the different differential equations of this system. Therefore we can choose n linearly independent combinations of $x_1, ...x_n$ in a convenient manner such that the matrix

$$\left((c_i)_j^k\right)$$

of coefficients has a standard form, in particular the Jordan canonical form.

We conclude this section by referring to some simple properties of the invariants, pointed out by Aballanos and Alonso [3].

The invariants found in the subsequent sections of this paper, of course, have these properties:

a)The number of independent invariants is n-r, where n is the dimension of the algebra and r is the rank of the commutator table, considered as a matrix, for the purpose of computing this rank, the generators $X_i$ are regarded as independent c-number variables.

b) Since an antisymmetric matrix has even rank the number of independent invariants is equal to the dimension of the algebra, modulo 2.

c) The invariants of nilpotent algebras may be chosen as homogeneous polynomials, i.e., Casimir operators.

This method gives the invariants of the algebra $g$. We have used here the base $\{X_1, ...X_n\}$ instead of $\{e_1, ...e_n\}$ . This method has been used to construct the partial differential equations which form a system.

**Definition** Let $Der(g)$ be the Lie algebra of derivations of $g$. The torus on $g$ is a commutative subalgebra of $Der(g)$ consisting of semi-simple endomorphismus. A torus $T$ is called *maximal*, if it is not contained strictly in any other torus.

A torus $T$ on $g$ defines a representation in $g$, that is

$$\varphi : T \times g \to g, \qquad \varphi : (t, x) \to tx$$

The elements of $T$ can be diagonalized simultaneously.

From the above we conclude that $g$ can be decomposed as follows

$$g = \underset{\beta \in T^*}{\oplus} g^\beta$$

where $T^*$ is the dual of $T$ and $g^\beta$ is defined by

$$g^\beta = \{x \in g / t(x) = \beta(t)x, \quad \forall t \in T\ \}$$

The subspace $g^\beta$ of $g$ is called root space associated to $\beta$.

Let $T$ be a maximal torus on $g$.The root system of $g$ associated to $T$, denoted by $R(T)$, is defined by

$$R(T) = \{\beta \in T^* / g^\beta \neq \{0\}\}$$

Let $R(T)$ be the root system of a nilpotent Lie algebra $g$ associated to its torus $T$. We assume that $R(T)$ has m roots :

$$\beta_1, \beta_2, ..., \beta_\nu, \beta_{\nu+1}, ..., \beta_m$$

from which only the $\beta_1, ..., \beta_\nu$ are independent and therefore the others can be expressed as linear combination of them with integer coefficients. In this case $g$ can be decomposed as follows

$$g = g^{\beta_1} \oplus ... \oplus g^{\beta_\nu} \oplus g^{\beta_{\nu+1}} \oplus ... \oplus g^{\beta_m}$$

This decomposition of $g$ is called weight system of $g$.

Let $\{x_1, ..., x_n\}$,be a base of $g$. The subset $\{x_1, ..., x_k\}$ $k < n$ of this base, whose elements do not appear in the second members of (1) is called minimal system of generators.

One can easily prove the following proposition:

**Proposition** *Let $g$ be a nilpotent Lie algebra. The following two statements are eqivalent:*

*I) The set $\{x_1, ..., x_i\}$is a minimal system of generators of $g$.*

*II) The set $\{x_1 + C^2g, ..., x_i + C^2g\}$ is a base of the vector space $g/C^2g$ , where* $C^2g = [g, g]$.

Let $T$ be a maximal torus on $g$. Let $\{x_1, ..., x_p\}$ be a minimal system of generators. Let $\{t_1, ..., t_p\}$ be root vectors of $T$ such that $t_i(x_i) = x_i$, then $\{t_1, ..., t_p\}$ or equivalently $\{x_1, ..., x_p\}$ is called $T - msg$ system.

**Proposition:** *Let $g$ be a nilpotent Lie algebra. For any torus $T$ on $g$ there exists a $T - msg$ system.*

*Proof***:** We take the root vectors $T$ such that they form a basis for $T$-supplement of $C^2g$.

Let $g$ be a nilpotent Lie algebra. The type of $g$ is defined by $\dim(g/C^2g)$.

Let $g$ be a nilpotent Lie algebra of type l. We can associate to $g$ a square matrix $A = (A_{ij})$ $i, j = 1, 2, ..., l$,which is called Generalized Cartan Matrix denoted briefly G.C.M.. This matrix is defined as follows. Let $T$ be a maximal torus on $g$ and $\{y_1, ..., y_n\}$ a $T$-*msg* system. Since $ady_i$ is a nilpotent, then there exists a unique negative integer $A_{ij} \in \mathbf{Z}$ such that

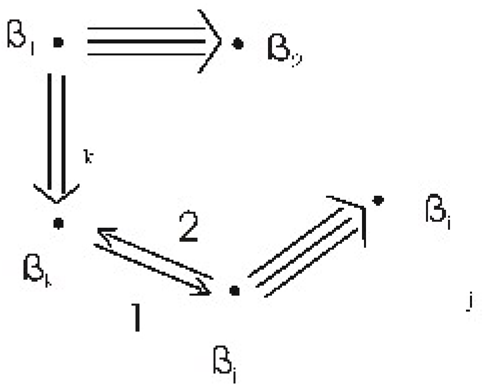
$$(ady_i)^{-A_{ij}} y_j \neq \{0\}, \quad (ady_i)^{-A_{ij}+1} y_j = \{0\}, \quad \text{i} \neq j$$

If we set $A_{ii} = 2$, $i = 1, ..., l$ then we obtain the G.C.M. $A = (A_{ij})$.

A universal property of G.C.M. can be expressed by the theorem:

**Theorem:** *Let $g$ be a nilpotent Lie algebra of type $r$ over the field K;, where K=$\mathbb{R}$ or K=$\mathbb{C}$ . The G.C.M. $A = (A_{ij})$ associated to $g$ is independent on the $T-msg$ system and the choice of the torus $T$ on $g$.*

The Dynkin diagram of a nilpotent Lie algebra can be presentes as follows. We write the roots of $g$ as points of a plane (figure 1). Then we connect the roots by segments of straight lines and put some directions. The number of segments of the straight lines between the roots $\beta_i$ and $\beta_j$ is equal to $A_{ij}A_{ji}$ and required direction is from $\beta_i$ to $\beta_j$ if $|A_{ij}|>|A_{ji}|$.




All the above characteristic elements will be given for the 10-dimensional nilpotent Lie algebras, over a field of characteristic zero , whose maximal abelian ideal is of dimension nine.

# 3 Nilpotent Lie algebras of dimension ten whose maximal abelian ideal is of dimension nine

The nilpotent Lie algebras of dimension ten whose maximal abelian ideal is of dimension nine are the following:

$g_{10,1}:$ $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_5] = e_6, [e_1, e_7] = e_8, [e_1, e_9] = e_{10},$

$g_{10,2}:$ $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_6] = e_7, [e_1, e_7] = e_8,$ $[e_1, e_9] = e_{10}$

$g_{10,3}:$ $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6, [e_1, e_7] = e_8,$ $[e_1, e_9] = e_{10}$

$g_{10,4}:$ $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_5] = e_6, [e_1, e_6] = e_7, [e_1, e_8] = e_9,$ $[e_1, e_9] = e_{10}$

$g_{10,5}:$ $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6, [e_1, e_7] = e_8,$ $[e_1, e_8] = e_9, [e_1, e_9] = e_{10}$

$g_{10,6}:$ $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6, [e_1, e_6] = e_7,$ $[e_1, e_8] = e_9, [e_1, e_9] = e_{10}$

$g_{10,7}:$ $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6, [e_1, e_6] = e_7,$ $[e_1, e_7] = e_8, [e_1, e_9] = e_{10}$

$g_{10,8}:$ $[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_1, e_5] = e_6, [e_1, e_6] = e_7,$ $[e_1, e_7] = e_8, [e_1, e_8] = e_9, [e_1, e_9] = e_{10}$

# 4 Characteristic Elements

We calculate the invariants of these Lie algebras.Firstly , we consider the Lie algebras $g_{10,7}$ and $g_{10,8}$.

We consider the system of partial differential equations.

$g_{10,7}$ :

$$e_3\frac{\partial F}{\partial e_2}+e_4\frac{\partial F}{\partial e_3}+e_5\frac{\partial F}{\partial e_4}+e_6\frac{\partial F}{\partial e_5}+e_7\frac{\partial F}{\partial e_6}+e_8\frac{\partial F}{\partial e_7}+e_{10}\frac{\partial F}{\partial e_9}=0$$

$$\begin{aligned}-e_3\frac{\partial F}{\partial e_2} &= 0,-e_4\frac{\partial F}{\partial e_3}=0,-e_5\frac{\partial F}{\partial e_4}=0,-e_6\frac{\partial F}{\partial e_5}=0\\ -e_7\frac{\partial F}{\partial e_6} &= 0,-e_8\frac{\partial F}{\partial e_7}=0,-e_{10}\frac{\partial F}{\partial e_9}=0\end{aligned}$$

$g_{10,8}$ :

$$e_3\frac{\partial F}{\partial e_2}+e_4\frac{\partial F}{\partial e_3}+e_5\frac{\partial F}{\partial e_4}+e_6\frac{\partial F}{\partial e_5}+e_7\frac{\partial F}{\partial e_6}+e_8\frac{\partial F}{\partial e_7}+e_9\frac{\partial F}{\partial e_8}+e_{10}\frac{\partial F}{\partial e_9}=0$$

$$\begin{aligned}-e_3\frac{\partial F}{\partial e_2} &= 0,-e_4\frac{\partial F}{\partial e_3}=0,-e_5\frac{\partial F}{\partial e_4}=0,-e_6\frac{\partial F}{\partial e_5}=0\\ -e_7\frac{\partial F}{\partial e_6} &= 0,-e_8\frac{\partial F}{\partial e_7}=0,-e_9\frac{\partial F}{\partial e_8}=0,-e_{10}\frac{\partial F}{\partial e_9}=0\end{aligned}$$

The solutions of the above linear systems of partial differential equations give the invariants, which are :

$$\begin{aligned}g_{10,7}\ :\ & e_8, e_{10}, e_{10}e_7-e_9e_8, 2e_6e_8-e_7^2, 3e_5e_8^2-3e_8e_6e_7+e_7^3, 2e_2e_8-2e_3e_7+2e_4e_6-e_5^2,\\ & 4e_4e_8^3-2e_6^2e_8^2-4e_5e_7e_8^2+4e_6e_7^2e_8-e_7^4,\\ & 5e_3e_8^4-5e_4e_7e_8^3-5e_5e_6e_8^3+5e_8^2e_5e_7^2+5e_7e_6^2e_8^2-5e_6e_8e_7^3+e_7^3.\end{aligned}$$

$$\begin{aligned}g_{10,8}\ :\ & e_{10}, 2e_{10}e_8-e_9^2, 3e_7e_{10}^2-3e_8e_9e_{10}+e_9^3, 2e_2e_{10}-2e_3e_9+2e_4e_8-2e_5e_7+e_6^2,\\ & 4e_6e_{10}^3-2e_{10}^2e_8^2-4e_9e_7e_{10}^2+4e_8e_9^2e_{10}-e_9^4,\\ & 5e_5e_{10}^4-5e_6e_9e_{10}^3-5e_7e_8e_{10}^3+5e_7e_{10}^2e_9^2+5e_9e_8^2e_{10}^2-5e_{10}e_8e_9^3+e_9^5,\\ & 36e_4e_{10}^5-36e_5e_9e_{10}^4+18e_6e_9^2e_{10}^3-6e_7e_{10}^2e_9^3+6e_{10}^3e_8^3-9e_8^2e_9^2e_{10}^2+6e_{10}e_8e_9^4-e_9^6,\\ & 210e_3e_{10}^6-210e_4e_9e_{10}^5+210e_5e_8e_{10}^5-210e_6e_7e_{10}^5+210e_9e_{10}^4e_7^2-210e_7e_8^2e_{10}^4+\\ & 210e_9e_8^3e_{10}^3-210e_8^2e_9^3e_{10}^2+84e_8e_{10}e_9^5-12e_9^7.\end{aligned}$$

From the above we have the theorem.

**Theorem**. *The invariants of the Lie algebras* $g_{10,i}, i = 1, 2, ...8$ *are the following.*

$g_{10,1}$ : $e_4, e_6, e_8, e_{10}, 2e_2e_4 - e_3^2, e_3e_6 - e_4e_5, e_5e_8 - e_7e_6, e_7e_{10} - e_8e_9$.

$g_{10,2}$ : $e_5, e_8, e_{10}, 2e_3e_5 - e_4^2, 2e_6e_8 - e_7^2, e_4e_{10} - e_5e_9, e_7e_{10} - e_8e_9$.

$g_{10,3}$ : $e_6, e_8, e_{10}, 2e_6e_4 - e_5^2, e_5e_{10} - e_6e_9, e_5e_8 - e_7e_6, e_7e_{10} - e_8e_9,$
$3e_3e_6^2 - 3e_4e_6e_5 + e_5^3$.

$g_{10,4}$ : $e_4, e_7, e_{10}, 2e_2e_4 - e_3^2, 2e_8e_{10} - e_9^2, 2e_5e_7 - e_6^2, e_3e_7 - e_4e_6, e_3e_{10} - e_4e_9$.

$g_{10,5}$ : $e_6, e_{10}, 2e_6e_4 - e_5^2, e_5e_{10} - e_6e_9, 2e_{10}e_8 - e_9^2, 3e_7e_{10}^2 - 3e_8e_9e_{10} + e_9^3,$
$3e_3e_6^2 - 3e_4e_6e_5 + e_5^3, 2e_2e_6 - 2e_3e_5 + e_4^2$.

$g_{10,6}$ : $e_7, e_{10}, 2e_5e_7 - e_6^2, e_6e_{10} - e_7e_9, 2e_{10}e_8 - e_9^2, 3e_4e_7^2 - 3e_5e_6e_7 + e_6^3,$
$4e_3e_7^3 - 2e_5^2e_7^2 - 4e_4e_6e_7^2 + 4e_5e_6^2e_7 - e_6^4,$
$5e_2e_7^4 - 5e_4e_5e_7^3 - 5e_3e_6e_7^3 + 5e_5^2e_6e_7^2 + 5e_4e_6^2e_7^2 - 5e_5e_6^3e_7 + e_6^5$.

$g_{10,7}$ : $e_8, e_{10}, 2e_{10}e_7 - e_9e_8, 2e_6e_8 - e_7^2, 3e_5e_8^2 - 3e_8e_6e_7 + e_7^3, 2e_2e_8 - 2e_3e_7 + 2e_4e_6 - e_5^2,$
$4e_4e_8^3 - 2e_6^2e_8^2 - 4e_5e_7e_8^2 + 4e_6e_7^2e_8 - e_7^4,$
$5e_3e_8^4 - 5e_4e_7e_8^3 - 5e_5e_6e_8^3 + 5e_8^2e_5e_7^2 + 5e_7e_6^2e_8^2 - 5e_6e_8e_7^3 + e_7^5$.

$g_{10,8}$ : $e_{10}, 2e_{10}e_8 - e_9^2, 3e_7e_{10}^2 - 3e_8e_9e_{10} + e_9^3, 2e_2e_{10} - 2e_3e_9 + 2e_4e_8 - 2e_5e_7 + e_6^2,$
$4e_6e_{10}^3 - 2e_{10}^2e_8^2 - 4e_9e_7e_{10}^2 + 4e_8e_9^2e_{10} - e_9^4,$
$5e_5e_{10}^4 - 5e_6e_9e_{10}^3 - 5e_7e_8e_{10}^3 + 5e_7e_{10}^2e_9^2 + 5e_9e_8^2e_{10}^2 - 5e_{10}e_8e_9^3 + e_9^5,$
$36e_4e_{10}^5 - 36e_5e_9e_{10}^4 + 18e_6e_9^2e_{10}^3 - 6e_7e_{10}^2e_9^3 + 6e_{10}^3e_8^3 - 9e_8^2e_9^2e_{10}^2 + 6e_{10}e_8e_9^4 - e_9^6,$
$210e_3e_{10}^6 - 210e_4e_9e_{10}^5 + 210e_5e_8e_{10}^5 - 210e_6e_7e_{10}^5 + 210e_9e_{10}^4e_7^2 - 210e_7e_8^2e_{10}^4 +$
$210e_9e_8^3e_{10}^3 - 210e_8^2e_9^3e_{10}^2 + 84e_8e_{10}e_9^5 - 12e_9^7$.

Now we shall determine all the other characteristic elements associated to the above eight nilpotent Lie algebras $g_{10,1}, ...g_{10,8}$.

We shall estimate these elements only for the Lie algebra $g_{10,1}$. With the same method we can estimate all these for the other Lie algebras

Let $T$ be the maximal torus on $g_{10,1}$such that $\{e_1, ...e_m\}$ are root vectors. If $t \in T$, then we have

$$t(e_i) = \beta_i(t)e_i \tag{6}$$

From the fact that $t \in T \subseteq Derg$ and if we apply $t$ to the Lie bracket $[e_1, e_2] = e_3$ of the Lie algebra $g_{10,1}$ we obtain

$$t(e_3) = [t(e_1), e_2] + [e_1, t(e_2)]$$

which by means of (6) implies:

$$\beta_3(t)e_3 = [\beta_1(t)e_1, e_2] + [e_1, \beta_2(t)e_2] = (\beta_1(t) + \beta_2(t))[e_1, e_2] = (\beta_1(t) + \beta_2(t))e_3$$

which finally gives:

$$\beta_3 = \beta_1 + \beta_2 \tag{7}$$

From the other Lie brackets of the Lie algebra $g_{10,1}$ we have the relations :

$$\beta_1+\beta_3=\beta_4,\ \beta_1+\beta_5=\beta_6,\ \beta_1+\beta_7=\beta_8,\ \beta_1+\beta_9=\beta_{10} \quad (8)$$

The solutions of (7) and (8) give

$$\beta_3=\beta_1+\beta_2\ ,\ \beta_4=\beta_1+\beta_3,\ \beta_6=\beta_1+\beta_5,\ \beta_8=\beta_1+\beta_7,\ \beta_{10}=\ \beta_1+\beta_9$$

Then the nilpotent Lie algebra $g_{10,1}$ can be written

$$\begin{aligned} g_{10,1} &= g^{\beta_1}\oplus g^{\beta_2}\oplus g^{\beta_1+\beta_2}\oplus g^{2\beta_1+\beta_2}\oplus g^{3\beta_1+\beta_2}\oplus g^{4\beta_1+\beta_2}\oplus g^{5\beta_1+\beta_2}\oplus \\ &\oplus g^{\beta_8}\oplus g^{\beta_1+\beta_8}\oplus g^{2\beta_1+\beta_8} \end{aligned}$$

which also can be written

$$g_{10,1}=\mathbf{C}e_1\oplus\mathbf{C}e_2\oplus\mathbf{C}e_3\oplus\mathbf{C}e_4\oplus\mathbf{C}e_5\oplus\mathbf{C}e_6\oplus\mathbf{C}e_7\oplus\mathbf{C}e_8\oplus\mathbf{C}e_9\oplus\mathbf{C}e_{10}$$

from which we obtain

$$[g,g]=\mathbf{C}e_3\oplus\mathbf{C}e_4\oplus\oplus\mathbf{C}e_6\oplus\mathbf{C}e_8\oplus\mathbf{C}e_{10}$$

Hence $T-msg=\{e_1,e_2,e_5,e_7,e_9\}=\{x_1,x_2,x_3,x_4,x_5\}$.
Therefore, the Generalized Cartan Matrix :

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35}\\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45}\\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \quad (9)$$

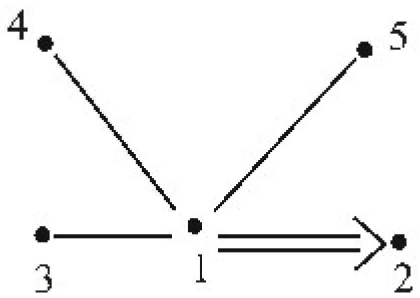
of $g_{10,1}$ can be computed as follows:

$$-a_{ij}=\min\{n\in N/(adx_i)^{n+1}x_j=0,x_i,x_j\in T-msg\}$$

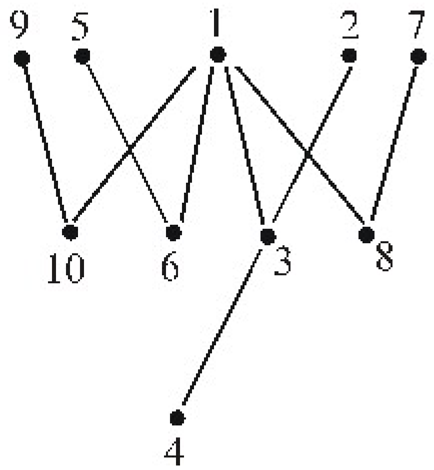
If we take under the consideration the $T-msg$ which is $\{x_1,x_2,x_3,x_4,x_5\}$ and the Lie brackets of the Lie algebra $g_{10,1}$ then the Generalized Cartan Matrix $A$ takes the form:

$$A=\begin{bmatrix} 2 & -2 & -1 & -1 & -1\\ -1 & 2 & 0 & 0 & 0\\ -1 & 0 & 2 & 0 & 0\\ -1 & 0 & 0 & 2 & 0\\ -1 & 0 & 0 & 0 & 2 \end{bmatrix}$$

From the above we conclude that the Dynkin diagram of $g_{10,1}$ has the form



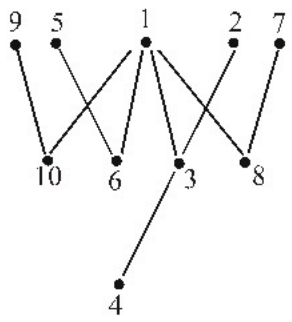
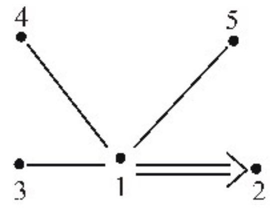
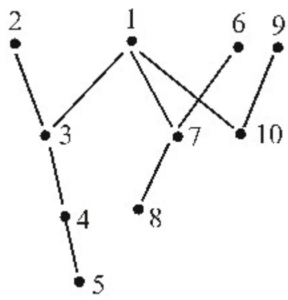
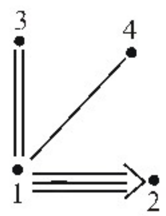
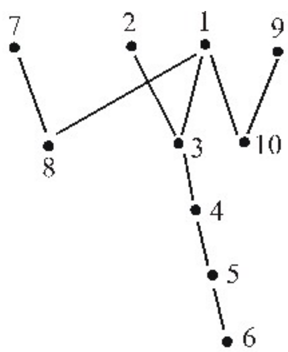
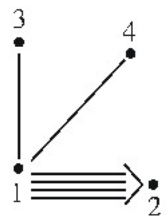
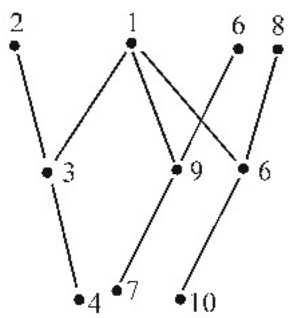
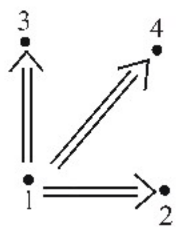

The root system can be expressed as follows:

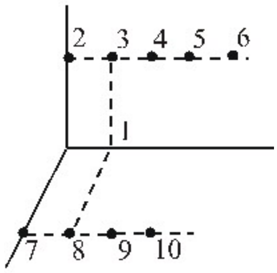
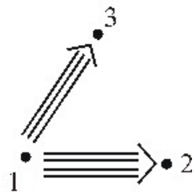
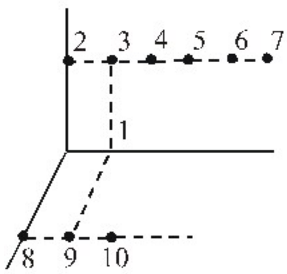
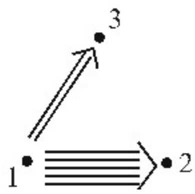
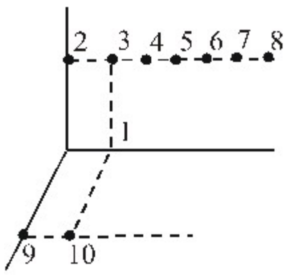
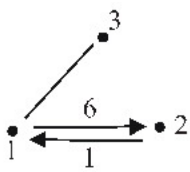
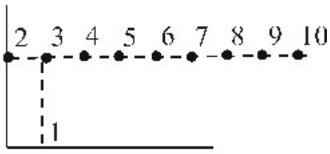
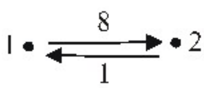



From the above we conclude the following theorem.

**Theorem.** *Let $g$ be a Nilpotent Lie algebra of dimension ten over* **C** *whose maximal abelian ideal $g_o$ is of dimension nine. The table below gives the characteristic elements of such Lie algebras.*

| $g$ | **type** | **dim** $z(g)$ | **T-msg** | **weight system** |
|---|---|---|---|---|
| $g_{10,1}$ | $10,5,$ $1$ | 4 | $e_1,e_2,$ $e_5,e_7,$ $e_9$ | $g = g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2} \oplus g^{\beta_5}\oplus$ $\oplus g^{\beta_1+\beta_5} \oplus g^{\beta_7} \oplus g^{\beta_1+\beta_7} \oplus g^{\beta_9} \oplus g^{\beta_1+\beta_9} =$ $= Ce_1 \oplus Ce_2 \oplus Ce_3 \oplus Ce_4 \oplus Ce_5\oplus$ $\oplus Ce_6 \oplus Ce_7 \oplus Ce_8 \oplus Ce_9 \oplus Ce_{10}$ |
| $g_{10,2}$ | $10,6,$ $3,1$ | 3 | $e_1,e_2,$ $e_6,e_9$ | $g = g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2} \oplus g^{3\beta_1+\beta_2}\oplus$ $\oplus g^{\beta_6} \oplus g^{\beta_1+\beta_6} \oplus g^{2\beta_1+\beta_6} \oplus g^{\beta_9} \oplus g^{\beta_1+\beta_9} =$ $= Ce_1 \oplus Ce_2 \oplus Ce_3 \oplus Ce_4 \oplus Ce_5\oplus$ $\oplus Ce_6 \oplus Ce_7 \oplus Ce_8 \oplus Ce_9 \oplus Ce_{10}$ |
| $g_{10,3}$ | $10,6$ $3,2,$ $1)$ | 3 | $e_1,e_2,$ $e_7,e_9$ | $g = g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2} \oplus g^{3\beta_1+\beta_2}\oplus$ $\oplus g^{4\beta_1+\beta_2} \oplus g^{\beta 7} \oplus g^{\beta_1+\beta_7} \oplus g^{\beta_9} \oplus g^{\beta_1+\beta_9} =$ $= Ce_1 \oplus Ce_2 \oplus Ce_3 \oplus Ce_4 \oplus Ce_5\oplus$ $\oplus Ce_6 \oplus Ce_7 \oplus Ce_8 \oplus Ce_9 \oplus Ce_{10}$ |
| $g_{10,4}$ | $10,6,$ $3$ | 3 | $e_1,e_2,$ $e_5,e_8$ | $g = g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2} \oplus g^{\beta_5}\oplus$ $\oplus g^{\beta_1+\beta 5} \oplus g^{2\beta_1+\beta 5} \oplus g^{\beta_8} \oplus g^{\beta_1+\beta_8} \oplus g^{2\beta_1+\beta_8} =$ $= Ce_1 \oplus Ce_2 \oplus Ce_3 \oplus Ce_4 \oplus Ce_5\oplus$ $\oplus Ce_6 \oplus Ce_7 \oplus Ce_8 \oplus Ce_9 \oplus Ce_{10}$ |
| $g_{10,5}$ | $10,7,$ $5,3,$ $1$ | 2 | $e_1,e_2,$ $e_5,e_8$ | $g = g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2} \oplus g^{3\beta_1+\beta_2}\oplus$ $\oplus g^{4\beta_1+\beta_2} \oplus g^{\beta_7} \oplus g^{\beta_1+\beta_7} \oplus g^{2\beta_1+\beta_7} \oplus g^{3\beta_1+\beta_7} =$ $= Ce_1 \oplus Ce_2 \oplus Ce_3 \oplus Ce_4 \oplus Ce_5\oplus$ $\oplus Ce_6 \oplus Ce_7 \oplus Ce_8 \oplus Ce_9 \oplus Ce_{10}$ |
| $g_{10,6}$ | $10,7,$ $5,3,$ $2,1$ | 2 | $e_1,e_2,$ $e_8$ | $g = g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2} \oplus g^{3\beta_1+\beta_2}\oplus$ $\oplus g^{4\beta_1+\beta_2} \oplus g^{5\beta_1+\beta_2} \oplus g^{\beta_8} \oplus g^{\beta_1+\beta_8} \oplus g^{2\beta_1+\beta_8} =$ $= Ce_1 \oplus Ce_2 \oplus Ce_3 \oplus Ce_4 \oplus Ce_5\oplus$ $\oplus Ce_6 \oplus Ce_7 \oplus Ce_8 \oplus Ce_9 \oplus Ce_{10}$ |
| $g_{10,7}$ | $10,7,$ $5,4,$ $3,2,$ $1$ | 2 | $e_1,e_2,$ $e_9$ | $g = g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2} \oplus g^{3\beta_1+\beta_2}\oplus$ $\oplus g^{4\beta_1+\beta_2} \oplus g^{5\beta_1+\beta_2} \oplus g^{6\beta_1+\beta_2} \oplus g^{\beta_9} \oplus g^{\beta_1+\beta_9} =$ $= Ce_1 \oplus Ce_2 \oplus Ce_3 \oplus Ce_4 \oplus Ce_5\oplus$ $\oplus Ce_6 \oplus Ce_7 \oplus Ce_8 \oplus Ce_9 \oplus Ce_{10}$ |
| $g_{10,8}$ | $10,8,$ $7,6,$ $5,4,$ $3,2,$ $1$ | 1 | $e_1,e_2$ | $g = g^{\beta_1} \oplus g^{\beta_2} \oplus g^{\beta_1+\beta_2} \oplus g^{2\beta_1+\beta_2} \oplus g^{3\beta_1+\beta_2}\oplus$ $\oplus g^{4\beta_1+\beta_2} \oplus g^{5\beta_1+\beta_2} \oplus g^{6\beta_1+\beta_2} \oplus g^{7\beta_1+\beta_2}\oplus$ $\oplus g^{8\beta_1+\beta_2} =$ $= Ce_1 \oplus Ce_2 \oplus Ce_3 \oplus Ce_4 \oplus Ce_5\oplus$ $\oplus Ce_6 \oplus Ce_7 \oplus Ce_8 \oplus Ce_9 \oplus Ce_{10}$ |

<table>
<tr><th>root system</th><th>Cartan matrix</th><th>Dynkin diagram</th></tr>
<tr><td></td><td>$$\begin{bmatrix} 2 & -2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{bmatrix}$$</td><td></td></tr>
<tr><td></td><td>$$\begin{bmatrix} 2 & -3 & -2 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}$$</td><td></td></tr>
<tr><td></td><td>$$\begin{bmatrix} 2 & -4 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}$$</td><td></td></tr>
<tr><td></td><td>$$\begin{bmatrix} 2 & -2 & -2 & -2 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}$$</td><td></td></tr>
</table>

| **root system** | **Cartan matrix** | **Dynkin diagram** |
|---|---|---|
|  | $\begin{bmatrix} 2 & -4 & -3 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ |  |
|  | $\begin{bmatrix} 2 & -5 & -2 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ |  |
|  | $\begin{bmatrix} 2 & -6 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ |  |
|  | $\begin{bmatrix} 2 & -8 \\ -1 & 2 \end{bmatrix}$ |  |

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