SOME REMARKS ON THE CHERNOFF'S THEOREM

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Abstract

We discuss in detail the Chernoff's proof of the Van Hove theorem.

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1 Introduction

Recently, in the paper [1], P. Chernoff has given a new and elegant proof of the following result: the geometric prequantization of an exact noncompact manifold gives rise to an irreducible representation of the space of physical observables $C^{\infty}(M, I\!R)$ to the Hilbert space $L^2(M, I\!R)$. This is an extension of an old result dues to Van Hove [2], namely the case $M = I\!R^{2n}$ with its canonical symplectic structure. The goal of our paper is to make this proof accesibile to physicists. We shall concentrate to the simplest case of Van Hove theorem, namely the case $M = I\!R^2$ with its canonical symplectic structure($\omega = dp \wedge dq$ and we shall explicitely verify that all conditions of the Chernoff's theorem are automatically satisfied.

2 Chernoff's theorem

Let M be a smooth, n-dimensional and paracompact manifold and (a smooth measure on M.

Definition 1 An *action* A of the Lie algebra $\mathcal{G}$ on M is an homomorphism $A : \mathcal{G} \rightarrow \chi(M)$, where $\chi(M)$ is the Lie algebra of smooth vector fields on M. We shall say that the action A is:
(*i*) *complete*, if $A(\chi)$ has a global flow for each $\chi \in \mathcal{G}$;

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(ii) *irreducible*, if the family

$$U_t = \exp(tA(\chi))$$

given by:

$$(U_t(f))(x) = f((\phi_t(x)), \tag{2.1}$$

where $\{\phi_t\}_{t\in I\!R}$ is the flow of $A(X)$ acts irreducible on $L^2(M, \mu)$, so it has no invariant subspaces except $\{0\}$;

(iii) *doubly transitive*, provided that for all pairs of points $x, y \in M$, $x \neq y$ the tangent space

$$T_{(x,y)}(M \times M) = \{(A(X))_x \oplus ((A(X))_y \,|X \in \mathcal{G}\}$$

or, in other words, if x and y are distinct points of M, and $v \in T_xM$, $w \in T_yM$ then there is an $X \in \mathcal{G}$ such that:

$$(A(X))_x = v$$

and

$$(A(X))_y = w.$$

Let A be an action of $\mathcal{G}$ on M and let us define a representation of $\mathcal{G}$ on $L^2(M, I\!R)$ by:

$$B(X) \overset{def}{=} A(X) + \delta((A(X)) + \rho(X), \tag{2.2}$$

where $\delta((A(X))$ is a function on M called the divergence of $A(X)$ with respect to a measure μ on M, i.e.,

$$L_{A(X)}\mu = \delta(A(X)).\mu,$$

and ρ is a linear mapping from $\mathcal{G}$ to $C^\infty(M, I\!R)$ satisfying the cocycle condition, namely

$$\rho([X, Y]) = A(X)\rho(Y) - A(Y)\rho(X) \tag{2.3}$$

for each $X, Y \in \mathcal{G}$.

For each $x \in M$ we define the stabilizer algebra at x, by the relation:

$$\mathcal{G}_x = \{X \in \mathcal{G} \,|(A(X))_x = 0\}\ .$$

It is immediately that $\mathcal{G}_x$ is a Lie subalgebra of $\mathcal{G}$. For $\in \mathcal{G}_x$ we define:

$$\rho_x(X) = \rho(X)(x).$$

Then

$$\rho_x : \mathcal{G} \to\ C$$

is a character of $\mathcal{G}_x$, this means that ρ_x is linear and

$$\rho_x([X, Y]) = 0,$$

for all $X, Y \in \mathcal{G}_x$.

Now, the Chernoff's theorem can be formulated as follows:

Theorem 1 *([1]) Let M be a smooth, n-dimensional, connected, paracompact manifold with a smooth measure μ,*

$$A : \mathcal{G} \rightarrow \ \chi(M)$$

a complete and doubly transitive action of the Lie algebra $\mathcal{G}$ on M, and ρ a cocycle for the action A satisfying the following condition:

(i) There are two point x and y of M such that ρ_x and ρ_y restrict to $\mathcal{G}_x \cap \mathcal{G}_y$ give rise to distinct characters of $\mathcal{G}_x \cap \mathcal{G}_y$.

Then the representation B given by (2.2) is irreducible on $L^2(M,\mu)$.

3 Van Hove's theorem

Let $I\!R^2(\approx T^* I\!R)$ be the Euclidian plane with its canonical symplectic structure $\omega = dp \wedge dq$. Then the Dirac problem can be formulated as follows:

Dirac's problem: *Quantize the infinite dimensional Lie algebra $C^\infty(I\!R^2, I\!R)$, i.e, find a mapping:*

$$\delta : C^\infty(I\!R^2, I\!R) \rightarrow \mathcal{O}(\mathcal{H})$$

from $C^\infty(I\!R^2, I\!R)$ to the self-adjoint operators, possibly unbounded, on some Hilbert space $\mathcal{H}$ with the following properties:

$(D_1) f \rightarrow \delta_f$,*is linear;*

$(D_2) \delta_{\{f,g\}} \rightarrow i\hbar[\delta_f, \delta_g]$;

$(D_3) \delta_{1_{I\!R^2}} = Id\ \mathcal{H}$.

In 1951 Van Hove has proved the following result:

Theorem 2 *([2]) Dirac's problem has a positive answer with:*

$$\begin{aligned} \delta_f &= -i\hbar \left(\frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial p}\frac{\partial}{\partial p} \right) - p\frac{\partial f}{\partial p} + f; \\ \mathcal{H} &= L^2(I\!R^2, \mathbb{C}). \end{aligned}$$

Moreover, the representation of $C^\infty(I\!R^2, I\!R)$ on $\mathcal{O}(L^2(I\!R^2, \mathbb{C}))$ is an irreducible one.

4 Chernoff's proof of Van Hove's theorem

Now we shall present in detail the Chernoff's proof of the Theorem 2. The proof will be a consequence of some preliminary results.

For beginig let us define the following differential operators:

$(i) A : C^\infty(I\!R^2, I\!R) \rightarrow \chi(I\!R^2)$,

$$A(f) \overset{def}{=} -i\hbar \left(\frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p} \right);$$

$(ii)\phi : C^\infty(\mathbb{R}^2, \mathbb{R}) \to C^\infty(\mathbb{R}^2, \mathbb{R}),$

$$\phi(f) \overset{def}{=} -p\frac{\partial f}{\partial p} + f;$$

$(iii)\, B : C^\infty(\mathbb{R}^2, \mathbb{R}) \to \mathcal{O}(L^2(\mathbb{R}^2, \mathbb{C})),$

$$B(f) \overset{def}{=} \delta_f = A(f) + \phi(f).$$

Proposition 3 *ϕ is a 1 - cocycle, i.e,*

$$\phi(\{f, g\}) = \{\phi(f), g\} + \{f, \phi(g)\},$$

for each $f, g \in C^\infty(\mathbb{R}^2, \mathbb{R})$, where

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial}{\partial q}.$$

Proof. We have succsively:

$$\begin{aligned}
\phi(\{f,g\}) &= \{f,g\} - p\frac{\partial}{\partial p}\{f,g\} = \\
&= \{f,g\} - p\frac{\partial}{\partial p}\left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right) = \\
&= \{f,g\} - p\left(\frac{\partial^2 f}{\partial p\partial q}\frac{\partial f}{\partial q}\frac{\partial^2 g}{\partial p^2} - \frac{\partial^2 f}{\partial p^2}\frac{\partial g}{\partial q} + \frac{\partial f}{\partial p}\frac{\partial^2 g}{\partial p\partial q}\right);
\end{aligned}$$

$$\begin{aligned}
\{\phi(f),g\}+\{f,\phi(g)\} &= \left\{f - p\frac{\partial f}{\partial p}, g\right\} + \left\{f, g - p\frac{\partial g}{\partial p}\right\} = \\
&= \{f,g\} - \left\{p\frac{\partial f}{\partial p}, g\right\} + \{f,g\} - \left\{f, p\frac{\partial g}{\partial p}\right\} =
\end{aligned}$$

$$= 2\{f,g\} - p\left\{\frac{\partial f}{\partial p}, g\right\} - \frac{\partial f}{\partial p}\{p,g\} - p\left\{f, \frac{\partial g}{\partial p}\right\} - \frac{\partial g}{\partial p}\{f,p\} =$$

$$= 2\{f,g\} - p\left\{\frac{\partial^2 f}{\partial p\partial q}\frac{\partial g}{\partial p} - \frac{\partial^2 f}{\partial p^2}\frac{\partial g}{\partial q}\right\} +$$

$$+\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - p\left(\frac{\partial f}{\partial q}\frac{\partial^2 q}{\partial p^2} - \frac{\partial f}{\partial p}\frac{\partial^2 g}{\partial p\partial q}\right) - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} =$$

$$= \{f,g\} - p\left(\frac{\partial^2 f}{\partial p\partial q}\frac{\partial g}{\partial p} - \frac{\partial^2 f}{\partial p^2}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial q}\frac{\partial^2 g}{\partial p^2} + \frac{\partial f}{\partial p}\frac{\partial^2 g}{\partial p\partial q}\right)$$

Making now the identifications we obtain the desired result.

Proposition 4 *For each$f \in C^{\infty}(I\!R^2, I\!R)$ we have:*

$$\delta(A(f)) = 0.$$

Proof. Indeed,

$$\delta(A(f)) = \delta(X_f) \overset{def}{=} div_{\mu}(X_f),$$

where

$$\mu = \frac{\omega^n}{n!}.$$

But

$$L_{X_f}\mu = \delta(X_f)\mu$$

and

$$L_{X_f}\omega^n = L_{X_f}(\omega \wedge \ldots \wedge \omega) = (L_{X_f}\omega) \wedge \omega \wedge \ldots \wedge \omega + \ldots + \omega \wedge \ldots \wedge \omega \wedge (L_{X_f}\omega).$$

Since

$$L_{X_f}\omega = di_{X_f}\omega + i_{X_f}d\omega = d(-df) + 0 = 0$$

It follows that:

$$\delta(A(f)) = 0.$$

Proposition 5 *The action*

$$f \to X_f$$

is double transitive.

Proof. This is a simple consequence of the fact that, given two distinct points $x, y \in I\!R^2$ and arbitrary tangent vectors $v_1 \in T_x I\!R^2$, $v_2 \in T_y I\!R^2$, there is a function $f \in C^{\infty}(I\!R^2, I\!R)$ such that:

$$X_f(x) = v_1$$

and

$$X_f(y) = v_2.$$

The function f can be constructed by patching together local functions with a partition of unity.

Proposition 6 *If x and y are distinct points in $I\!R^2$, there exists a function $f \in C^{\infty}(I\!R^2, I\!R)$ such that:*
$(i)(X_f)_x = 0$ and $(X_f)_y = 0$;
(ii) $\phi(f)(x) \neq \phi(f)(y)$.

Proof. Let us observe that condition (i) is the same as requiring that:

$$d_f = 0$$

at x and y, and this is true if f is constant in a neighborhood of each of these points. Moreover, from the relation:

$$\phi(f) = f - p\,dq(X_f)$$

it follows that, since

$$(X_f)_x = 0,$$

then

$$(pdq)(X_f)(x) = (pdq)(X_f)(y) = 0$$

and so

$$\phi(f)(x) = f(x).$$

Similarly

$$\phi(f)(y) = f(y).$$

Hence, we may take any $f \in C^{\infty}(I\!R^2, I\!R)$ such that:

$$f(x) \neq f(y)$$

and f is constant near x and y respectively.

Proof (of the Theorem 2, [1]) Using the above propositions it is easy to see that all conditions of Theorem 2 are satisfied and so we obtain the desired result.

References

[1] Chernoff P.R., *Irreducible representations of infinite-dimensional transformation groups and Lie algebra*, I, Journal of Functional Analysis, vol. 130, no2 (1995), 255-282.

[2] Van Hove L., *Sur certains representations unitaires d'une groupe infini de transformations,* Acad. Roy. Belg. Cl. Sci. Mem. Collect. 8 (2), 29 (1951), 1-102.

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