NILPOTENT LIE ALGEBRAS OF MAXIMAL RANK AND OF TYPE F_4 AS AN ASSOCIATED GENERALIZED CARTAN MATRIX

Gr. Tsagas

Abstract

The aim of the present paper is to determined all Nilpotent Lie algebras of the maximal rank and rank F_4. The number of such algebras is 43.

Key words: Nilpotent Lie algebras Kac-Moody Algebra, Generalized Cartan Matrix

1 INTRODUCTION

Let $A = (A_{ij})$, $i, j = 1, \ldots, n$ be a Generalized Cartan Matrix denoted briefly by G.C.M. From this matrix and a given root system we can construct Nilpotent Lie algebras of maximal rank having $A = (A_{ij})$, $i, j = 1, \ldots, n$ as a G.C.M. In order to obtain these we consider the positive part $L^+(A)$ of the Kac-Moody Lie algebras $L(A)$ taken by the G.C.M., $A = (A_{ij})$ and the given root system Δ.

The aim of the present paper is to obtain all the Nilpotent Lie algebras of maximal rank whose G.C.M. is the Cartan matrix of the exceptional Lie algebras F_4. Each of them is called of type F_4. The cases A_n, B_n, C_n, D_n and G_2 have been studied in ([9]) and ([21]). The cases for E_6, E_7 and E_8 are studied in ([24]), ([25]) and ([26]).

The whole paper contains seven paragraphs each of them is analyzed as follows. The second paragraph gives the general theory of Kac- Moody Lie algebras. The basic elements and properties of Nilpotent Lie algebras are given in the fourth paragraph. The relation between Kac- Moody Lie algebras and Nilpotent Lie algebras is given in the fourth paragraph. The fifth paragraph contains estimates and constructions of Nilpotent Lie algebras of maximal rank and of type F_4. The sixth paragraph includes the determination of the ideals. The structure constants and some other properties of the fourtythree Nilpotent Lie algebra of maximal rank and of type F_4 are included in the last paragraph.

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2 Kac- Moody Lie algebra

Let $A = (A_{ij})$, $i,\ j = 1, \dots, n$, be a square matrix of order n with entries in Z satisfaying:

$$(i)\ A_{ii} = 2, i = 1, \dots, n;$$

$$(ii)\ A_{ij} \leq 0, \text{ if } i \neq j,\ i,\ j = 1, \dots, n;$$

$$(iii) \text{ if } A_{ij} = 0,\ i \neq j, \text{ then } A_{ij} = 0.$$

$A = (A_{ij})$ is called Generalized Cartan Matrix denoted briefly by G. C. M.

All through this pare the G. C. M. will be of order n .

Two G. C. M. A and D are called equivalent if there exists $\sigma \in G_n$, where G_n is the group of permutatios of $\{1, \dots, n\}$, such that:

$$B_{ij} = A_{\sigma i \sigma j}, \forall i, j = 1, ..., n.$$

We consider the Lie algebra $L(A)$, associated to the G. C. M. $A(A_{ij})$, generated by the set $\{e_1, \dots, e_n, h_1, \dots, h_n, f_1, \dots, f_n\}$ satisfying:

$$[h_i, h_j] = 0, [e_i, f_j] = \delta_{ij} h_i, i, j = 1, \dots, n, \delta_{ij} = \begin{cases} 1,\ i = j \\ 0,\ i \neq j \end{cases}$$

$$[h_i, e_j] = A_{ij} e_j, [h_i, f_j] = -A_{ij} f_j, i, j = 1, \dots, n$$

$$(ade_i)^{-A_{ij}+1} e_j = 0, (adf_i)^{-A_{ij}+1} f_j = 0, i, j = 1, \dots, n, i \neq j.$$

Let $\{a_1, \dots, a_n\}$ be the canonical base of Z^n. If $a \in N^n - \{0\}$, then $a = \sum\limits_{i=1}^{n} d_i a_i$, where $d_i \in N$ and at last one of them is different zero. We denote by L_a (resp. L_{-a}) the subroctor space of $L(A)$ generated by the elements $([e_{i_1}, \dots, e_{i_r}])$ (resp. f_i) appears d_i times and the meaning between the two brackets is the following:

$$([X_1, \dots, X_k]) = (X_1[X_2, \dots,])$$

We assume that if $a = \sum d_i a_i \in Z^n$ and all the $d_i s$ are not the same sing, then $L_a = (0)$. We denote:

$$L_0 = H = Lh_1 \oplus Kh_2 \oplus \dots \oplus Kh_n$$

The root system of $L(A)$, denoted bbbby Δ, is defined by:

$$\Delta = \{a \in Z^n \diagup a \neq 0,\ L_a \neq (0)\}$$

The Lie algebra $L(A)$, by means of $\Delta \sqcup \{0\}$, is grade, that means:

$$L(A) = \underset{a \in \Delta \sqcup \{0\}}{\oplus} L_a [L_a, L_\beta] \subset L_{a+\beta}, \forall a, \beta \in \Delta \sqcup \{0\}\,.$$

The positive root system is defined as follows:

$$\Delta_+ = \{a \in N^n \diagup a \neq 0,\ L_a \neq (0)\}$$

The negative roots, denoted by Δ, are defined by:

$$\Delta_- = -\Delta_+ = \{-a/a \in N^n, a \neq 0, L_a = (0)\}$$

It is obvious that:

$$\Delta = \Delta_- \sqcup \{0\} \sqcup \Delta_+.$$

From the above we conclude that the Lie algebra $L(A)$ can be written:

$$L(A) = L_-(A) \oplus H \oplus L_+(A)$$

where:

$$L_-(A) = \underset{a \in \Delta_+}{\oplus} L_a, L_+(A) = \underset{a \in \Delta_-}{\oplus} L_a$$

The Lie algebra $L(A)$, associated to G. C. M., generated by:

$$\{e_1, \ldots, e_n, h_1, \ldots, h_n, f_1, \ldots, f_n\}$$

and defined above, is called Kac- Moody Lie algebra.

Remark 1 *If $a = \sum d_i a_i$, then we denote by $|a| = \sum d_i$ which is called height of a. We denote by:*

$$\Delta_+^k = \{a \in \Delta_+ / |a| = k\}, \Delta_\rho = \{a \in \triangle_+ / |a| \leq p\}$$

and therefore we have $\Delta_+^n = \{a_1, \ldots, a_n\}$.

3 Nilpotent Lie algebras.

Let g be a Nilpotent Lie algebra of dimension m over the algebraically closed field K of characteristic zero. We denote by Derg and Autg the derivation Lie algebra and automorphism group of g respectively.

A torus T on g is a commutative subalgebra of Derg consisting of semi- simple endomorphisms. A torus T is called maximal, if it is not contained strictly in any other torus. A torus defines a representation in g, that means:

$$Txg \to g, (t, x) \to tx$$

From the fact that T is a commutative family of semi- simple endomorphisms and the properties of K, we conclude that the elements of T can be diagonalized simultaneously. Therefore g is decomposed into a direct sum of root spaces, that means:

$$g = \underset{\beta \in T^B}{\oplus} g^\beta$$

where T^* is the dual of the vector space T and:

$$g^\beta = \{x \in g / tx = \beta(t)x, \forall t \in T\}.$$

The root system of g associated to T, denoted by $R(T)$, is defined by:

$$R(T) = \{\beta \in T^* \diagup g^\beta \neq (0)\}.$$

From now on we assume that g is a Nilpotent Lie algebra. We suppose that T is a maximal torus on g and $\dim T = k$. Let $\{\beta_1, ..., \beta_k\}$ be a base of T^* whose dual base of T is $\{t_1, \ldots, t_n\}$, that means:

$$\beta_i(t_j) = \delta_{ij}.$$

The vectors $\{x_1, \ldots, x_k\}$ of g with the property:

$$t_i(x_j) = \delta_{ij} x_j$$

is called T minimal system of generators or briefly $T - msg$. Hence we have:

$$g^{\beta_i} = K x_i$$

and therefore $\{x_1, \ldots, x_k\}$ are root vectors for T. β_i, $i = 1, \ldots, k$, is called root of x_i, $i = 1, \ldots, k$, respectively.

We have the following propositions ([21]).

Proposition 1 *If g is a Nilpotent Lie algebra, then the following two statements are equivalent:*

(1) $\{x_1, \ldots, x_k\}$ *is a minimal system of generators;*

(2) $\{x_1 + c^2 g, \ldots, x_k + c^2 g\}$ *is a base for the vector space $g/c^2 g$, where $c^2 g = [g, g]$. The type of g is defined the dimension of $g/c^2 g$.*

Proposition 2 *Let g be a Nilpotent Lie algebra of type (1). Let T be a maximal torus on g, $\{x_1, \ldots, x_k\}$ $T - mg$ system, β_i the root of x_i. The dimension of T is equal to the rank of $\{\beta_1, \ldots, \beta_k\}$.*

Proposition 3 *Let gbe a Nilpotent Lie algebra of type s. The dimension of the maximal torus T on g is called rank of g. If k is the rank, then we have $k \leq s$.*

4 Connection between Nilpotent Lie algebras and Kac- Moody Lie algebras

Let g be a Nilpotent Lie algebra of type n which is the dimension of the Lie algebra $g \diagup c^2 g$. If the rank of g is n, then g is called maximal rank.

Let g be a Nilpotent Lie algebra. Let T be a maximal torus on g. Let $(x_1, \ldots, x_n)$ be a $T - mg$, for those elements we have:

$$(adx_i)^{-A_{ij}+1}, x_j = 0, \text{ then } A_{ij} \in Z_- \sqcup \{0\}.$$

If we put $A_{ij} = 2$, then using A_{ij} $i = 1j$ from the above we have the matrix:

$$A = (A_{ij})$$

with the properties:

$$(1) A_{ij} = 2,\ i = 1, \ldots, n$$

$$(2) A_{ij} \leq 0,\ i,\ j = 1, \ldots, n,\ i \neq j,$$

$$(3)\text{ If } A_{ij} = 0, \text{ then } A_{ij} = 0, \text{ when } i \neq j, i, j = 1, \ldots, n.$$

This is the G. C. M. associated to g.

Let $A = (A_{ij})$ be a G. C. M. we assume that the of positive roots $\triangle_+$ are given, whose number is finite, that means:

$$\triangle_+ = \left\{ a^v = \sum_{i=1}^{n} d_i^v a_i \diagup d_i^v \in N \right\},\ \{a_1, \ldots, a_n\} \text{ base of } Z^n$$

the number of roots a^v, $v = 1, ..., m$, is finite.

We denote by $L_+(A)$ the positive part of the Kac- Moody Lie algebra $L(A)$ associated to A and $\triangle_+$. Therefore $L_+(A)$ is a Lie algebra generated by $\{e_1, \ldots, e_n\}$ satisfying only the relations:

$$(1)\ (ade_i)^{-A_{ij}+1} e_j = 0,\ i \neq j,\ i = 1, ..., n$$

$$(2)\ L_+(A) \text{ is grated by: } L_+(A) = \underset{a \in \triangle_+}{\oplus} L_a,\ [L_a, L_\beta] \subset L_{a+\beta},\ \forall a, \beta \in \Delta\triangle_+$$

We refer the following propositions ([21]):

Proposition 4 Let $L_+(A)$ *be the positive part of the Kac- Moody Lie algebra* $L(A)$ *associated to G. C. M. A. Then we have:*

$$C^m L_+(A) = \underset{|\alpha| \geq m}{\oplus} L_a,$$

where $C^m L_+(A)$ *is the nth. term of central descending series.*

Proposition 5 *Let* $\triangle_+$ *be the set of positive roots of the Kac- Moody Lie algebra* $L(A)$ *associted to the G. C. M.* $A = (A_{ij})$, $i,\ j = 1, ..., n$. *Then for all* $aa \in \triangle\triangle - \{a_1, ..., a_n\}$, *there exists* $i \in \{1, ..., n\}$ *such that* $a - a_i \in \triangle_+$.

Proposition 6 *Let* $\triangle_+$ *be the set of positive roots of the Kac- Moody Lie algebra* $L(A)$ *associted to the G. C. M.* $A = (A_{ij})$, $i,\ j = 1, ..., n$. *If* $\triangle_+^p = \{a \in \triangle_+ \diagup |a| = p\} = \oslash$ *for some* $p \in N^*$, *then* $\triangle_+^{p+m} = \oslash$ *for all* $m \in N^*$.

Proposition 7 *Let* $L(A)$ *be the Kac- Moody Lie algebra associted to the G. C. M* $A = (A_{ij})$, $i,\ j = 1, \ldots, n$, *then we have:*

$$L(A) = \underset{a \in \triangle \sqcup \{0\}}{\oplus} L_a.$$

If $\{a_1, \ldots, a_n\}$ *is the natural base of* Z^1, *then we have:*

$$L_{a_i + va_i} = K(ada_i)^v a_j$$

We consider the conditions H_1 *and* H_2 *for the* $p \in N^*$.

Inorder to construct the Nilpotent Lie algebras $m_p(A)$ from the positive part $L_+(A)$ of the Kac- Moody Lie algebra $L(A)$ associated to the G. C. M. $A = (A_{ij})$, i, $j = 1, \dots, n$, then the $p \in N^*$ satisfies the inequalities:

H_1 : The number $p \le p_A$, where p_A is the height of the highest root of $L_+(A)$

$$H_2 : p \ge Sup\,\{-A_{ij} + 1/i,\, j = 1, \dots, n\}$$

Now, we can state the basic theorem ([21]).

Theorem 8 *We consider the quotient Lie algebra:*

$$m = m_p(A) = L_+(A)/C^{p+1}L_+(A),\ p > 1$$

$$\mu : L_+(A) \to m_p(A), \mu : x \to \mu(x) = \overline{x},\ \textit{where } \mu \textit{ is the canonical map.}$$

The following are valid:

(I) The restriction of μ *to the vector space* L_a*, such that* $|a| \le p$*, is an isomorphism from* L_a *into* I_a *and* $m_p(A)$ *is graded by:*

$$\{a \in \triangle_+ : |a| \le p\} : m_p(A) = \underset{|a|<p}{\oplus} I_a,\ [I_a, I_\beta] \subset I_{a+\beta}$$

(II) The Lie algebra $m_p(A)$ *is Nilpotent and its* ρ *is obtained from hypothesis* H_1*.*

(III) The set $\{\overline{e}_1, \dots, \overline{e}_n\}$ *is a minimal system of generators of* $m_p(A)$*.*

(IV) Let $t_i \in Der(m_p(A))$*,* $i = 1, \dots, n$*, be* n *derivationson* $m_p(A)$ *defined by:*

$$t_1\overline{e}_j = \delta_{ij}\overline{e}_n,\ \textit{then } T = \underset{i=1}{\oplus} Kt_1$$

is a maximal torus on $m_p(A)$ *and the Nilpotent Lie algebra* $m_p(A)$ *is of maximal rank. Furthermore* $\{\overline{e}_1, \dots, \overline{e}_n\}$ *is a* $T - msg$*.*

(V) Let $(t^{*-1}, \dots, t^{*n})$ *be the dual basis of* $(t_1, \dots, t_1)$*. If we identify* t^{*i} *and* a_i*,* $i = 1, \dots, 1$*, then the root space decomposition of* $m_p(A)$ *with respect to* T *is identical to the decomposition*

$$m_p(A) \underset{a\in\triangle_+}{\oplus} ,\ \underset{|a|\le p}{I_a} .$$

(VI) Under the hypothesis H_2*:* $A = (A_{ij})$*,* $i, j = 1, \dots, 1$*, is a G.C.M. associated to* $m_p(A)$ *and* $(\overline{e}_1, \dots, \overline{e}_n)$ *is order relative to* $A = (A_{ij})$*.*

Now, we can obtain from $L_+(A)$ the following Nilpotent Lie algebras

$$m_p(A) = L_+(A)/C^{p+1}L_+(A)$$

where $p_0 = \{Sup - A_{ij} + 1/i, j = 1, \dots, n\} \le p \le p_A$, p_A the height of the highest root.

The number of these Nilpotent Lie algebras of maximal rank of Nilpotent p and type n, that means having $A = (A_{ij})$ a G.C.M., is:

$$p_A - p_0 + 1.$$

These Nilpotency Lie algebras are the following:

$$L_+(A)/C^{p_0+1}(L_+(A)),\ L_+(A)/C^{p_0+2}(L_+(A)), \dots, L_+(A)/C^{p_A+1}L_+(A)).$$

It can be easily proved the following proposition.

Proposition 9 *Let g be a Nilpotent Lie algebra of finite dimension over an algebraically closed field K of characteristic zero. If v is an ideal of g, then the quotient Lie algebra g/v is a Nilpotent.*

Let β be an ideal of the Nilpotent Lie algebras $m_p(A)$, where p satisfies the conditions H_1 and H_2. We consider the Lie algebra:

$$g = m_p(A)/\beta \text{ and } \pi : m_p(A) \to g,$$

where π is the canonical map and the Nilpotency of gm is less than p.

We have the following propositions {[2]}.

Proposition 10 *Let β be an ideal of $m_p(A)$. From these we obtain the Nilpotent Lie algebra $g = m_p(A)/\beta$. The following statements are equivalent:*

$$(I)\ \beta \subset C^2 m_p(A)$$

$$(II)\ (\pi\overline{e}_1, \dots, \pi\overline{e}_n) \text{ is a minimal system of generators of } g.$$

Proposition 11 *Let β be an ideal of $m_p(A) = L_+(A)/C^{p+1}L_+(A)$ contained in $C^2 m_p(A)$. Let T be the maximal torus on $m_p(A)$. Then we have:*

(I) For any $t \in T$ there exists $\overline{\pi}t \in Derg$ such that:

$$\pi \circ t = \pi \circ \overline{\pi}(t) : m_p(A) \to g : \text{ Comutive diagram,}$$

where $\overline{\pi} : Der\ m_p(A) \to Der(g)$.

(II) The Nilpotent Lie algebra g is of maximal rank with $\overline{\pi}(T)$ as maximal torus and $(\pi\overline{e}_1, \dots, \pi\overline{e}_n)$ is a $\overline{\pi}(T) - msg$.

(III) If $(y_1, \dots, y_n)$ is a $T{-}msg$ of g, then there exists a unique $T{-}msg$ $(x_1, \dots, x_n)$ of $m_p(A)$ such that $\pi x_j = y_i$, $i = 1, \dots, n$. We must notice that β is called maximal ideal of $m_p(A)$, if and only if, is $T - invariant$, that is:

$$t = \beta \to t(\beta) \equiv \beta\ \forall t \in T$$

Proposition 12 *Let β be the* hom*ogeneous ideal of $m_p(A)$. Then $g = m_p(A)/\beta$ is a maximal rank and having $A = (A_{ij})$, $i, j = 1, \dots, n$, as the G.C.M. , if:*

$$(ade_i)^{-A_{ij}} e_j \notin b\ \forall i, j = 1, \dots, n \text{ and } i = 1, j$$

Proposition 13 *Let $g = m_p(A)/\beta$ be the quotient Lie algebra, where β is a maximal ideal of $m_p(A)$. Then the following statements are equivalent:*

(I) g is Nilpotency p

(II) $C^p m_p(A) \not\subseteq \beta$.

Let $A = (A_{ij})$, $i, j = 1, \dots, n$ be a G.C.M. The group

$$G_n(A) = \{\sigma \in G_n / A_{\sigma i\ \sigma j} = A_{ij},\ \forall i, j = 1, \dots, n\} \qquad ((4.1))$$

is called automorphism group of $A = (A_{ij})$.

Proposition 14 *Let $m_p(A)$ be the Nilpotent Lie algebra defined above. There exists $\overline{\sigma} \in Aut\ m_p(A)$ with the property $\overline{\sigma e_i} = \overline{e_{\sigma i}}$, $\forall i = 1, \dots, n$, if, and only if, $\sigma \in G_1(A)$.*

Now, we define:

$$\overline{G}_1(A) = \{\overline{\sigma} \in Aut\ m_p(A) /\ \overline{\sigma \varepsilon}_i = \overline{\varepsilon}_{\sigma i},\ \forall i = 1, \dots, n,\ \exists \sigma \in G_1(A)\} \qquad ((4.2))$$

We also define:

$$\begin{aligned} J = J_p(A) = \{\beta \text{ homogeneous ideas of } m_p(A)/C^p m_p(A) \subsetneq \beta, \qquad ((4.3)) \\ (ad\ \overline{e}_i)^{-A_{ij}} e_j \notin \beta,\ \forall\ i, j = 1, \dots, n\ i \neq j\} \end{aligned}$$

Proposition 15 *The set $J_p(A)$ is stable under the action of $G_1(A)$.*

Proposition 16 *Let $m_p(A)$ be the Nilpotent Lie algebra defined above. Let g be a Nilpotent Lie algebra of maximal rank, of Nilpotency p such that $A = (A_{ij})$, $i, j = 1, \dots, n$, is an associated G.C.M. Then we have:*

(I) There exists $\beta \in J$ such that $g \cong m_p(A)\beta$.

(II) If $\beta^{'} \in J$ such that $g \cong m_p(A)\beta^{'}$, then there exists $\overline{\sigma} \in \overline{G}_n(A)$ such that $\overline{\sigma}\beta = \beta^{'}$.

Theorem 17 *Let $m_p(A)/C^{p+1}L_+(A)$ be the Nilpotent Lie algebra given in theorem 4.5 $J_p(A)$ is the set of homogeneous ideals of $m_p(A)$ defined by (4.3). Then the isomorphism classes of Nilpotent Lie algebras of maximal rank, of Nilpotency p such that $A = (A_{ij})$, $i, j = 1, \dots, n$ is an associated G.C.M. , are in bijection with the orbits of $J_p(A)$ under the action of $\overline{G}_n(A)$ defined by (3.2).*

From this theorem we conclude that the construction of Nilpotent Lie algebras of maximal rank, of Nilpotency p and such that $A = (A_{ij})$, $i, j = 1, \dots, n$, is a G.C.M., is the following:

We determine all the ideals of $m_p(A)$ which are stable under the action of $\overline{G}_n(A)$. If the number of these ideals is equal to $\lambda(p)$, then we obtain $\lambda(p)$ Nilpotent Lie algebras with these properties. Since there exist:

$$p_0, p_0 + 1, \dots, p_0 + (p_A - p_0) = p_A,$$

we conclude that the number of Nilpotent Lie algebras of maximal rank and of type:

$$g \text{ is } \sum_{p=p_0}^{p_A} \lambda(p)$$

Because the determination of the ideas of $m_p(A)$, with the properties defined in (4.3), is difficult for this reason we reduce this problem to study a similar notion in $\Delta_\rho = \{a \in \Delta \in +/ \, |a| \leq p\}$.

Now, we explain this theory.

Let β be a homogeneous ideal of $m_p(A)$. Then we have:

$$\beta = \underset{a\in\Delta_\rho}{\oplus} \beta \cap I_a \beta \cap I_a \tag{4.4}$$

Since we have:

$$\beta \cap I_a = \begin{cases} (0) \\ I_a \end{cases} \tag{4.5}$$

we conclude that:

$$\beta = \underset{a\in\Delta_\pi(\beta)}{\oplus} I_a \tag{4.6}$$

where:

$$\Delta_\rho(\beta) = \{a \in \Delta / I_a \neq (0)\} \tag{4.7}$$

we have the following:

$$\text{(I) } C^p m_p(A) \notin \beta \Longleftrightarrow \Delta_+^\rho \subsetneqq \Delta_p(\beta) \tag{4.8}$$

$$\text{(II) } (ada_i)^{-A_{ij}} a_j \notin \beta \Longleftrightarrow a_j - A_{ij} a_i \in \Delta_p(\beta) \tag{4.9}$$

Let E be a subset of Δ_p. E is called ideal of Δ_p, if for all $a \in E$ and some $a_i, i = 1, \ldots, n$, such that $a + a_i \in \Delta_p$ we have $a + a_i \in E$. Therefore, β is an ideal of $m_p(A)$, if, and only if, $\Delta_p(\beta)$ is an ideal of Δ_p.

Now, we define:

$$j_p(A) = \big\{E \text{ ideal of } \Delta_p / \Delta_+^p \subsetneqq E \text{ and } a_j - A_{ij} \bullet a_i \notin E\big\} \tag{4.10}$$

From (4.3) and (4.10) we obtain the mapping:

$$\psi : J_p(A) \to j_p(A),\ \psi : \beta \to \Delta_p(\beta) \tag{4.11}$$

which is a bijection with inverse:

$$\psi^{-1} : j_p(A) \to J_p(A),\ \psi^{-1} : E \to \beta_E = \underset{a\in E}{\oplus} I_a \tag{4.12}$$

The group $G_n(A)$ operates on Δ_p by:

$$\sigma(\Sigma d_i a_i) = \Sigma d_i a_{\sigma i} \tag{4.13}$$

We define the following sets:

$$\overline{J}_p(A) = \{\text{set of orbits from the action } G_n(A) \text{ on } J_p(A)\} \tag{4.14}$$

$\overline{N}_p(A) = \{$isomorphism classes of Nilpotent Lie algebras of maximal rank, of Nilpotency p such that

$$A = (A_{ij}),\ i, j = 1, \ldots, n, \text{is an associated G.C.M.} \tag{4.15}$$

From the above we have the theorem ([2.1]).

Theorem 18 *If $A = (A_{ij})$, $i, j = 1, \ldots, n$, is a G.C.M. and if p satisfies H_1 and H_2, then the $G_n(A)$ -orbits of $\overline{J}_p(A)$ classify canonically the elements of $\overline{N}_p(A)$. More precisely, the map*

$$\Phi : \overline{N}_p(A) \to \overline{J}_p(A),\ \Phi : \overline{g} \to G_n(A).\ \Delta_p(\beta)$$

is bijection and

$$\Phi^{-1} : \overline{J}_p(A) \to \overline{N}_p(A),\ \Phi^{-1} : G_n(A).\ E \to \overline{m_p(A)}\beta_E$$

is a inverse.

Therefore in order to find the Nilpotent Lie algebras of maximal rank, of Nilpotency p such that $A = (A_{ij})$, $i, j = 1, \ldots, n$,is an associated G.C.M. we estimate the elements of $\overline{J}_p(A)$.

5 Constraction of Lie algebras by means of F_4

Now, we consider the Cartan matrix F_4 of the exceptional Lie algebras denoted also F_4. Therefore F_4 has the form

$$F_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{5.1}$$

The positive root system Δ_+ of F_4 is following:

$$\begin{aligned} \Delta_+ = {} & \{a_1, a_2, a_3, a_4, a_1 + a_2, a_2 + a_3, a_3 + a_4, a_1 + a_2 + a_3, a_2 + 2a_3, \\ & a_2 + a_3 + a_4, a_1 + a_2 2a_3, a_1 + a_2 + a_3 + a_4, a_2 + 2a_3 + a_4, \\ & a_1 + 2a_2 + a_4, a_2 + 2a_3 + 2a_4, a_1 + 2a_2 2a_3 + a_4, \\ & a_1 + a_2 + 2a_3 + 2a_4, a_1 + 2a_2 + 3a_3 + a_4, a_1 + 2a_2 + 2a_3 + 2a_4, \\ & a_1 + 2a_2 + 3a_2 + 2a_4, a_1 + 2a_2 + 2a_3 + 2a_2, a_1 + 3a_2 + 4a_3 + 2a_4, \\ & 2a_1 + 3a_2 + 4a_2 + 2a_4\}. \end{aligned} \tag{5.2}$$

These roots, if we use the canonical base $\{a_1 = (1, 0, 0, 0)$, $a_2 = (0, 1, 0, 0)$, $a_3 = (0, 0, 1, 0)$, $a_4 = (0, 0, 0, 1)\}$ of Z^4, can be written

$$\begin{aligned} a_1 &= X_1 = (1, 0, 0, 0), \\ a_2 &= X_2 = (1, 0, 0, 0), \\ a_3 &= X_3 = (1, 0, 0, 0), \\ a_4 &= X_4 = (1, 0, 0, 0), \\ a_1 + a_2 &= X_5 = (1, 1, 0, 0), \\ a_2 + a_3 &= X_6 = (0, 1, 1, 0), \end{aligned} \tag{5.3}$$

$$\begin{aligned}
a_3 + a_4 &= X_7 = (0,0,1,1),\\
a_1 + a_2 + a_3 &= X_8 = (1,1,1,0),\\
a_2 + 2a_3 &= X_9 = (0,1,2,),\\
a_2 + a_3 + a_4 &= X_{10} = (0,1,1,1),\\
a_1 + a_2 + 2a_3 &= X_{11} = (1,1,2,0),\\
a_2 + 2a_3 + a_4 &= X_{12} = (0,1,2,1),\\
a_1 + a_2 + a_3 + a_4 &= X_{13} = (1,1,1,1),\\
a_1 + 2a_2 + 2a_4 &= X_{14} = (1,2,0,2),\\
a_1 + a_2 + 2a_3 + a_4 &= X_{15} = (1,1,2,1),\\
a_2 + 2a_3 + 2a_4 &= X_{16} = (0,1,2,2),\\
a_1 + 2a_2 + 2a_3 + a_4 &= X_{17} = (1,2,2,1),\\
a_1 + a_2 + 2a_3 + 2a_4 &= X_{18} = (1,1,2,2),\\
a_1 + 2a_2 + 3a_3 + a_4 &= X_{19} = (1,2,3,1),\\
a_1 + 2a_2 + 2a_3 + 2a_4 &= X_{20} = (1,2,2,2),\\
a_1 + 2a_2 + 3a_3 + 2a_4 &= X_{21} = (1,2,3,2),\\
a_1 + 2a_2 + 4a_3 + 2a_2 &= X_{22} = (1,2,4,2),\\
a_1 + 2a_2 4a_3 + 2a_4 + &= X_{23} = (1,3,4,2),\\
2a_1 + 3a_2 + 4a_3 + 2a_2 + &= X_{24} = (2,3,4,2)
\end{aligned}$$

In order to construct the ideals of $m_p(F_4)$, where p satisfies the conditions H_1 and H_2 or to construct the ideals of Δ_p and having the properties of (4.10) we must write the root system explicitly using the relation $\beta \to a$, it means, that there exists one element a_i, $i = 1,2,3,4$, of the base

$$\{a_1, a_2, a_3, a_4\} \text{ of } Z^4 \text{ such that} \tag{5.4}$$

$$\beta = a + a_i, \text{ where } i \text{ one of } \{1,2,3,4\} \tag{5.5}$$

Proposition 19 *5.4 Let Δ_4 be the root system defined by (5.2) or equivalently by (5.3). Write Δ_4 using the relation (5.5).*

Proof: From the form of the root system of Δ_+ we obtain the following diagram:

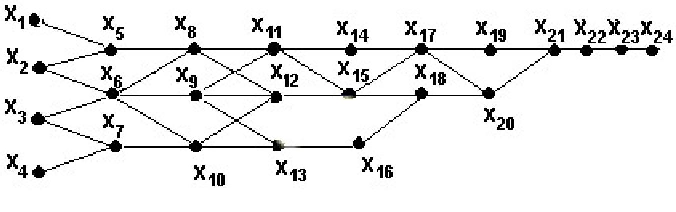


Figure 5.1

This figure allows us to construct the ideals of Δ_p, where p takes the values defined by the conditions H_1 and H_2. It is known:

$$H_1 : \text{ that } p \leq p_{F_4}$$

$$H_2 : Sup(-A_{ij} + 1/i, j = 1, \ldots, 1) \leq p_{F_4} \tag{5.6}$$

Where p_{F_4} the height of the heighest root. From (5.1) and (5.3) we conclude that:

$$3 \leq p \leq 11 \tag{5.7}$$

Therefore we can construct nine Nilpotent Lie algebras $m_p(F_4)$, which are:

$$\begin{gathered} m_3(F_4), m_4(F_4), m_5(F_4), m_6(F_4), m_7(F_4), \\ m_p(F_4), m_9(F_4), m_{10}(F_4) \text{ and } m_{11}(F_4), \end{gathered} \tag{5.8}$$

Proposition 20 *Let F_4 be the G.C.M. defined by (5.1). Describe the Nilpotent Lie algebras $m_p(F_4)$, $p = 3, \ldots, 11$.*

Proof: The Nilpotent Lie algebra $m_3(F_4)$ is given by

$$m_3(F_4) = L_+(F_4)/C^4(L_+(F_4))$$

and by means of the theorem 4.5 takes the form:

$$\begin{aligned} m_3(F_4) &= \underset{|a|\leq 3}{\oplus} I_a = KX_1 \oplus KX_2 \oplus KX_3 \oplus \\ & KX_4 \oplus KX_5 \oplus KX_6 \oplus \\ & KX_7 \oplus KX_8 \oplus KX_9 \oplus \\ KX_{10} &= \overset{10}{\underset{i=1}{\oplus}} K_{X_i} \text{ whose dimensionis 10, that is } \dim m_3(F_4) = 10. \end{aligned}$$

The Nilpotent Lie algebra $m_4(F_4)$ is described as follows

$$m_4(A) = \overset{13}{\underset{i=1}{\oplus}} K_{X_i}$$

whose dimension is 13.

The other Nilpotent Lie algebra $m_5(F_4)$, $m_6(F_4)$, $m_7(F_4)$, $m_8(F_4)$, $m_9(F_4)$, $m_{10}(F_4)$ and $m_{11}(F_4)$, have the form

$$\begin{aligned} m_5(F_4) &= \overset{16}{\underset{i=1}{\oplus}} K_{X_i},\ m_6(F_4) = \overset{13=8}{\underset{i=1}{\oplus}} K_{X_i},\ m_7(F_4) = \overset{20}{\underset{i=1}{\oplus}} K_{X_i},\ m_8(F_4) = \overset{21}{\underset{i=1}{\oplus}} K_{X_i}, \\ m_9(F_4) &= \overset{22}{\underset{i=1}{\oplus}} K_{X_i},\ m_{10}(F_4) = \overset{23}{\underset{i=1}{\oplus}} K_{X_i},\ m_{11}(F_4) = \overset{24}{\underset{i=1}{\oplus}} K_{X_i}. \end{aligned}$$

The dimensions of these Nilpotent Lie algebras are:

$$\begin{aligned} \dim(m_5(F_4)) &= 16, \dim(m_6(F_4)) = 19, \dim(m_7(F_4)) = 20, \\ \dim(m_8(F_4)) &= 21, \dim(m_9(F_4)) = 22, \dim(m_{10}(F_4)) = 23, \\ \dim(m_{11}(F_4)) &= 24. \end{aligned}$$

Let $m_v(F_4), v = 3, 4, \ldots, 11$, be the Nilpotent Lie algebras. The homogeneous iideas β of $m_v(F_4), v = 3, 4, \ldots, 11$, which the properties described in (4.5), will give the Nilpotent Lie algebras:

> $\{m_v(F_4)/\beta$ having properties described in (4.5) of maximal rank with F_4 as an associated G.C.M.$\}$.

This problem is equivalent to determine the ideals E of

$$\Delta_p,\ p = 3, 4, \ldots, 11,$$

with the properties $\Delta_+^p \notin E$, $a_j - A_{ij}a_i \notin E$.

Problem 5.5 Determine the ideal E of Δ_p, $p = 3, 4, \ldots, 11$, with the properties $\Delta_+^p \notin E$ and $a_j - A_{ij}a_i \notin E$.

Solution. Firstly, we define the basic chain N for F_4, which has the form

$$a_j - A_{ij}a_i,\ i \neq j,\ i, j = 1, 2, 3, 4$$

where A_{ij} are the entries of the matrix F_4 given by (5.1). After some estimates we have:

$$N = \{a_1, a_2, a_3, a_4, a_1 + a_2, a_2 + a_3, a_3 + a_4, a_2 + 2a_3\}.$$

Now, we consider Δ_p, $p = 3, 4, \ldots, 11$

$$\Delta_3 = N \cup \{a_1 + a_2 + a_3, a_2 + a_3 + a_2\} = N \cup \Delta_+^3 - \{a_2 + 2a_3\} = N \cup T_3$$

where

$$T = \{a_1 + a_2 + a_3, a_2 + a_3 + a_4\}$$

$$\Delta_4 = N \cup T \cup \{a_1 + a_2 + 2a_3, a_1 + a_2 + a_3 + a_4\} = N \cup T_3 \cup \Delta_+^3$$

$$\begin{aligned} \Delta_5 &= N \cup T_3 \cup \Delta_+^4 \cup \{a_1 + 2a_2 + 2a_3, a_1 + a_2 + 2a_3 + a_1, a_2 + 2a_3 + 2a_4\} = \\ &= N \cup T_3 \cup \Delta_+^4 \cup \Delta_+^5 \end{aligned}$$

$$\begin{aligned} \Delta_6 &= N \cup T_3 \cup \Delta_+^4 \cup \Delta_+^5 \cup \{a_1 2a_2 + 2a_3 + a_4, a_1 + a_2 + 2a_3 + 2a_4\} = \\ &= N \cup T_3 \cup \Delta_+^4 \cup \Delta_+^5 \cup \Delta_+^6 \end{aligned}$$

$$\Delta_7 = N \cup T_3 \cup \Delta_+^4 \cup \Delta_+^5 \cup \Delta_+^6 \cup \{a_1 + 2a_2 + 3a_3 + a_4, a_1 + 2a_2 + 3a_3 + a_4,$$
$$a_1 + 2a_2 + 3a_3 + a_4\} = N \cup T_3 \bigcup_{\mu=4}^{7} \Delta_+^\mu$$

$$\Delta_8 = N \cup T_3 \bigcup_{\mu=4}^{7} \Delta_+^\mu \cup \{a_1 + 2a_2 + 3a_3 + a_4\} = N \cup T_3 \bigcup_{\mu=4}^{8} \Delta_+^\mu$$

$$\Delta_9 = N \cup T_3 \bigcup_{\mu=4}^{8} \Delta_+^\mu \cup \{a_1 + 2a_2 + 3a_3 + a_4\} = N \cup T_3 \bigcup_{\mu=4}^{9} \Delta_+^\mu$$

$$\Delta_{10} = N \cup T_3 \bigcup_{\mu=4}^{9} \Delta_+^\mu \cup \{a_1 + 2a_2 + 4a_3 + 2a_4\} = N \cup T_3 \bigcup_{\mu=4}^{10} \Delta_+^\mu$$

$$\Delta_{11} = N \cup T_3 \bigcup_{\mu=4}^{10} \Delta_+^\mu \cup \{2a_2 + 3a_3 + 4a_3 + 2a_3\} = N \cup T_3 \bigcup_{\mu=4}^{11} \Delta_+^\mu.$$

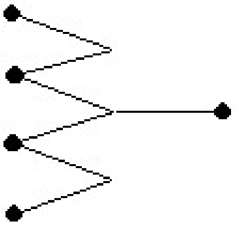
6 Calculations of the ideals

We calculate the ideals of Δ_v, $v = 3, 4, \ldots, 11$, and using the notation of (5.3) we obtain

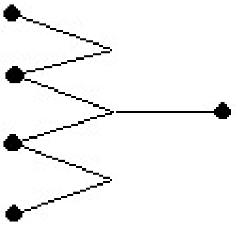
$E_1 = \{X_9, X_{10}\}, E_2 = \{X_9\}, E_3 = \{X_{10}\}, E_3 = \{\emptyset\}$ for Δ_3.

We can proceed with the same method as for Δ_3 for calculations the ideals of the others Δ_v, $v = 4, \ldots, 11$ and take under the consideration some of these ideals, used for the quotient Lie algebras, give the same Lie algebras, then we represent only these ideals which give the non-isomorphic 43 Nilpotent Lie algebras of maximal rank and of type F_4.

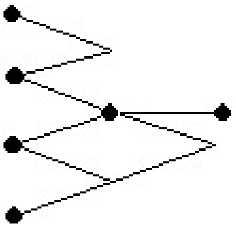
We list now the ideals and the generators of the corresponding Lie algebra respectively.



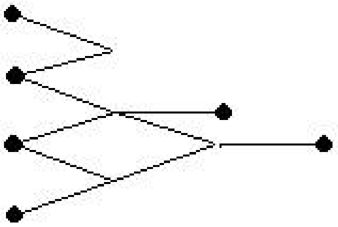
$\{X_8 = (1,1,1,0)$,
$X_{10} = (0,1,1,1)\}$ $\{X_1, \ldots, X_7\}\, F_4^1$



$\{X_{10} = (0,1,1,1)\}$ $\{X_1, \ldots, X_9\}\, F_4^2$

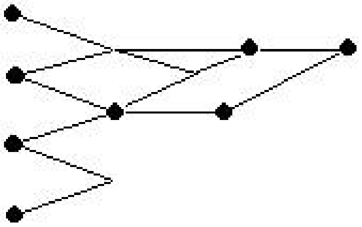


$\{X_8 = (1,1,1,0)\}$ $\{X_1, \ldots, X_7, X_9, X_{10}\}\, F_4^3$



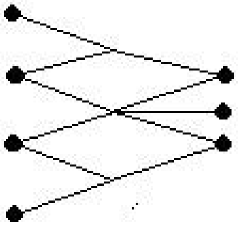
$\{X_8 = (1,1,1,0)\,,$
$X_{11} = (1,1,2,0)\,,$
$X_{12} = (1,1,1,1)\}$

$\{X_1, \ldots, X_9, X_{10}, X_{13}\}\; F_4^4$



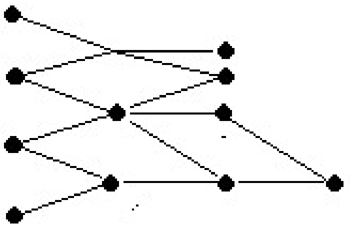
$\{X_{10} = (0,1,1,1)\,,$
$X_{12} = (1,1,1,1)\,,$
$X_{13} = (0,1,2,1)\}$

$\{X_1, \ldots, X_9, X_{11}\}\; F_4^5$



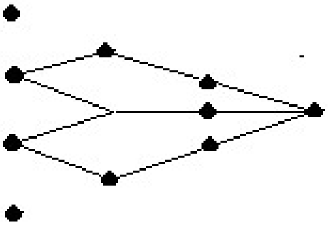
$\{X_{11} = (1,1,2,0)\,,$
$X_{12} = (1,1,1,1)\,,$
$X_{13} = (0,1,2,1)\}$

$\{X_1, \ldots, X_{10}\}\; F_4^6$



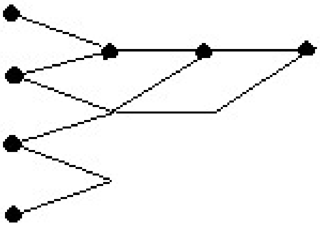
$\{X_{11} = (1,1,2,0)\,,$
$X_{12} = (1,1,1,1)\}$

$\{X_1, \ldots, X_{10}, X_{13}\}\; F_4^7$



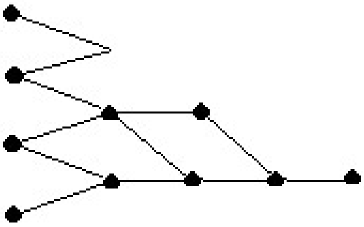
$\{X_{11} = (1,1,2,0)\,,$
$X_{13} = (0,1,2,1)\}$

$\{X_1, \ldots, X_{10}, X_{12}\}\, F_4^8$



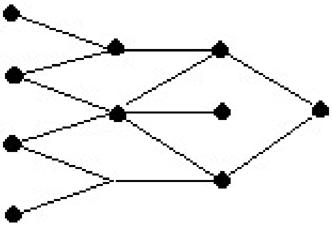
$\{X_{10} = (0,1,1,1)\,,$
$X_{11} = (1,1,2,0)\,,$
$X_{13} = (1,1,1,1)\,,$
$X_{14} = (0,2,2,0)\,,$
$X_{16} = (0,1,2,2)\}$

$\{X_1, \ldots, X_9, X_{12}, X_{15}\}\, F_4^9$



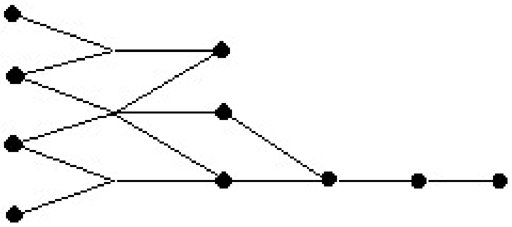
$\{X_9 = (1,1,1,0)\,,$
$X_{11} = (1,1,2,0)\,,$
$X_{12} = (0,1,2,1)\,,$
$X_{15} = (1,1,2,1)\,,$
$X_{16} = (0,1,2,2)\}$

$\{X_1, \ldots, X_9, X_{12}, X_{15}\}\, F_4^{10}$



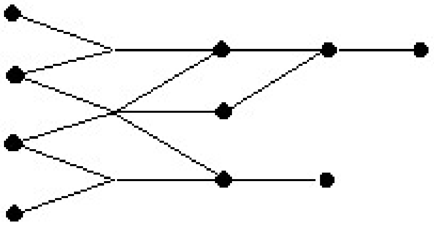
$\{X_{12} = (0,1,2,1)\,,$
$X_{13} = (1,1,1,1)\}$

$\{X_1, \ldots, X_{11}, X_{14}, \ldots, X_{16}\}\, F_4^{11}$



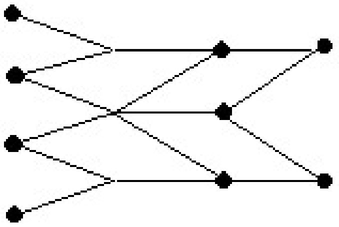
$\{X_{11} = (1,1,2,0)\,,$
$X_{12} = (0,1,2,1)\,,$
$X_{14} = (1,2,2,0)\,,$
$X_{15} = (0,1,2,1)\}$

$\{X_1, \ldots, X_{10}, X_{13}, X_{16}\}\, F_4^{12}$



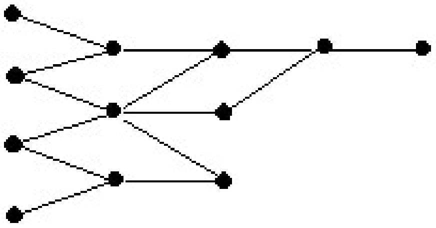
$\{X_{11} = (1,1,2,0)\,,$
$X_{14} = (1,2,2,0)\,,$
$X_{15} = (1,1,2,1)\,,$
$X_{16} = (0,1,2,2)\}$

$\{X_1, \ldots, X_{10}, X_{12}, X_{13}\}\, F_4^{13}$



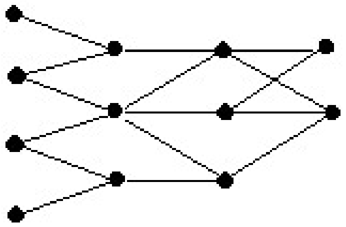
$\{X_{12} = (1,1,1,1)\,,$
$X_{13} = (0,1,2,1)\,,$
$X_{15} = (1,1,2,1)\,,$
$X_{16} = (0,1,2,2)\}$

$\{X_1, \ldots, X_{11}, X_{14}\}\, F_4^{14}$

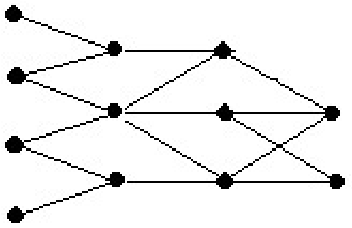


$\{X_{13} = (0,1,2,1)\,,$
$X_{15} = (1,1,2,1)\,,$
$X_{16} = (0,1,2,2)\}$

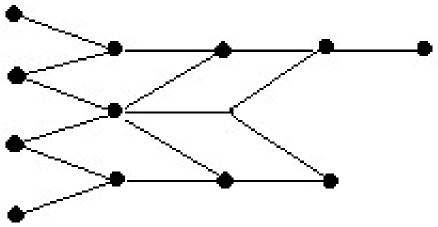
$\{X_1, \ldots, X_{12}, X_{14}\}\, F_4^{15}$



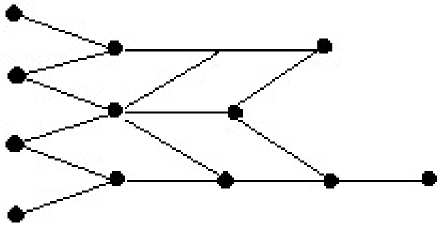
$\{X_{13}=(0,1,2,1)\}$ $\{X_1,\dots,X_{12},X_{14},\dots,X_{16}\}\,F_4^{16}$



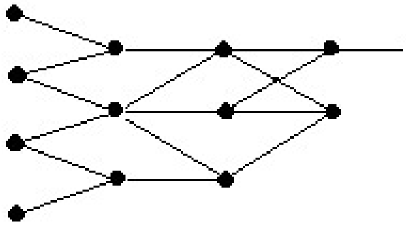
$\{X_{11}=(1,1,2,0)\,,$
$X_{14}=(1,2,2,0)\,,$
$X_{15}=(1,1,2,1)\}$ $\{X_1,\dots,X_{10},X_{12},X_{13},X_{16}\}\,F_4^{17}$



$\{X_{12}=(1,1,1,1)\,,$
$X_{15}=(1,1,2,1)\,,$
$X_{16}=(0,1,2,2)\}$ $\{X_1,\dots,X_{11},X_{13},X_{14}\}\,F^{18}$

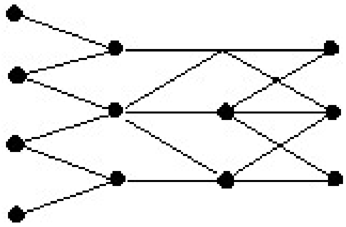


$\{X_{12}=(1,1,1,1)\,,$
$X_{14}=(1,2,2,0)\,,$
$X_{15}=(1,1,2,1)\}$ $\{X_1,\dots,X_{11},X_{13},X_{16}\}\,F_4^{19}$



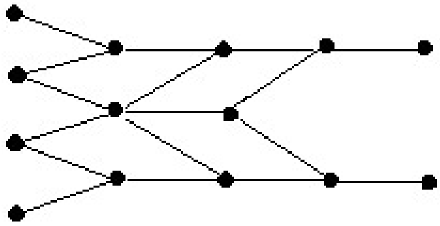
$\{X_{13} = (0,1,2,1)\,,$
$X_{15} = (1,1,2,1)\,,$
$X_{16} = (0,1,2,2)\}$

$\{X_1, \ldots, X_{12}, X_{14}\}\, F_4^{20}$



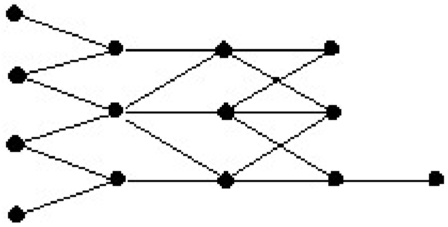
$\{X_{14} = (1,2,2,0)\,,$
$X_{15} = (1,1,2,1)\,,$
$X_{16} = (0,1,2,2)\}$

$\{X_1, \ldots, X_{13}\}\, F_4^{21}$



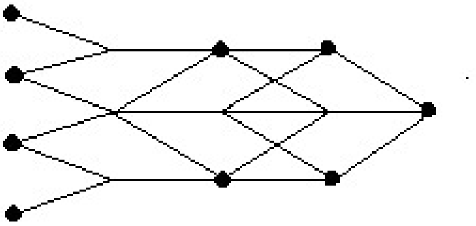
$\{X_{12} = (1,1,1,1)\,,$
$X_{15} = (1,1,2,1)\}$

$\{X_1, \ldots, X_{11}, X_{13}, X_{14}, X_{16}\}\, F_4^{22}$



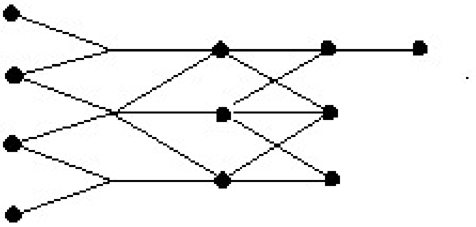
$\{X_{14} = (1,2,2,0)\,,$
$X_{15} = (1,1,2,1)\}$

$\{X_1, \ldots, X_{13}, X_{16}\}\, F_4^{23}$



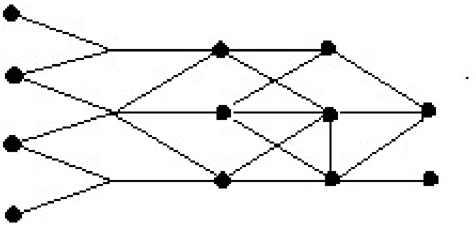
$\{X_{14} = (1,2,2,0)\,,$
$X_{16} = (1,1,2,1)\}$

$\{X_1, \ldots, X_{13}, X_{15}\}\, F_4^{24}$



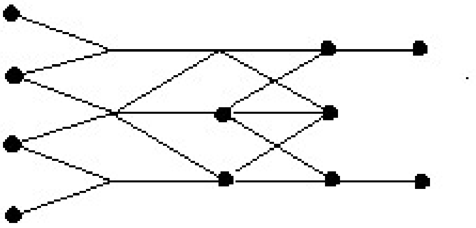
$\{X_{15} = (1,1,2,1)\,,$
$X_{16} = (0,1,2,2)\}$

$\{X_1, \ldots, X_{14}\}\, F_4^{25}$



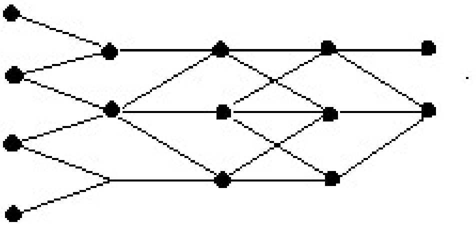
$\{X_{14} = (1,2,2,0)\}$

$\{X_1, \ldots, X_{13}, X_{15}, X_{16}\}\, F_4^{26}$



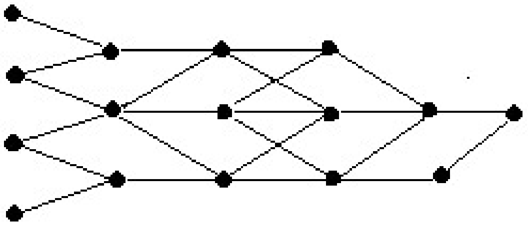
$\{X_{15} = (1,1,2,1)\}$

$\{X_1, \ldots, X_{14}, X_{16}\}\, F_4^{27}$



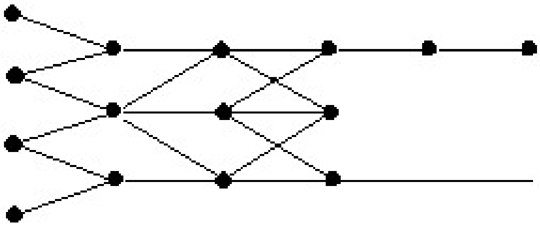
$\{X_{16} = (0,1,2,2)\}$

$\{X_1, \ldots, X_{15}\}\, F_4^{28}$



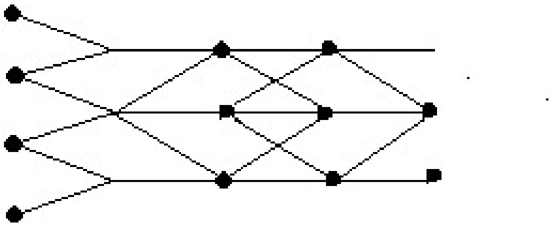
$\{X_{14} = (1,2,2,0)\,,$
$X_{17} = (1,2,2,1)\}$

$\{X_1, \ldots, X_{13}, X_{15}, X_{16}, X_{18}\}\, F_4^{29}$



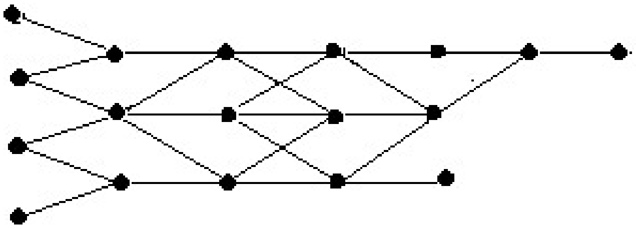
$\{X_{16} = (0,1,2,2)\,,$
$X_{18} = (1,2,2,2)\}$

$\{X_1, \ldots, X_{15}, X_{17}\}\, F_4^{30}$



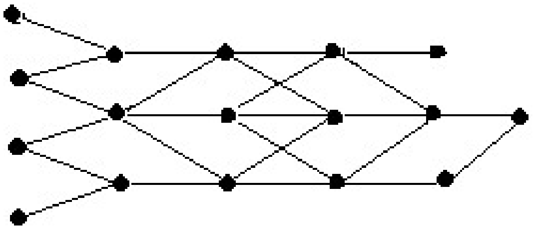
$\{X_{17} = (1,2,2,1)\,,$
$X_{18} = (1,1,2,2)\}$

$\{X_1, \ldots, X_{16}\}\, F_4^{31}$



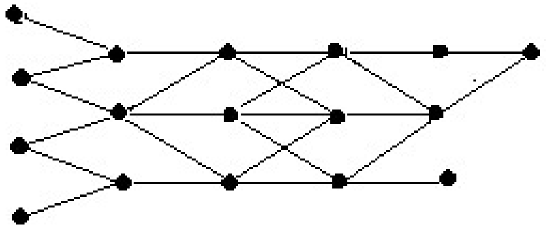
$\{X_{16} = (0,1,2,2)\,,$
$X_{18} = (1,1,2,2)\,,$
$X_{20} = (1,2,2,2)\}$

$\{X_1, \ldots, X_{15}, X_{17}, X_{19}\}\, F_4^{32}$

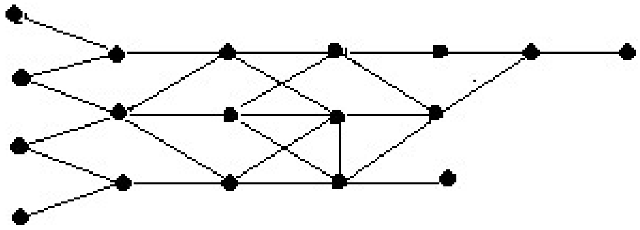


$\{X_{17} = (1,2,2,1)\}$

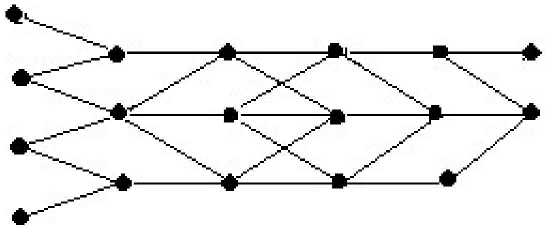
$\{X_1, \ldots, X_{16}, X_{18}, \ldots, X_{20}\}\, F_4^{33}$



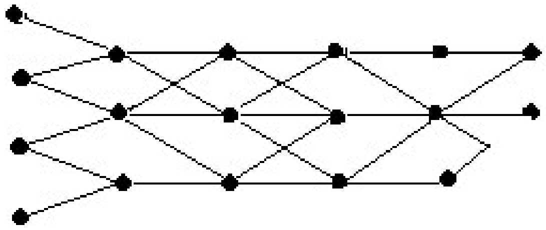
$\{X_{18} = (1,1,2,2)\}$ $\{X_1, \dots, X_{17}, X_{19}, X_{20}\}$ F_4^{34}



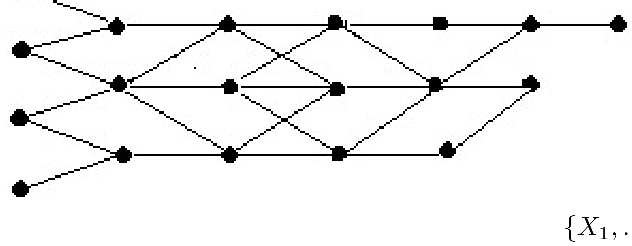
$\{X_{18} = (1,1,2,2)\,,$
$X_{20} = (1,2,2,2)\}$ $\{X_1, \dots, X_{17}, X_{19}\}$ F_4^{35}



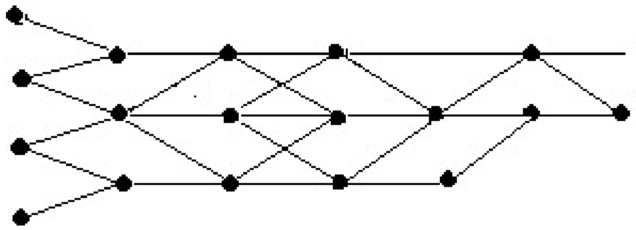
$\{X_{19} = (1,2,3,1)\,,$
$X_{20} = (1,2,2,2)\}$ $\{X_1, \dots, X_{18}\}$ F_4^{36}



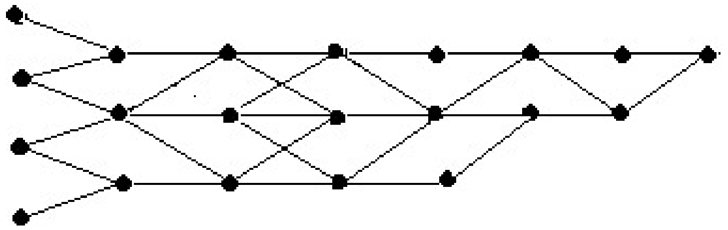
$\{X_{19} = (1,2,3,1)\}$ $\{X_1, \dots, X_{18}, X_{20}\}$ F_4^{37}



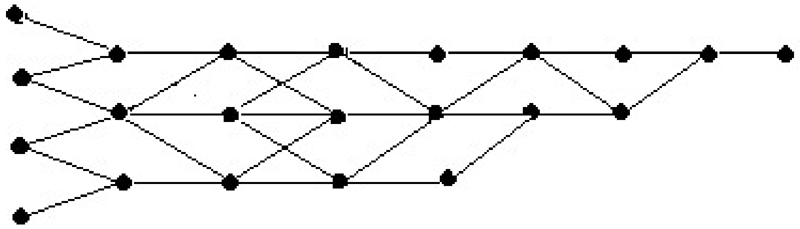
$\{X_{20} = (1,2,2,2)\}$ $\{X_1, \dots, X_{19}\}$ F_4^{38}



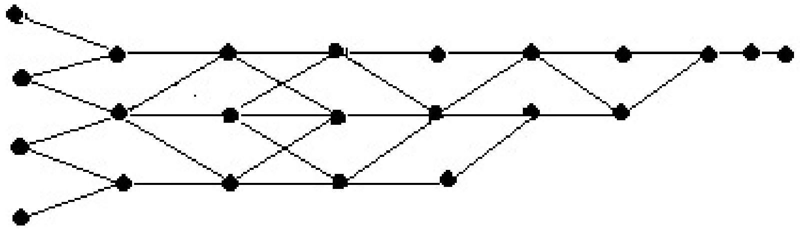
$\{X_{21} = (1,2,3,2)\}$ $\{X_1, \dots, X_{20}\}\, F_4^{39}$



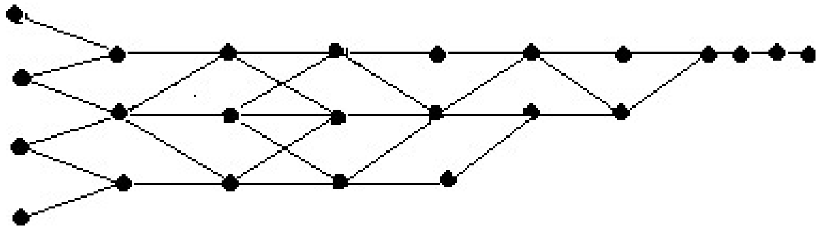
$\{X_{22} = (1,2,4,2)\}$ $\{X_1, \dots, X_{21}\}\, F_4^{40}$



$\{X_{23} = (1,3,4,2)\}$ $\{X_1, \dots, X_{22}\}\, F_4^{41}$



$\{X_{24} = (2,3,4,2)\}$ $\{X_1, \dots, X_{22}\}\, F_4^{42}$



$\{\emptyset\}$ $\{X_1, \dots, X_{24}\}\, F_4^{43}$

7 Elements of the Lie algebras F_4^λ

We have estimated the above 43 Nilpotent Lie algebras of maximal rank and of type F_4. In this section we determine the structure constants of these Lie algebras and give some other elements, which are the following.

(I) We write each F_4^λ, $\lambda = 1, \dots, 43$, as a quotient Lie algebras that means

$$F_4^\lambda = m_a(A)/\beta$$

where $a = 3, \dots, 11$ and β an ideal of $m_a(a)$.

(II) We give the dimension each of the Lie algebras F_4^λ, $\lambda = 1, \dots, 43$.

(III) We compute the sequence

$$\{v_1^\lambda, v_2^\lambda, \dots, v_p^\lambda\}$$

where $v_i^\lambda = \dim(C_{i+1}F_4^\lambda)\, i = 1, \dots, p$

$p + 1$ is the nolpotency of F_4^λ, $\lambda = 1, \dots, 43$.

(IV) Briefly we note the Lie brackets by form $[t]$ where t is a positive integer which runs from 1 to 60. Therefore for F_4, whose Lie brackets are

$$\begin{aligned}
[1] &= [X_1, X_2] = -X_5, [2] = [X_2, X_3] = -X_6, [3] = [X_3, X_4] = -X_7, \\
[6] &= [X_3, X_5] = -2X_9
\end{aligned}$$

we have its representations by

$$F_4^1 : \{[1], \dots, [3], [6]\}.$$

The Lie brackets of F_4^1 are valid for F_4^2 having two new non-zero Lie brackets denoted by

$$[7] = [X_2, X_7] = -X_{10}, [8] = [X_1, X_9] = X_{10}.$$

Therefore F_4^2 is characterized by

$$F_4^2 : \{[1], \dots, [8]\}.$$

For thee Lie algebra F_4^3 some of the previous Lie brackets do not appear however for this new Lie brackets appear:

$$[4] = [X_1, X_6] = -X_8, [5] = [X_3, X_5] = X_8.$$

Hence F_4^3 is characterized by

$$F_4^3 : \{[1], \dots, [6]\}.$$

Therefore for each Lie algebra F_4^λ, $\lambda = 1, \dots, 43$, we write the Lie brackets in the form:

$$\{[1], [2], \dots, \}.$$

Now, we give the list of F_4^λ, $\lambda = 1, \ldots, 43$, with all the elements which have been referred above.

$$F_4^1 : F_4^1 = m_3(F_4)/L_{X_8} \oplus L_{X_{10}},\ (8,4,1),\ \dim F_4^1 = 8.$$

$$\begin{aligned}[1] &= [X_1, X_2] = -X_5, [2] = [X_2, X_3] = -X_6, [3] = [X_3, X_4] = -X_7,\\ [6] &= [X_3, X_5] = -2X_9\end{aligned}$$

$$F_4^1 : \{[1], \ldots, [3], [6]\}$$

$$F_4^2 : F_4^2 = m_4(F_4)/L_{X_8},\ (9,5,2),\ \dim F_4^2 = 9$$

$$[7] = [X_2, X_7] = -X_{10}, [8] = [X_1, X_9] = X_{10}$$

$$F_4^2 : \{[1], \ldots, [3], [6], \ldots, [8]\}$$

$$F_4^3 : F_4^3 = m_3(F_4)/L_{X_{10}},\ (9,5,2),\ \dim F_4^3 = 9$$

$$[4] = [X_1, X_6] = -X_8, [5] = [X_3, X_5] = X_8$$

$$F_4^3 : \{[1], \ldots, [6]\}$$

$$F_4^4 : F_4^4 = m_4(F_4)/L_{X_9} \oplus L_{X_{11}} \oplus L_{X_{12}},\ (10,6,3,1),\ \dim F_4^4 = 10$$

$$[14] = [X_3, X_{10}] = -X_{13}, [15] = [X_4, X_9] = X_{13}, [16] = [X_6, X_7] = X_{13}$$

$$F_4^4 : \{[1], \ldots, [3], [7], [14], [16]\}$$

$$F_4^5 : F_4^5 = m_4(F_4)/L_{X_{10}} \oplus L_{X_{12}} \oplus L_{X_{13}},\ (10,6,3),\ \dim F_4^5 = 10$$

$$[9] = [X_4, X_9] = -X_{11}, [10] = [X_3, X_8] = -2X_{11}$$

$$F_4^5 : \{[1], \ldots, [6], [7], [9], [10]\}$$

$$F_4^6 : F_4^6 = m_4(F_4)/L_{X_{11}} \oplus L_{X_{12}} \oplus L_{X_{13}},\ (10,6,3),\ \dim F_4^6 = 10$$

$$F_4^6 : \{[1], \ldots, [8]\}$$

$$F_4^7 : F_4^7 = m_4(F_4)/L_{X_{11}} \oplus L_{X_{12}},\ (11,7,4,1),\ \dim F_4^7 = 11$$

$$F_4^7 : \{[1], \ldots, [8], [14], \ldots, [16]\}$$

$$F_4^8 : F_4^8 = m_4(F_4)/L_{X_{11}} \oplus L_{X_{13}},\ (11,7,4,1),\ \dim F_4^8 = 11$$

$$[11] = [X_1, X_{10}] = -X_{12}, [12] = [X_4, X_8] = X_{12}, [13] = [X_5, X_7] = -X_{12}$$

$$F_4^8 : \{[1], \ldots, [8], [11], \ldots, [13]\}$$

$$F_4^9 : F_4^9 = m_5(F_4)/L_{X_{10}} \oplus L_{X_{12}} \oplus L_{X_{13}} \oplus L_{X_{15}} \oplus L_{X_{16}},\ (11,7,4,2),\ \dim F_4^9 = 11$$

$$[17] = [X_2, X_{11}] = -X_{14}, [18] = [X_5, X_7] = X_{14}, [19] = [X_6, X_8] = -X_{14}$$

$$F_4^9 : \{[1], \ldots, [6], [9], [10], [11], [14]\}$$

$$F_4^{10} : F_4^{10} = m_5(F_4)/L_{X_8} \oplus L_{X_{11}} \oplus L_{X_{13}} \oplus L_{X_{14}} \oplus L_{X_{15}},\ (11,7,4,2,1),\ \dim F_4^{10} = 11$$

$$F_4^{10} : \{[1], \ldots, [3], [6], \ldots, [8], [14], \ldots, [16]\}$$

$$F_4^{11} : F_4^{11} = m_5(F_4)/ \oplus L_{X_{12}} \oplus L_{X_{13}},\ (11,7,4,1),\ \dim F_4^{11} = 11$$

$$-[X_5, X_7] = -[X_4, X_9] = [X_1, X_{10}] = X_{11}$$

$$F_4^{11} : \{[1], [2], \ldots, [10]\}$$

$$F_4^{12} : F_4^{12} = m_5(F_4)/L_{X_{11}} \oplus L_{X_{12}} \oplus L_{X_{14}} \oplus L_{X_{15}},\ (12,8,5,2,1),\ \dim F_4^{12} = 12$$

$$[24] = [X_4, X_{13}] = -2X_{16}, [25] = [X_7, X_{10}] = 2X_{16}$$

$$F_4^{12} : \{[1], \ldots, [8], [14], [15], [24], [25]\}$$

$$F_4^{13} : F_4^{13} = m_5(F_4)/L_{X_{11}},\ (12,8,5,2,1),\ \dim F_4^{13} = 12$$

$$F_4^{13} : \{[1], \ldots, [8], [11], \ldots, [16]\}$$

$$F_4^{14} : F_4^{14} = m_5(F_4)/L_{X_{12}},\ (12,8,5,2),\ \dim F_4^{14} = 12$$

$$F_4^{14} : \{[1], \ldots, [10], [14], \ldots, [16]\}$$

$$F_4^{15} : F_4^{15} = m_5(F_4)/L_{X_{12}} \oplus L_{X_{13}} \oplus L_{X_{15}} \oplus L_{X_{16}},\ (12,8,5,2)\ \dim F_4^{15} = 12$$

$$F_4^{15} : \{[1], \ldots, [10], [17], \ldots, [19]\}$$

$$F_4^{16} : F_4^{16} = m_5(F_4)/L_{X_{13}},\ (12,8,5,2),\ \dim F_4^{16} = 12$$

$$F_4^{16} : \{[1], \ldots, [13]\}$$

$$F_4^{17} : F_4^{17} = m_5(F_4)/L_{X_{11}} \oplus L_{X_{14}} \oplus L_{X_{15}},\ (13,9,6,3,1),\ \dim F_4^{17} = 13$$

$$F_4^{17} : \{[1], \ldots, [8], [11], \ldots, [16], [24], [25]\}$$

$$F_4^{18} : F_4^{18} = m_5(F_4)/L_{X_{12}} \oplus L_{X_{15}} \oplus L_{X_{16}},\ (13,9,6,3,1),\ \dim F_4^{18} = 13$$

$$F_4^{18} : \{[1], \ldots, [10], [14], \ldots, [19]\}$$

$$F_4^{19} : F_4^{19} = m_5(F_4)/L_{X_{12}} \oplus L_{X_{14}} \oplus L_{X_{15}},\ (13,9,6,3,1),\ \dim F_4^{19} = 13$$

$$F_4^{19} : \{[1], \ldots, [10], [14], [16], [24], [25]\}$$

$$F_4^{20} : F_4^{20} = m_5(F_4)/L_{X_{13}} \oplus L_{X_{15}} \oplus L_{X_{16}},\ (13,9,6,3,1),\ \dim F_4^{20} = 13$$

$$F_4^{20} : \{[1], \ldots, [13], [17], \ldots, [19]\}$$

$$F_4^{21} : F_4^{21} = m_5(F_4)/L_{X_{14}} \oplus L_{X_{15}} \oplus L_{X_{16}},\ (13,9,6,3),\ \dim F_4^{21} = 13$$

$$F_4^{21} : \{[1], \ldots, [16]\}$$

$$F_4^{22} : F_4^{22} = m_5(F_4)/L_{X_{12}} \oplus L_{X_{15}},\ (14,10,7,4,2),\ \dim F_4^{22} = 14$$

$$F_4^{22} : \{[1], \ldots, [10], [14], \ldots, [16]\}$$

$$F_4^{23} : F_4^{23} = m_5(F_4)/L_{X_{14}} \oplus L_{X_{15}},\ (14,10,7,4,1),\ \dim F_4^{23} = 14$$

$$F_4^{23} : \{[1], \ldots, [16], [24], [25]\}$$

$$F_4^{24} : F_4^{24} = m_5(F_4)/L_{X_{14}} \oplus L_{X_{16}},\ (14,10,7,4,1),\ \dim F_4^{24} = 14$$

$$\begin{aligned}
[20] &= [X_1, X_{13}] = -X_{15}, [21] = [X_3, X_{12}] = -X_{15}, [22] = [X_4, X_{11}] = X_{15},\\
[23] &= [X_7, X_8] = X_{15}
\end{aligned}$$

$$F_4^{24} : \{[1], \ldots, [16], [21], \ldots, [23]\}$$

$$F_4^{25} : F_4^{25} = m_5(F_4)/L_{X_{15}} \oplus L_{X_{16}},\ (14, 10, 7, 4, 1),\ \dim F_4^{25} = 15$$

$$F_4^{25} : \{[1], \ldots, [17]\}$$

$$F_4^{26} : F_4^{26} = m_5(F_4)/L_{X_{14}},\ (15, 11, 8, 5, 2, 1),\ \dim F_4^{26} = 15$$

$$F_4^{26} : \{[1], \ldots, [16], [20], \ldots, [25]\}$$

$$F_4^{27} : F_4^{27} = m_5(F_4)/L_{X_{15}}, (15, 11, 8, 5, 2, 1),\ \dim F_4^{27} = 15$$

$$F_4^{27} : \{[1], \ldots, [19], \ldots [24], [25], \}$$

$$F_4^{28} : F_4^{28} = m_5(F_4)/L_{X_{16}}, (15, 11, 8, 5, 2), \dim F_4^{28} = 15$$

$$F_4^{28} : \{[1], \ldots, [23], \}$$

$$F_4^{29} : F_4^{29} = m_6(F_4)/L_{X_{14}} \oplus L_{X_{17}}, (16, 12, 9, 6, 3, 1), \dim F_4^{29} = 16$$

$$[31] = [X_1, X_{16}] = -X_{18}, [32] = [X_4, X_{15}] = -2X_{18}, [33] = [X_7, X_{12}] = -2X_{18}$$

$$F_4^{29} : \{[1], \ldots, [16], [20], \ldots, [25], [31], \ldots, [33]\}$$

$$F_4^{30} : F_4^{30} = m_6(F_4)/L_{X_{16}} \oplus L_{X_{18}},\ (16, 12, 9, 6, 2),\ \dim F_4^{30} = 16$$

$$\begin{aligned}
[26] &= [X_2, X_{15}] = -X_{17}, [27] = [X_4, X_{14}] = X_{17}, [28] = [X_5, X_{13}] = X_{17}\\
[29] &= [X_6, X_{12}] = -X_{17}, [30] = [X_8, X_{10}] = X_{17}
\end{aligned}$$

$$F_4^{30} : \{[1], \ldots, [23], [26], \ldots, [30]\}$$

$$F_4^{31} : F_4^{31} = m_6(F_4)/L_{X_{17}} \oplus L_{X_{18}},\ (16, 12, 9, 6, 3),\ \dim F_4^{31} = 16$$

$$F_4^{31} : \{[1], \ldots, [25]\}$$

$$F_4^{32} : F_4^{32} = m_6(F_4)/L_{X_{16}} \oplus L_{X_{18}} \oplus L_{X_{20}},\ (17, 13, 10, 7, 4, 3, 1),\ \dim F_4^{32} = 17$$

$$\begin{aligned}
[34] &= [X_3, X_{17}] = X_{19}, [35] = [X_7, X_{14}] = X_{19}, [36] = [X_9, X_{12}] = -X_{19}\\
[37] &= [X_{10}, X_{11}] = -X_{19}
\end{aligned}$$

$$F_4^{32} : \{[1], \ldots, [23], [26], \ldots, [30], [34], \ldots, [37]\}$$

$$F_4^{33} : F_4^{33} = m_7(F_4)/L_{X_{17}},\ (17, 13, 10, 7, 4, 2, 1),\ \dim F_4^{33} = 17$$

$$F_4^{33} : \{[1], \ldots, [25], [31], \ldots, [33]\}$$

$$F_4^{34} : F_4^{34} = m_7(F_4)/L_{X_{18}},\ (17, 13, 10, 7, 4, 2),\ \dim F_4^{34} = 17$$

$$F_4^{34} : \{[1], \ldots, [30], [28], \ldots, [37]\}$$

$$F_4^{35} : F_4^{35} = m_7(F_4)/L_{X_{18}} \oplus L_{X_{20}},\ (18, 14, 11, 8, 5, 3, 1),\ \dim F_4^{35} = 18$$

$$F_4^{35} : \{[1], \ldots, [30], [34], \ldots, [37]\}$$

$$F_4^{36} : F_4^{36} = m_7(F_4)/L_{X_{19}} \oplus L_{X_{20}},\ (18,14,11,8,5,3),\ \dim F_4^{36} = 18$$

$$F_4^{36} : \{[1], \ldots, [33]\}$$

$$F_4^{37} : F_4^{37} = m_7(F_4)/L_{X_{19}},\ (19,12,9,6,4,2,1),\ \dim F_4^{37} = 19$$

$$\begin{array}{rcl} [38] & = & [X_2, X_{18}] = -X_{20}, [39] = [X_4, X_{17}] = -2X_{20}, [40] = [X_5, X_{16}] = X_{20} \\ [41] & = & [X_{10}, X_{12}] = 2X_{20} \end{array}$$

$$F_4^{37} : \{[1], \ldots, [33], [38], \ldots, [41]\}$$

$$F_4^{38} : F_4^{38} = m_7(F_4)/L_{X_{20}},\ (19,15,12,9,6,4,2),\ \dim F_4^{38} = 19$$

$$F_4^{38} : \{[1], \ldots, [37]\}$$

$$F_4^{39} : F_4^{39} = m_8(F_4)/L_{X_{21}},\ (20,16,13,10,7,4,2),\ \dim F_4^{39} = 20$$

$$F_4^{39} : \{[1], \ldots, [37]\}$$

$$F_4^{40} : F_4^{40} = m_9(F_4)/L_{X_{22}},\ (21,17,14,11,8,5,3,1),\ \dim F_4^{40} = 20$$

$$\begin{array}{rcl} [42] & = & [X_3, X_{20}] = X_{21}, [43] = [X_4, X_{19}] = X_{21}, [44] = [X_6, X_{18}] = X_{21} \\ [45] & = & [X_7, X_{17}] = -X_{21}, [46] = [X_8, X_{16}] = -X_{21}, [47] = [X_{10}, X_{15}] = -X_{21} \\ [48] & = & [X_{12}, X_{13}] = -X_{21} \end{array}$$

$$F_4^{40} : \{[1], \ldots, [48]\}$$

$$F_4^{41} : F_4^{41} = m_{10}(F_4)/L_{X_{23}},\ (23,18,15,12,9,6,4,2,1),\ \dim F_4^{41} = 22$$

$$\begin{array}{rcl} [49] & = & [X_3, X_{21}] = -2X_{22}, [50] = [X_7, X_{19}] = -2X_{22}, [51] = [X_9, X_{18}] = X_{22} \\ [52] & = & [X_{11}, X_{16}] = -X_{22}, [53] = [X_{13}, X_{15}] = X_{22} \end{array}$$

$$F_4^{41} : \{[1], \ldots, [53]\}$$

$$F_4^{42} : F_4^{42} = m_{11}(F_4)/L_{X_{24}},\ (23,19,16,13,10,7,5,3,2,1),\ \dim F_4^{42} = 23$$

$$\begin{array}{rcl} [54] & = & [X_2, X_{22}] = -2X_{23}, [55] = [X_6, X_{21}] = -2X_{23}, [56] = [X_9, X_{20}] = X_{23} \\ [57] & = & [X_{10}, X_{19}] = -2X_{23}, [58] = [X_{13}, X_{17}] = -2X_{23}, [59] = [X_{14}, X_{16}] = -X_{23} \end{array}$$

$$F_4^{42} : \{[1], \ldots, [59]\}$$

$$F_4^{43} : F_4^{43} = m_{11}(F_4)/\{0\},\ (24,20,17,14,11,8,6,4,3,2,1),\ \dim F_4^{43} = 24$$

$$\begin{array}{rcl} [60] & = & [X_{11}, X_{23}] = -X_{24}, [61] = [X_5, X_{22}] = -X_{24}, [62] = [X_8, X_{21}] = -2X_{24} \\ [63] & = & [X_{11}, X_{20}] = X_{24}, [64] = [X_{12}, X_{19}] = -2X_{24}, [65] = [X_{14}, X_{18}] = -X_{24} \\ [66] & = & [X_{14}, X_{18}] = 2X_{24} \end{array}$$

$$F_4^{43} : \{[1], \ldots, [66].$$

From the above we have the following theorem.

Theorem 21 *Up to isomorphism,* F_4^v, $v = 1, \ldots, 43$, *defined above, are the only Nilpotent Lie Algebras of maximal rank with* F_4 *as an associated G.C.M.*

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Author's address:

Gr. Tsagas
Division of Mathematics
Departement of Mathematics and Physics
Aristotle University of Thessaloniki
Thessaloniki 54006, Greece