NEW LIFTS OF SASAKI TYPE OF THE RIEMANNIAN METRICS

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Abstract

One defines new elliptic and hyperbolic lifts to tangent bundle TM of a Riemann metric g given on the base manifold M. They are homogeneous on the fibres of the bundle (TM, π, M). This property allows to study the Riemannian spaces determined by the manifold TM and one of these lifts. These new lifts of the Riemannian metric are useful in geometrical models of gauge theories.

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1 Introduction

The Sasaki lift $\overset{o}{G}$ to tangent bundle TM of a Riemannian metric g defined on the differentiable manifold M determines a nonhomogeneous Riemannian metric on TM. This fact is not convenient for study of global geometric properties of the space $(TM, \overset{o}{G})$. (See, for instance, a theorem of Gauss–Bonnet type, [2,3].)

Recently, [6,7] the author introduced an homogeneous lift $\mathbb{G}$ to TM of g, which together with the natural almost complex structure $\mathbb{F}$ give rise to a conformal Kählerian structure on TM.

In the following we introduce an homogeneous hyperbolic lift $\check{\mathbb{G}}$ to TM of g. There exists a natural almost product structure $\mathbb{P}$ such that the pair $(\check{\mathbb{G}}, \mathbb{P})$ is a conformal almost parakählerian structure.

Some new geometrical properties of these structures are pointed out.

The homogeneous lifts of the Riemannian structure are important for the geometrical models of the gauge fields theory in theoretical physics.

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2 Homogeneous elliptic lift

Let M be a real n–dimensional smooth manifold and (TM, π, M) its tangent bundle. A point $u \in TM$ has the canonical coordinates $(x^i, y^i),\ (i = 1, .., n)$ and it is denoted by $u = (x, y),\ x = \pi(u),\ y \in T_xM$.

Assuming that g is a Riemannian metric on the manifold M, expressed in local coordinates by $g_{ij}(x)$, we consider "the length" of Liouville vector field $y^i \dfrac{\partial}{\partial y^i}$ (the Einstein convention of summarising one applies), defined by:

$$||y|| = \sqrt{g_{ij}(x)y^i y^j}. \tag{1.1}$$

On the manifold $\widetilde{TM} = TM \setminus \{0\}$, of the nonvanish vectors, we have a nonlinear connection N with the property

$$T_u\widetilde{TM} = N(u) \oplus V(u), \quad \forall u \in \widetilde{TM}, \tag{1.2}$$

$V(u)$ being the vertical linear space on TM and N having the local coefficients $N^i_j(x, y) = \gamma^i_{jh}(x)y^h$. So, the nonlinear connection N is determined only by the metric g since $\gamma^i_{jh}(x)$ are the Christoffel symbols of g.

Then, an adapted basis of the direct decomposition (1.2), [1,4], is

$$\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x, y)\frac{\partial}{\partial y^j}, \quad \frac{\partial}{\partial y^i}\}, \quad (i = 1, \dots, n). \tag{1.3}$$

The dual basis is

$$\{dx^i, \quad \delta y^i = dy^i + N^i_j(x, y)dx^j\}. \tag*{(1.3)$'$}$$

Definition 1 The *homogeneous elliptic lift* of the Riemann metric g is:

$$\mathbb{G} = g_{ij}(x)dx^i \otimes dx^j + \frac{1}{||y||^2}g_{ij}(x)\delta y^i \otimes \delta y^j. \tag{1.4}$$

In the paper [6], we proved:

Theorem 1 *1. $\mathbb{G}$ is a Riemannian metric on $\widetilde{TM}$ depending only on g.*

2. $\mathbb{G}$ is homogeneous of 0-degree on the fibres of TM.

3. The distribution N and V are orthogonal with respect to $\mathbb{G}$.

Clearly, on the sphere bundle, with the fiberes $g_{ij}(x)y^i y^j = 1$, the homogeneous lift $\mathbb{G}$ coincides to the classical Sasaki lift [2,4].

Let $I\!F : \chi(\widetilde{TM}) \longrightarrow \chi(\widetilde{TM})$ be the $\mathcal{F}(\widetilde{TM})$-linear mapping defined by:

$$I\!F(\frac{\delta}{\delta x^i}) = -||y||\frac{\partial}{\partial y^i}; \quad I\!F(\frac{\partial}{\partial y^i}) = \frac{1}{||y||}\frac{\delta}{\delta x^i}, \quad (i = 1, \dots, n).$$

The following theorems hold, [6]:

Theorem 2 *1. $\mathbb{F}$ is an almost complex structure on the manifold $\widetilde{TM}$ depending only on the Riemannian metric g.*

2. $\mathbb{F}$ preserve the property of homogeneity of the vector fields on $\widetilde{TM}$.

And, finally:

Theorem 3 *1. The pair $(\mathbb{G}, \mathbb{F})$ is an almost Kählerian structure on the manifold $\widetilde{TM}$ depending only on the Riemannian metric g.*

2. $(\mathbb{G}, \mathbb{F})$ is a conformal almost Kählerian structure.

Of course, we can determine the linear connections on $\widetilde{TM}$ which are compatible to the structure $(\mathbb{G}, \mathbb{F})$.

3 Homogeneous hyperbolic lift

We investigate the previous problems for a hyperbolic lift determinated by a Riemannian metric g, defined on the base manifold M.

Let us consider again the canonical nonlinear connection N on $\widetilde{TM}$, determinated by g. The adapted basis to the direct decomposition (1.2) is, of course, (1.3) and dual basis is (1.3).

Definition 2 The *hyperbolic lift* to TM of the Riemannian metric g is given by:

$$\check{\mathbb{G}} = g_{ij}(x)dx^i \otimes dx^j - \frac{1}{||y||^2} g_{ij}(x)\delta y^i \otimes \delta y^j. \tag{2.1}$$

Some important results:

Theorem 4 *1. $\check{\mathbb{G}}$ is a pseudo-Riemannian metric on $\widetilde{TM}$ depending only on the Riemannian structure g.*

2. $\check{\mathbb{G}}$ is 0–homogeneous on the fibres of tangent bundle TM.

3. The distribution N and V are orthogonal with respect to $\check{\mathbb{G}}$.

Proof. 1. Since g_{ij} is a covariant tensor and $(dx^i), (\delta y^i)$ are vector fields, it follows that $\check{\mathbb{G}}$ is globally defined on $\widetilde{TM}$. Since N depend only on g it follows from (2.1) that $\check{\mathbb{G}}$ has the same property.

2. Every of two terms of $\check{\mathbb{G}}$ is 0–homogeneous with respect to y^i.

$\check{\mathbb{G}}(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = 0, \quad (i, j = 1, \ldots, n)$ implies 3°. □

Corollary 3.1 *If the Riemannian metric g is given on a paracompact manifold M, then on $\widetilde{TM}$ the pseudo-Riemannian structure $\check{\mathbb{G}}$, (2.1), exists.*

Indeed, it follows that $\widetilde{TM}$ is paracompact and $\check{\mathbb{G}}$ is globally defined on $\widetilde{TM}$.

Proposition 5 *1. The isotropic con of the space $(\widetilde{TM},\check{\mathbb{G}})$ in every point $u \in \widetilde{TM}$, is:*

$$g_{ij}X^iX^j - \frac{1}{||y||^2}g_{ij}\dot{X}^i\dot{X}^j = 0. \tag{2.2}$$

2. The distributions generated by the system of linear independent vector fields

$$\frac{\delta}{\delta x^i} + ||y||\frac{\partial}{\partial y^i} \quad and \quad \frac{\delta}{\delta x^i} - ||y||\frac{\partial}{\partial y^i}, \quad (i = 1, \dots, n)$$

are isotropic.

Let us determine the Levi–Civita connection D of the metric tensor $\check{\mathbb{G}}$. So, $\check{\mathbb{G}}$ must satisfies the following equations:

$$\begin{array}{l} (D_X\check{\mathbb{G}})(Y,Z) = X\check{\mathbb{G}}(Y,Z) - \check{\mathbb{G}}(D_XY,Z) - \check{\mathbb{G}}(Y,D_XZ) = 0 \\ T(X,Y) = D_XY - D_YX - [X,Y] = 0, \ \forall X,Y,Z \in \chi(\widetilde{TM}). \end{array} \tag{2.3}$$

The previous equations, uniquely determine the Levi–Civita connection. Therefore it is sufficiently to determine its coefficients in the adapted basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$, given by

$$\begin{array}{ll} D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j} = \overset{H}{L^i_{jk}}\frac{\delta}{\delta x^i} + \widetilde{L^i_{jk}}\frac{\partial}{\partial y^i}, & D_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial y^j} = \widetilde{\widetilde{L^i_{jk}}}\frac{\delta}{\delta x^i} + \overset{V}{L^i_{jk}}\frac{\partial}{\partial y^i}, \\ D_{\frac{\partial}{\partial y^k}}\frac{\delta}{\delta x^j} = \overset{H}{C^i_{jk}}\frac{\delta}{\delta x^i} + \widetilde{C^i_{jk}}\frac{\partial}{\partial y^i}, & D_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^j} = \widetilde{\widetilde{C^i_{jk}}}\frac{\delta}{\delta x^i} + \overset{V}{C^i_{jk}}\frac{\partial}{\partial y^i}. \end{array} \tag{2.4}$$

The following theorem can be proved by means of (2.3) and (2.4):

Theorem 6 *In the adapted basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$, the Levi–Civita connection D of the pseudo-Riemannian metric $\check{\mathbb{G}}$ has the coefficients:*

$$\begin{array}{lll} \overset{H}{L^i_{jk}} = \overset{V}{L^i_{jk}} = \gamma^i_{jk}(x), & \widetilde{L^i_{jk}} = -\frac{1}{2}R^i_{jk}, & \widetilde{\widetilde{L^i_{jk}}} = -\frac{1}{2||y||^2}g^{is}g_{mj}R^m_{sk}, \\ \overset{H}{C^i_{jk}} = -\frac{1}{2||y||^2}g^{is}g_{mk}R^m_{sj}, & \widetilde{C^i_{jk}} = \widetilde{\widetilde{C^i_{jk}}} = 0, & \overset{V}{C^i_{jk}} = -\frac{1}{||y||^2}(\delta^i_j y_k + \delta^i_k y_j - g_{jk}y^i), \end{array} \tag{2.5}$$

where $y_i = g_{ij}(x)y^j$, and $R^i_{jk} = y^m\rho_{m\ jk}^{\ i}(x)$, $\rho_{m\ jk}^{\ i}(x)$ being the curvature tensor of the Riemann metric $g_{ij}(x)$.

4 The almost parahermitian structure $(\check{\mathbb{G}}, \check{\mathbb{P}})$

To the hyperbolic metric $\check{\mathbb{G}}$, (2.1), we can associate an almost product structure, $\check{\mathbb{P}} : \chi(\widetilde{TM}) \longrightarrow \chi(\widetilde{TM})$, defined by:

$$\check{\mathbb{P}}(\frac{\delta}{\delta x^i}) = ||y||\frac{\partial}{\partial y^i}, \quad \check{\mathbb{P}}(\frac{\partial}{\partial y^i}) = \frac{1}{||y||}\frac{\delta}{\delta x^i}. \tag{3.1}$$

We can prove, without difficulties:

Theorem 7 *1. $\check{\mathbb{P}}$ is a tensor field of type $(1,1)$ on $\widetilde{TM}$.*

2. $\check{\mathbb{P}}$ is an almost product structure on $\widetilde{TM}$, i.e.

$$\check{\mathbb{P}} \circ \check{\mathbb{P}} = I \tag{3.2}$$

3. $\check{\mathbb{P}}$ depend only by the Riemannian metric g.

4. The mapping $\check{\mathbb{P}} : \chi(\widetilde{TM}) \longrightarrow \chi(\widetilde{TM})$ preserve the property of homogeneity of vectors fields on $\widetilde{TM}$.

Let us consider the distributions W_1 and W_2, local generated by the vector fields $\frac{\delta}{\delta x^i} + ||y||\frac{\partial}{\partial y^i}$ and $\frac{\delta}{\delta x^i} - ||y||\frac{\partial}{\partial y^i}$, $(i = 1, .., n)$, respectively.

Evidently, $dimW_1 = dimW_2 = n$, $rank||\check{\mathbb{P}}|| = 2n$ and the eigenvalues of $\check{\mathbb{P}}$ are +1, -1.

Since we have
$\check{\mathbb{P}}(\frac{\delta}{\delta x^i} + ||y||\frac{\partial}{\partial y^i}) = \frac{\delta}{\delta x^i} + ||y||\frac{\partial}{\partial y^i}$; $\check{\mathbb{P}}(\frac{\delta}{\delta x^i} - ||y||\frac{\partial}{\partial y^i}) = -(\frac{\delta}{\delta x^i} - ||y||\frac{\partial}{\partial y^i})$
and remarking Proposition 2.1, we get:

Theorem 8 *1. In every point $u \in \widetilde{TM}$, the linear eigenspace of the operator $\check{\mathbb{P}}$, corresponding to the eigenvalue +1 is W_1 and the eigenspace corresponding to the eigenvalue -1 is W_2.*

2. W_1 and W_2 are the isotropic distribution of the hyperbolic metric $\check{\mathbb{G}}$.

Now, let us consider the Nijenhuis tensor of the almost product structure $\check{\mathbb{P}}$:

$$\mathcal{N}_{\check{\mathbb{P}}}(X,Y) = \check{\mathbb{P}}^2[X,Y] + [\check{\mathbb{P}}X, \check{\mathbb{P}}Y] - \check{\mathbb{P}}[\check{\mathbb{P}}X, Y] - \check{\mathbb{P}}[X, \check{\mathbb{P}}Y].$$

$\mathcal{N}_{\check{\mathbb{P}}}$ vanishes if and only if the almost product structure $\check{\mathbb{P}}$ is integrable. So, we get:

Theorem 9 *The almost product structure $\check{\mathbb{P}}$ is integrable if and only if the Riemannian structure g on the manifold M is of constant sectional curvature.*

Now, taking into account that the pair of structures $\check{\mathbb{G}}, \check{\mathbb{P}}$ satisfy the following condition of compatibility:

$$\check{\mathbb{G}}(\check{\mathbb{P}}X, \check{\mathbb{P}}Y) = -\check{\mathbb{G}}(X, Y), \quad \forall X, Y \in \chi(\widetilde{TM}) \tag{3.4}$$

it follows:

Theorem 10 *The pair $(\check{\mathbb{G}}, \check{\mathbb{P}})$ is a Riemannian almost parahermitian structure on the manifold $\widetilde{TM}$, depending only by the Riemannian metric g on the base manifold M.*

Let us consider the 2-form $\check{\theta}$ associated to the pair $(\check{\mathbb{G}}, \check{\mathbb{P}})$:

$$\check{\theta}(X, Y) = \check{\mathbb{G}}(\check{\mathbb{P}}X, Y), \quad \forall X, Y \in \chi(\widetilde{TM}). \tag{3.5}$$

In the adapted basis, $\check{\theta}$ is given by:

$$\check{\theta} = -\frac{1}{||y||} g_{ij} \delta y^i \wedge dx^j. \tag{3.5$'$}$$

But the 2-form $\overset{o}{\theta}$:

$$\overset{o}{\theta} = g_{ij} \delta y^i \wedge dx^j \tag{3.6}$$

is closed [4,6]. So, $\check{\theta} = -\frac{1}{||y||} \overset{o}{\theta}$ is conformal with $\overset{o}{\theta}$.

It follows:

Theorem 11 *The space $\check{K}^{2n} = (\widetilde{TM}, \check{\mathbb{G}}, \check{\mathbb{P}})$ is conformal with an almost parakählerian space and depend only on the Riemannian structure g on the manifold M.*

$\check{K}^{2n}$ is the hyperbolic model on $\widetilde{TM}$ of the Riemannian space $\mathcal{R}^n = (M, g)$.

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