ON NON-COMMUTATIVE MODULES WITH DIFFERENTIALS

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Abstract

The aim of the paper is to point out some aspects of modules with differentials in the non-commutative case. The categories of algebras, of anchored modules, preinfinitesimal modules and pseudoalgebras are defined.

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1 Anchored and preinfinitesimal modules over non-associative algebras

Let $\mathcal{A}$ be an algebra, non necessarily associative, over a commutative ring $\mathbf{k}$ and $\mathcal{M}$ a left $\mathcal{A}$-module. We suppose that the center $\mathcal{Z}(\mathcal{A}) = \{a \in \mathcal{A}\colon\ ab = ba,\ (\forall) b \in \mathcal{A}\}$ is a subalgebra of $\mathcal{A}$ and $\mathcal{A}$ is a bimodule over $\mathcal{Z}(\mathcal{A})$ (these conditions follows automatically if $\mathcal{A}$ is associative). Denote by $\mathcal{D}er(\mathcal{A}) = \{D : \mathcal{A} \to \mathcal{A}\colon\ D$ is $\mathbf{k}$-linear and $D(a \cdot b) = D(a) \cdot b + a \cdot D(b),\ (\forall) a, b \in \mathcal{A}\}$ the $\mathcal{Z}(\mathcal{A})$-module of derivations.

A *covariant left morphism* (*covl-morphism*) of two modules $(\mathcal{A}, \mathcal{M})$ and $(\mathcal{A}', \mathcal{M}')$ is a couple (φ, ψ) denoted by $(\varphi, \psi) : (\mathcal{A}, \mathcal{M}) \longrightarrow (\mathcal{A}', \mathcal{M}')$ or $(\mathcal{A}, \mathcal{M}) \overset{(\varphi,\psi)}{\longrightarrow} (\mathcal{A}', \mathcal{M}')$, where $\varphi : \mathcal{A} \to \mathcal{A}'$ is an algebra morphism and $\psi : \mathcal{M} \to \mathcal{M}'$ is a left $\mathcal{A}$-module morphism.

A *contravariant left morphism of first kind* (*$conl_I$-morphism*) of two modules $(\mathcal{A}', \mathcal{M}')$ and $(\mathcal{A}, \mathcal{M})$ is a couple (φ, ψ) denoted by $(\varphi, \psi) : (\mathcal{A}', \mathcal{M}') \longleftarrow (\mathcal{A}, \mathcal{M})$ or $(\mathcal{A}', \mathcal{M}') \overset{(\varphi,\psi)}{\longleftarrow}_I (\mathcal{A}, \mathcal{M})$, where $\varphi : \mathcal{A} \to \mathcal{A}'$ is an algebra morphism and $\psi : \mathcal{M}' \to \mathcal{M}$ is a left $\mathcal{A}$-module morphism.

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A *contravariant left morphism of second kind* (*conl$_{II}$-morphism*) of two modules $(\mathcal{A}',\mathcal{M}')$ and $(\mathcal{A},\mathcal{M})$ is a couple (φ,ψ) denoted by $(\varphi,\psi):(\mathcal{A}',\mathcal{M}')\longleftarrow(\mathcal{A},\mathcal{M})$ or $(\mathcal{A}',\mathcal{M}')\overset{(\varphi,\psi)}{\longleftarrow}_{II}(\mathcal{A},\mathcal{M})$, where $\varphi:\mathcal{A}\to\mathcal{A}'$ is an algebra morphism and $\psi:\mathcal{M}'\to\mathcal{A}'\otimes_{\mathcal{Z}(\mathcal{A})}\mathcal{M}$ is a left $\mathcal{A}'$-module morphism.

Example 1 Let $(\mathcal{A}',\mathcal{M}')$ be a left module and $\varphi:\mathcal{A}\to\mathcal{A}'$ be an algebra morphism. Considering $\mathcal{M}'$ as an $\mathcal{A}$-module, then $(\varphi,id_{\mathcal{M}}):(\mathcal{A},\mathcal{M}')\to(\mathcal{A},\mathcal{M})$ is a covl-morphism.

Let $\xi=(E,\pi,M)$ and $\xi'=(E',\pi',M')$ be vector bundles and denote by $E_x=\pi^{\leftarrow}(x)$ the fibre in $x\in M$. We denote by $(f_0,f):\xi\to\xi'$ a morphism and by $(f_0,g):\xi\leftarrow\xi'$ a comorphism of vector bundles, where $f_0:M\to M'$, $f:E\to E'$ and $g:E'\to E$ are differentiable maps and for every $x\in M$, $f_x=f_{|E_x}:E_x\to E'_{f_0(x)}$ and $g_x=g_{|E'_{f_0(x)}}:E'_{f_0(x)}\to E_x$ are linear maps. It is well known that $f_0^*:\mathcal{F}(M')\to\mathcal{F}(M)$, $f_0^*(u)=u\circ f_0$, is a commutative algebra morphism and concerning sections: $f^*:\Gamma(\xi)\to\mathcal{F}(M)\otimes_{\mathcal{F}(M')}\Gamma(\xi')$ is an $\mathcal{F}(M)$-module morphism and $g_*:\Gamma(\xi')\to\Gamma(\xi)$ is an $\mathcal{F}(M')$-module morphism.

Example 2 If $\xi=(E,\pi,M)$ and $\xi'=(E',\pi',M')$ are vector bundles and $(f_0,g):\xi\leftarrow\xi'$ is a comorphism of vector bundles, then $(f_0^*,g_*):(\mathcal{F}(M),\Gamma(\xi))\leftarrow_I(\mathcal{F}(M'),\Gamma(\xi'))$ is a con$_I$-morphism of modules.

Example 3 If $\xi=(E,\pi,M)$ and $\xi'=(E',\pi',M')$ are vector bundles and $(f_0,f):\xi\to\xi'$ is a morphism of vector bundles, then $(f_0^*,f_*):(\mathcal{F}(M),\Gamma(\xi))\leftarrow_{II}(\mathcal{F}(M'),\Gamma(\xi'))$ is a con$_{II}$-morphism of modules.

A *left anchor (l-anchor)* on $\mathcal{M}$ is a left $\mathcal{Z}(\mathcal{A})$-morphism $\mathfrak{a}:\mathcal{M}\to\mathcal{D}er(\mathcal{A})$. For $X\in\mathcal{M}$ and $a\in\mathcal{A}$ we denote $\mathfrak{a}(X)(a)=[X,a]_{\mathcal{M}}$. Notice that $(i,\mathfrak{a}):(\mathcal{A},\mathcal{M})\longleftarrow_I(\mathcal{Z}(\mathcal{A}),\mathcal{D}er(\mathcal{A}))$, where $i:\mathcal{Z}(\mathcal{A})\to A$ is the inclusion, is a con$_I$-morphism of modules. A left module $(\mathcal{A},\mathcal{M})$ together with an l-anchor is an *l-anchored module.*

A *covariant morphism* (*covl-morphism*) of two l-anchored modules $(\mathcal{A},\mathcal{M})$ and $(\mathcal{A}',\mathcal{M}')$ is a covl-morphism (φ,ψ) of modules, where $\varphi:\mathcal{A}\to\mathcal{A}'$ and $\psi:\mathcal{M}\to\mathcal{M}'$ such that $[\psi(X),\varphi(a)]_{\mathcal{M}'}=\varphi([X,a]_{\mathcal{M}})$, $(\forall)X\in\mathcal{M}$ and $a\in\mathcal{A}$.

A *contravariant morphism of first kind* (*conl$_I$-morphism*) of two anchored left modules $(\mathcal{A}',\mathcal{M}')$ and $(\mathcal{A},\mathcal{M})$ is a covl$_I$-morphism (φ,ψ) of modules, where $\varphi:\mathcal{A}\to\mathcal{A}'$ and $\psi:\mathcal{M}'\to\mathcal{M}$ fulfills the condition $[\psi(X'),a]_{\mathcal{M}}=[X',\varphi(a)]_{\mathcal{M}'}$, $(\forall)X'\in\mathcal{M}'$ and $a\in\mathcal{A}$

A *contravariant morphism of second kind* (*conl$_{II}$-morphism*) of two anchored left modules $(\mathcal{A}',\mathcal{M}')$ and $(\mathcal{A},\mathcal{M})$ is a covl$_{II}$-morphism (φ,ψ) of modules, where $\varphi:\mathcal{A}\to\mathcal{A}'$ and $\psi:\mathcal{M}'\to\mathcal{A}'\otimes_{\mathcal{Z}(\mathcal{A})}\mathcal{M}$ fulfills the condition $[X',\varphi(a)]_{\mathcal{M}'}=\sum\limits_i a'_i\varphi([X_i,a]_{\mathcal{M}})$, $(\forall)X'\in\mathcal{M}'$ and $a\in\mathcal{A}$, where $\psi(X')=\sum\limits_i a'_i\otimes X_i$.

Example 4 Let us suppose that the algebra $\mathcal{A}$ has the property

$$(ab)\,c + b\,(ac) + (bc)\,a = a\,(bc) + (ba)\,c + b\,(ca)\,, \tag{AD}$$

$(\forall)a, b, c \in \mathcal{A}$. We say that an algebra $\mathcal{A}$ which fulfills the condition (AD) is an *algebra with interior derivations*. Then algebra $\mathcal{A}$ itself is a covl-anchored module according to the anchor $ad : \mathcal{A} \to \mathcal{I}nt(\mathcal{A}) \subset \mathcal{D}er(\mathcal{A})$, $u \to ad(u) = ad_u$, where $ad_u : \mathcal{A} \to \mathcal{A}$ is the interior derivation $ad_u(v) = u \cdot v - v \cdot u$. An algebra morphism $\varphi : \mathcal{A} \to \mathcal{A}'$ of two algebras with interior derivations induces a covl-morphism (φ, φ) of underlying anchored modules since $[\varphi(a), \varphi(b)]_{\mathcal{A}'} = \varphi(a) \cdot \varphi(b) - \varphi(b) \cdot \varphi(a) = \varphi(a \cdot b - b \cdot a) = \varphi([a, b]_{\mathcal{A}})$.

Notice that associative algebras, commutative algebras and Lie algebras over $\mathbf{k}$ are algebras with interior derivations.

Example 5 Let $(\mathcal{A}, \mathcal{M})$ be an anchored module which has the anchor $\mathfrak{a} : \mathcal{M} \to \mathcal{D}er(\mathcal{A})$ and consider the anchored module $(\mathcal{Z}(\mathcal{A}), \mathcal{D}er(\mathcal{Z}(\mathcal{A})))$, the inclusion $i : \mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ and the restriction $i' : \mathcal{D}er(\mathcal{A}) \to \mathcal{D}er(\mathcal{Z}(\mathcal{A}))$. Using the definition of the anchor, for every $X \in \mathcal{D}er(\mathcal{A})$ and $a \in \mathcal{Z}(\mathcal{A})$, we have $[X, i(a)]_{\mathcal{M}} = [\mathfrak{a}(X), a] = [i' \circ \mathfrak{a}(X), a]$, thus $(i' \circ \mathfrak{a}, i) : (\mathcal{A}, \mathcal{M}) \leftarrow_I (\mathcal{Z}(\mathcal{A}), \mathcal{D}er(\mathcal{Z}(\mathcal{A})))$ is a conl$_I$-anchored morphism.

Example 6 If $f : M \to M'$ is a differentiable map and τM and $\tau M'$ are the tangent bundles, then $(f, \tau f) : (M, \tau M) \to (M', \tau M')$ induces the conl$_{II}$-anchored morphism $(f^*, \tau f_*) : \mathcal{X}(M) \to \mathcal{F}(M) \otimes_{\mathcal{F}'(M)} \mathcal{X}(M')$. This example can be extended for morphisms of relative tangent spaces.

If $(\mathcal{A}, \mathcal{M})$ is an anchored module, then a *bracket* on $\mathcal{M}$ is a skew symmetric $\mathbf{k}$-bilinear map $[\cdot, \cdot]_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ which enjoys the property:

$$[X, u \cdot Y]_{\mathcal{M}} = [X, u]_{\mathcal{M}} \cdot Y + u \cdot [X, Y],$$

$(\forall)X, Y \in \mathcal{M}$, $u \in \mathcal{A}$.

A *preinfinitesimal left module* (*pl-module*) $(\mathcal{A}, \mathcal{M})$ is a left module together with an anchor and a bracket on $\mathcal{M}$. In analogous way a *preinfinitesimal right module* (*pr-module*) and a *preinfinitesimal bi-module* (*pb-module*) can be defined.

If $(\mathcal{A}', \mathcal{M}')$ and $(\mathcal{A}, \mathcal{M})$ are pl-modules then:

A *covariant morphism of preinfinitesimal left module* (*cov-morphism of pl-module*) is a covl-morphism $(\varphi, \psi)$of anchored modules, where $\varphi : \mathcal{A} \to \mathcal{A}'$ and $\psi : \mathcal{M} \to \mathcal{M}'$, such that for every $X, Y \in \mathcal{M}$ then $\psi([X, Y]_{\mathcal{M}}) = [\psi(X), \psi(Y)]$.

A *contravariant morphism of first type of preinfinitesimal left module* (*con$_I$-morphism of pl-module*) is a conl$_I$-morphism (φ, ψ) of l-anchored modules, where $\varphi : \mathcal{A} \to \mathcal{A}'$ and $\psi : \mathcal{M}' \to \mathcal{M}$ such that $[\psi(X'), \psi(Y')]_{\mathcal{M}} = \psi([X', Y']_{\mathcal{M}'})$, $(\forall)X', Y' \in \mathcal{M}'$.

A *contravariant morphism of second type of preinfinitesimal left module* (*con$_{II}$-morphism of pl-module*) is a conl$_{II}$-morphism (φ,ψ) of l-anchored modules, where $\varphi : \mathcal{A} \to \mathcal{A}'$ and $\psi : \mathcal{M}' \to \mathcal{A}' \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{M}$ fulfill the condition that for every $X', Y' \in \mathcal{M}'$ which allow the ψ-decompositions $\psi(X') = \sum\limits_i a'_i \otimes X_i$ and $\psi(Y') = \sum\limits_\alpha b'_\alpha \otimes Y_\alpha$, then one has $\psi([X',Y']_{\mathcal{M}'}) = \sum\limits_\alpha [X', b'_\alpha] \otimes Y_\alpha - \sum\limits_i [Y', a'_i] \otimes X_i + \sum\limits_{i,\alpha} a'_i b'_\alpha \otimes [X_i, Y_\alpha]$.

Let $(\mathcal{A},\mathcal{M})$ be a pl-module, $i : \mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ be the inclusion and $\mathfrak{a}$ be its anchor. The couple $(\mathcal{Z}(\mathcal{A}), \mathcal{D}er(\mathcal{A}))$ is a pl-module. If $(i,\ \mathfrak{a}) : (\mathcal{A},\mathcal{M}) \leftarrow_I (\mathcal{Z}(\mathcal{A}), \mathcal{D}er(\mathcal{A}))$ is a con$_I$-morphism of pl-module, then we say that $(\mathcal{A},\mathcal{L})$ is a *pseudoalgebra.* A pseudoalgebra $(\mathcal{A},\mathcal{L})$ is a *Lie pseudoalgebra* if the bracket on $\mathcal{L}$ fulfills the Jacobi identity

$$\mathcal{J}_{\mathcal{L}}(X,Y,Z) \overset{not.}{=} [[X,Y]_{\mathcal{L}}, Z]_{\mathcal{L}} + [[Y,Z]_{\mathcal{L}}, X]_{\mathcal{L}} + [[Z,X]_{\mathcal{L}}, Y]_{\mathcal{L}} \equiv 0.$$

Let $(\mathcal{A},\mathcal{M})$ be a pl-module. Then

$$\mathcal{D} : \mathcal{M} \times \mathcal{M} \to \mathcal{D}er(\mathcal{M}),\ \mathcal{D}(X,Y) = [D(X), D(Y)] - D\left([X,Y]_{\mathcal{M}}\right)$$

is an anchor for $\mathcal{M} \wedge \mathcal{M}$, i.e. $(\mathcal{M} \wedge \mathcal{M}, \mathcal{D})$ is an l-anchored module.

Let $(\mathcal{A},\mathcal{M})$ be a pl-module with an l-anchor $D : \mathcal{M} \to \mathcal{D}er(\mathcal{A})$ and

$$\mathcal{D} : \mathcal{M} \times \mathcal{M} \to \mathcal{D}er(\mathcal{M}),$$

$$\mathcal{D}(X,Y) = [D(X), D(Y)]_{\mathcal{D}er(\mathcal{A})} - D\left([X,Y]_{\mathcal{M}}\right).$$

Then $\mathcal{D}$ is an l-anchor for $\mathcal{M} \wedge \mathcal{M}$, i.e. $(\mathcal{M} \wedge \mathcal{M}, \mathcal{D})$ is an l-anchored module, where the exterior product is taken over $\mathcal{A}$.

A *linear connection* on the preinfintesimal module $(\mathcal{A},\mathcal{M})$ is a map $\nabla : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ which satisfies the Koszul conditions:

$$\nabla_{aX} Y = a \nabla_X Y,\ \nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z,$$

$$\nabla_X (aY) = [X,a]_{\mathcal{M}} \cdot Y + a \cdot \nabla_X Y,\ \nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z,$$

$(\forall) a \in \mathcal{A},\ X, Y \in \mathcal{M}$.

The *torsion* of a linear connection ∇ is the $\mathcal{A}$-bilinear map $T : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ defined by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. The *curvature* of ∇ is $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z$.

Notice that the curvature of a linear connection is a linear connection on the anchored module $(\mathcal{M} \wedge \mathcal{M}, \mathcal{D})$ defined above. Using a linear connection ∇ on the preinfinitesimal module $(\mathcal{A},\mathcal{M})$, the formula

$$[X \wedge Y, U \wedge V]_{\mathcal{M} \wedge \mathcal{M}} = \nabla_{X \wedge Y} (U \wedge V) - \nabla_{U \wedge V} (X \wedge Y)$$

defines a bracket on $\mathcal{M} \wedge \mathcal{M}$, thus $(\mathcal{A}, \mathcal{M} \wedge \mathcal{M})$ becomes a preinfinitesimal module; additionally, the formulas

$$[X,Y]_{\mathcal{D}'(\mathcal{M})} = [X,Y]_{\mathcal{M}} + X \wedge Y,$$

$$[X\wedge Y,Z]_{\mathcal{D}'(\mathcal{M})}=\nabla_{X\wedge Y}Z-\nabla_{Z}\left(X\wedge Y\right), \tag{1}$$
$$[X\wedge Y,Z\wedge T]_{\mathcal{D}'(\mathcal{M})}=\nabla_{X\wedge Y}\left(Z\wedge T\right)-\nabla_{Z\wedge T}\left(X\wedge Y\right),$$

define a bracket on $\mathcal{D}(M)=\mathcal{M}\oplus(\mathcal{M}\wedge\mathcal{M})$, where the anchor is given by $\mathcal{D}'(X+(Y\wedge Z))=D(X)+\mathcal{D}(Y,Z)$. We call this preinfinitesimal module $(\mathcal{A},\mathcal{D}(M))$ as the *derived* preinfinitesimal module of $(\mathcal{A},\mathcal{M})$, given by ∇.

The corresponding notions for right modules and bimodules can be defined in an analogous way and the above properties remain true, under suitable changes.

Let $\mathcal{A}$ be an algebra with interior derivations, $(\mathcal{A},\mathcal{M})$ be a module and $\alpha:\mathcal{M}\to\mathcal{A}$ be an $\mathcal{A}$-linear map. Then $\mathfrak{a}:\mathcal{M}\to\mathcal{D}er(\mathcal{A})$, $\mathfrak{a}(X)=ad_{\alpha(X)}$ is an anchor on $\mathcal{M}$, where $ad_u:\mathcal{A}\to\mathcal{A}$ is the commutator $ad_u(v)=u\cdot v-v\cdot u$. The bracket $[X,Y]_{\mathcal{M}}=\alpha(X)Y-\alpha(Y)X+\beta(X,Y)$ defines on $\mathcal{M}$ a pl-module structure, where $\beta:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ is an $\mathcal{A}$-linear and a skew symmetric map. Since $\mathcal{D}(X,Y)=-ad_{\alpha(\beta(X,Y))}$, it follows that a pl-module is a pseudoalgebra iff β takes values in $\mathcal{Z}(\mathcal{A})$.

2 Lie pseudoalgebra structure

The algebra $\mathcal{A}$ in this section is associative. Then $\mathcal{A}$ itself is a covb-anchored module according to the covb-anchor $ad:\mathcal{A}\to\mathcal{I}nt(\mathcal{A})\subset\mathcal{D}er(\mathcal{A})$, $u\to ad(u)=ad_u$.

Proposition 1 *Let us suppose that $\mathcal{A}$ has a unit* 1. *Then the only bracket on $\mathcal{A}$ of the covl-anchor ad is the commutator $(a,b)\to[a,b]=a\cdot b-b\cdot a$.*

Proof. Let as denote by $[\cdot,\cdot]$ the bracket. We must have $[a,b\cdot c]=ad_a(b)\cdot c+b\cdot[a,c]$. Taking in this relation $a=c=1$, we have $[1,b]=0$, then taking $c=1$, it follows $[a,b]=ad_a(b)+b\cdot[a,1]=ad_a(b)$. □

A Lie covb-pseudoalgebra $(\mathcal{A},\mathcal{A})$ is obtained, which we call the *canonical Lie covb-pseudoalgebra* structure on $\mathcal{A}$.

Proposition 2 *The canonical Lie covb-pseudoalgebra on $\mathcal{A}$ allows a linear connection defined by $\nabla_a b=a\cdot b$, which has a null torsion.*

If the algebra is associative then the connection ∇ is flat, i.e. it has also a null curvature.

Proof. We have $\nabla_{ab}c=(ab)c=a(bc)=a\nabla_b c$ and $\nabla_a(bc)=(ab-ba)c+b(ac)=(ad_a b)c+b\nabla_a c$, thus ∇ is a linear connection. It is obvious that $T=0$ and that the curvature of ∇ is $R(a,b)c=a(bc)-b(ac)-(ab-ba)c=0$. □

This example shows some different aspects of the non-commutative case, compared with the commutative one:

Given an anchor, it is possible that a linear connection or a bracket, if there exists, to be unique.

The equality $[aX,b]_{\mathcal{M}}=a\cdot[X,b]_{\mathcal{M}}$, for $a,b\in A$ and $X\in\mathcal{M}$ does not always hold.

3 Non-associative and associative algebras

Let $\mathcal{A}$ be an algebra, non necessarily associative, over a commutative ring $\mathbf{k}$. Consider two linear subspaces $V_1, V_2 \subset \mathcal{A}$, where $V_1 = \{\sum_{i=1}^{n} ((a_i \cdot b_i) \cdot c_i) - a_i \cdot (b_i \cdot c_i)) |\, a_i, b_i, c_i \in \mathcal{A}\}$ and $V_2 = \{\sum_{i=1}^{n} ((a_i \cdot b_i) \cdot c_i) - a_i \cdot (b_i \cdot c_i)) \cdot d_i |\, a_i, b_i, c_i, d_i \in \mathcal{A}\}$.

Proposition 3 *The linear subspace $\mathcal{J} = V_1 + V_2 \subset \mathcal{A}$ is an ideal of $\mathcal{A}$.*

Proof. We prove the condition of ideal on additive generators of V_1 and V_2. For $a, b, c \in \mathcal{A}$, denote by $\varphi(a, b, c) = (a \cdot b) \cdot c - a \cdot (b \cdot c)$.

Consider $x_1 = \varphi(a, b, c) \in V_1$, $y_1 = \varphi(a, b, c) \cdot d \in V_2$ and $e \in \mathcal{A}$. It is obvious that $x_1 \cdot e \in V_2$. We have:

$y_1 \cdot e = (\varphi(a, b, c) \cdot d) \cdot e = \varphi(\varphi(a, b, c), d, e) + \varphi(a, b, c) \cdot (d \cdot e) \in V_1 + V_2$;

$e \cdot x_1 = e \cdot ((a \cdot b) \cdot c) - e \cdot (a \cdot (b \cdot c)) = (e \cdot (a \cdot b)) \cdot c - \varphi(e, a \cdot b, c) - (e \cdot a) \cdot (b \cdot c) + \varphi(e, a, b \cdot c) = (e \cdot (a \cdot b) - (e \cdot a)) \cdot c - \varphi(e, a \cdot b, c) + \varphi(e, a, b \cdot c) - \varphi(e \cdot a, b, c) \in V_1 + V_2$;

$e \cdot y_1 = e \cdot (\varphi(a, b, c) \cdot d) = (e \cdot \varphi(a, b, c)) \cdot d - \varphi(e, \varphi(a, b, c), d) \in V_1 + V_2$, using the previous steps. □

The quotient algebra $\mathcal{A}' = \mathcal{A}/\mathcal{J}$ is an associative algebra, the canonical projection $\pi : \mathcal{A} \to \mathcal{A}'$ is an algebra morphism and $\mathcal{A}'$ is an $\mathcal{A}$-module. Let us suppose that $\mathcal{J} \subset \mathcal{Z}(\mathcal{A})$. Then every $a' = a + \mathcal{J} \in \mathcal{A}'$ defines $ad_{a'} : \mathcal{A} \to \mathcal{A}$, $ad_{a'}(b) = [a, b]$ which is an anchor on the module $(\mathcal{A}, \mathcal{A}')$. Since $\mathcal{A}'$ is an associative algebra, $(\mathcal{A}', \mathcal{A}')$ is an anchored module. If $\mathcal{A}$ has a unit, then $\psi : \mathcal{A}' \to \mathcal{A}' \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A}'$, $\psi(a') = 1 \otimes a'$ defines a contravariant morphism $(\pi, \psi) : (\mathcal{A}', \mathcal{A}') \leftarrow (\mathcal{A}, \mathcal{A}')$ of anchored modules, since if $a' = \pi(a)$, then $[a', \pi(b)] = [\pi(a), \pi(b)] = \pi([a, b]) = \pi(ad_{a'}(b))$. But $(\mathcal{A}, \mathcal{A}')$ and $(\mathcal{A}', \mathcal{A}')$ are also Lie pseudoalgebras and (π, ψ) is a Lie pseudoalgebra morphism.

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