CONFARMALLY FLAT MANIFOLDS AND SPECTRA OF ELLIPTIC OPERATORS

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Abstract

Let (M,g) be a compact Riemannian manifold of dimension n. We consider a special self-adjoint elliptic differential operator on the cross sections of $\Lambda^q(M,\Re)$. We study the influence of the spectra of these operators on the conformally flat structure on M.

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Key words: Riemannian manifold, conformally flat manifold, Weyl curvature tensor, elliptic differential operator and spectrum.

1 Introduction

Let (M,g) be a compact Riemannian manifold of dimension n. We consider the vector bundle $V = \Lambda^1(M,\Re)$ from which we obtain the vector space of the cross sections $C^\infty(V)$ of V. Let D be a self-adjoint elliptic differential operator on $C^\infty(V)$. This operator has a spectrum. The aim of the present paper is to study the influence of $Sp(D,M)$, when D is one parameter family of operators, on the conformally flat structure on M.

The whole paper contains three sections. The first section contains the introduction. The basic notions for a Riemannian manifold and conformally flat structure are given in the second section. The third section includes the main results of this paper. It gives the conditions under which the conformally flat structure on (M,g) are detemined by two spectra.

2 Conformally flat manifold

Let (M,g) be a compact Riemannian manifold of dimension n. We consider a chart (U,ϕ) with local coordinate system $(x_1,...,x_n)$. The Riemannian metric g on U has the form

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$$g/U = g_{ij}dx^i dx^j. \tag{1}$$

The inverse matrix of (g_{ij}) is (g^{ij}). It is known that on the tangent bunlde $T(M)$ there exist the Levi-Civita connection ∇ which in local coordinate system

$$(x_1, ..., x_n, \ \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}) \tag{2}$$

can be expressed by the Cristoffel's symbols

$$\{\Gamma^k_{ij}\}, \qquad i, j, k = 1, ..., n, \tag{3}$$

where

$$\Gamma^k_{ij} = \frac{1}{2} g^{hk} \left(\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jh}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m} \right). \tag{4}$$

On this Riemannian manifold we have the curvature tensor R, the Ricci tensor Ric and the scalar curvature T which are defined in the known matter. The conformal curvature tensor, which is called *Weyl's curvature tensor*, is defined by

$$C(X, Y)Z = R(X, Y)Z - (X \wedge Y)\left(L(Z) - L(X \wedge Y)Z\right), \tag{5}$$

where

$$L(W) = \frac{1}{n-2} Ric(W) - \frac{T}{2(n-1(n-2)} W \tag{6}$$

and

$$L(X \wedge Y)W = g(Y, W)X - g(X, W)Y. \tag{7}$$

It is known that M admits locally a conformal mapping into some open set of $I\!R^n$, which is the Euclidean space of n dimension, if and only, if

$$(I) \quad C = 0 \quad for \ \ n > 3 \tag{8}$$

$$(II) \quad C = 0 \text{ and } L(D_X L)(Y) = (D_Y L)(X) \text{ for } n = 3. \tag{9}$$

The second condition of (9) which is the Codazzi equation for L, is satisfied automatically for $n > 3$.

Let (U, ϕ) be a chart on the manifold with local coordinate system $(x_1, ..., x_n)$. The conformal curvature tensor C on U can be expressed as follows:

$$C_{ijk\ell} = R_{ijk\ell} - \frac{1}{n-2}\left(R_{jk}\delta_{ij} - R_{j\ell}\delta_{ik} + g_{jk}R_{i\ell} - g_{j\ell}R_{ik}\right) + \tag{10}$$

$$\frac{T}{(n-1)(n-2)}(g_{jk}\delta_{i\ell} - g_{j\ell}\delta_{ik}) \tag{11}$$

and the same time the second of (9) in local level can be expressed as follows

$$\nabla_k L_{j\ell} = \nabla_j L_{k\ell}, \tag{12}$$

where

$$L_{ij} = \frac{1}{n-2} R_{ij} - \frac{T}{2(n-1)(n-2)} g_{ij}. \tag{13}$$

We denote below p_{ij} instead of R_{ij}.

The aim of this paper is to study conformally flat manifold of dimensions $n > 3$, because the case for $n = 3$ has been studied in the ([6]). Hence we take under the consideration only the case (8).

3 Influence of the spectrum of $D^1(\varepsilon)$ on the conformally flat structure

We consider a compact Riemannian manifold of dimension $n > 3$. Let $\Lambda^1(M, I\!R)$ be the vector space of exterior 1-forms on M. On $\Lambda^1(M, I\!R)$ we have two differential operators, which are the Laplace operator Δ and the Bochner Laplace operator B, that is:

$$\Delta : \Lambda^1(M, I\!R) \to \Lambda^1(M, I\!R) \text{ and } B : \Lambda^1(M, I\!R) \to \Lambda^1(M, I\!R). \tag{14}$$

From these two operators we construct the one parameter family of operators:

$$D^1(\varepsilon) = \varepsilon\Delta + (1-\varepsilon)B : \Lambda^1(M, I\!R) \to \Lambda^1(M, I\!R), \tag{15}$$

$$D^1(\varepsilon) : a \to D^1(\varepsilon)a, \tag{16}$$

where $0 \preceq \varepsilon \preceq 1$.

The 1-form a is called *eigen 1-form*, if the following condition is satisfied

$$D^1(\varepsilon)a = \lambda a \qquad \lambda \in I\!R, \tag{17}$$

and λ is called *eigen-value* associated to eigenvalue 1-form α. The set of all eigenvalues is denoted by $Sp(D^1(\varepsilon))$ and has the form:

$$Sp(D^1(\varepsilon)) = \{0 = 0... =< \lambda_1 = ... = \lambda_1 < \lambda_2 = ... = \lambda_2 < ... < \infty\}. \tag{18}$$

From (13) we form the function

$$f(t) = \sum_{i=0}^{\infty} m_i e^{-\lambda_1 t}, \tag{19}$$

where m_i is the multiplicity of λ_i $i = 1, 2, ...$ and m_0 is the multiplicity of $\lambda_0 = 0$.

The asymptotic expansion of the formula (18) yields

$$f(t) \simeq (4\pi t)^{-\frac{n}{2}} \left\{a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ...\right\}, \tag{20}$$

where

$$a_0 = nVo\ell(M), \tag{21}$$

$$a_1 = \frac{1}{6} \int_M (6\varepsilon - n) T dM, \tag{22}$$

$$a_2 = \frac{1}{360} \int_M \left[(5n - 60\varepsilon) T^2 + \left(-2n + 180\varepsilon^2\right) |p|^2 - (2n - 30) |R|^2 \right] dM, \tag{23}$$

where T, p and R are the scalar curvature, the Ricci tensor field and the Riemannian tensor field respectively.

Let G be the Einstein tensor field on the Riemannian manifold (M, g). The components (G_{ij}) of G with respect to the local coordinate system $(x_1, ..., x_n)$ are given by:

$$G_{ij} = p_{ij} - \frac{1}{n} g_{ij} T, \qquad i, j = 1, 2, ..., n. \tag{24}$$

From (10) and (23) we obtain:

$$|C|^2 = |R|^2 - \frac{4\,|p|^2}{n-2} + \frac{2T^2}{(n-1)(n-2)}, \quad |G|^2 = |p|^2 - \frac{1}{n} T^2. \tag{25}$$

The relation (22), by means of (24), has the form:

$$a_2 = \frac{1}{180} \int_M \left[B_1(n,\varepsilon) T^2 + B_2(n,\varepsilon) |G|^2 + B_3(n) |C|^2 \right] dM, \tag{26}$$

where

$$B_1(n,\varepsilon) = \frac{180(n-1)\varepsilon^2 - 60n(n-1)\varepsilon + 5n^3 - 7n^2 + 6n - 60}{n(n-1)}, \tag{27}$$

$$B_2(n,\varepsilon) = \frac{180n(n-1)\varepsilon^2 - 2n^3 - 4n^2 + 8n - 12}{n(n-2)}, \tag{28}$$

$$B_3(n) = 2(n - 15). \tag{29}$$

Now, we prove the following theorem.

Theorem 1 *Let (M, g), (M, g') be two compact Riemannian manifolds of dimension n. There are two values ε_1 and ε_2 of the parameter ε such that $Sp\left(D^1_{\varepsilon_1}(M,g)\right) = Sp\left(D^1_{\varepsilon_1}(M',g')\right)$ and $Sp\left(D^1_{\varepsilon_2}(M,g)\right) = Sp\left(D^1_{\varepsilon_2}(M',g')\right)$ then they imply that (M, g) is conformally flat with constant scalar curvature T if, and only if, (M', g') is conformally flat with constant scalar curvature T' and $T = T'$*

Proof. From the conditions

$$Sp\left(D^1_{\varepsilon_1}(M,g)\right) = Sp\left(D^1_{\varepsilon_1}(M',g')\right) \text{ and } Sp\left(D^1_{\varepsilon_2}(M,g)\right) = Sp\left(D^1_{\varepsilon_2}(M',g')\right), \quad (30)$$

we obtain:

$$\begin{aligned} a_0(M,g,\varepsilon_1) &= a_0(M',g',\varepsilon_1),\ a_1(M,g,\varepsilon_1) = a_1(M',g',\varepsilon_1), \\ a_2(M,g,\varepsilon_1) &= a_2(M',g',\varepsilon_1) \end{aligned}$$

and

$$\begin{aligned} a_0(M,g,\varepsilon_2) &= a_0(M',g',\varepsilon_2),\ a_1(M,g,\varepsilon_2) = a_1(M',g',\varepsilon_2), \\ a_2(M,g,\varepsilon_2) &= a_2(M',g',\varepsilon_2), \end{aligned} \quad (31)$$

which by means of (20), (21) and (22) yield:

$$Vo\ell(M) = Vo\ell(M'), \quad (32)$$

Theorem 2

$$\int\limits_M (6\varepsilon_1 - n)T dM = \int\limits_{M'} (6\varepsilon_1 - n)T' dM', \quad (33)$$

$$\int\limits_M \left[B_1(n,\varepsilon_1)T^{\,2} + B_2(n,\varepsilon_1)\left|G\right|^2 + B_3(n)\left|C\right|^2\right] dM =$$

$$\int\limits_M \left[B_1(n,\varepsilon_1)T'^2 + B_2(n,\varepsilon_1)\left|G'\right|^2 + B_3(n)\left|C'\right|^2\right] dM\,', \quad (34)$$

$$\int\limits_M (6\varepsilon_2 - n)T dM = \int\limits_{M'} (6\varepsilon_2 - n)T\,' dM\,', \quad (35)$$

$$\int\limits_M \left[B_1(n,\varepsilon_2)T^{\,2} + B_2(n,\varepsilon_2)\left|G\right|^2 + B_3(n)\left|C\right|^2\right] dM =$$

$$\int\limits_M \left[B_1(n,\varepsilon_2)T\,'^2 + B_2(n,\varepsilon_2)\left|G'\right|^2 + B_3(n)\left|C'\right|^2\right] dM\,', \quad (36)$$

From (33) and (35) we obtain

$$\int\limits_M T dM = \int\limits_{M'} T' dM'. \quad (37)$$

The relations (34) and (36) imply

$$\int_M \left[A_1\,(n,\varepsilon_1,\varepsilon_2)\,T^{\,2} + A_2(n,\varepsilon_1,\varepsilon_2)\,|C|^2\right] dM =$$

$$\int_{M'} \left[A_1\,(n,\varepsilon_1,\varepsilon_2)\,T'^{\,2} + A_2(n,\varepsilon_1,\varepsilon_2)\,|C|^2\right] dM', \tag{38}$$

where

$$A_1(n,\varepsilon_1,\varepsilon_2) = B_1(n,\varepsilon_1)B_2(n,\varepsilon_2) - B_1(n,\varepsilon_2)B_2(n,\varepsilon_1), \tag{39}$$

$$A_2(n,\varepsilon_1,\varepsilon_2) = B_3(n)[B_2(n,\varepsilon_2) - B_2(n,\varepsilon_1)]. \tag{40}$$

The roots of the equations

$$180n(n-1)\varepsilon^2 - (2n^3 + 4n^2 - 8n + 12) = 0, \tag{41}$$

$$180(n-1)\varepsilon^2 - 60n(n-1)\varepsilon + 5n^3 - 7n^2 + 6n - 60 = 0,$$

are the following:

$$x_1 = \sqrt{\frac{E}{180n(n-1)}},\ x_2 = -\sqrt{\frac{E}{180n(n-1)}},\ y_1 = \frac{Z+\sqrt{H}}{360(n-1)},\ y_2 = \frac{Z-\sqrt{H},}{360(n-1)}, \tag{42}$$

where

$$E = 2n^3 + 4n^2 - 8n + 12,\quad Z = 60n(n-1),$$

$$H = 3600n^2(n-1)^2 - 720(n-1)(5n^3 - 7n^2 + 6n - 60).$$

From these roots we have the inequalities

$$x_2 < y_2 < 0 < x_1 < y_1. \tag{43}$$

Now, if we take

$$\varepsilon_1 > y_1 \quad \textit{and} \quad x_1 < \varepsilon_2 < y_1, \tag{44}$$

then we have

$$B_1(n,\varepsilon_1) > 0 \quad B_2(n,\varepsilon_2) > 0, \tag{45}$$

$$B_1(n,\varepsilon_2) < 0 \quad B_2(n,\varepsilon_1) > 0 \tag{46}$$

and therefore

$$A_1(n,\varepsilon_1,\varepsilon_2) > 0 \quad A_2(n,\varepsilon_1,\varepsilon_2) > 0. \tag{47}$$

We assume that the manifold (M',g') is conformally flat, that is $C'=0$ and has constant scalar curvature T'.

Therefore the relation (38) becomes

$$\int\limits_{M} \left[A_1\,(n,\varepsilon_1,\varepsilon_2)\,T^{\,2} + A_2(n,\varepsilon_1,\varepsilon_2)\,|C|^2\right] dM = \int\limits_{M'} A_1\,(n,\varepsilon_1,\varepsilon_2)\,T'^{\,2} dM'. \tag{48}$$

From (37) and since $T\,'$ is constant we obtain:

$$\int\limits_{M} T^{\,2} dM \succeq \int\limits_{M'} T'^{\,2} dM'. \tag{49}$$

If we choose the two values ε_1 and ε_2 of the parameter ε such that they satisfy the inequalities (45) and (46) then (48), by means of (47) and (49), imply

$$|C|^2 = 0 \ \textit{ which yields } \ C = 0.$$

We also obtain $T = T' = constant$. $\square$

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