

SPECIAL SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Let $g(\overline{F}_4)$ be the nilpotent Lie algebra of maximal rank and of type \overline{F}_4 . The aim of the present paper is to construct all the systems of partial differential equations determined by all the nilpotent Lie algebras of maximal rank and of type \overline{F}_4 .

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1. Introduction

Let \overline{F}_4 be the Cartan matrix of the exceptional Lie algebra F_4 . From \overline{F}_4 and using the positive roots of the F_4 we can determine all the nilpotent Lie algebras of maximal rank and of type \overline{F}_4 .

For each of these Lie algebras we can associate a system of partial differential equations, from them we can obtain the invariants of this Lie algebra by means of the enveloping algebra and the Casimir operator.

The aim of the present paper is to solve some systems of partial differential equations obtained by the Lie algebra $g(\overline{F}_4)$, which have some applications in Nuclear Physics.

The whole paper contains four sections. The content of each section is the following: The first section is the introduction.

The system of partial differential equations, which is determined by a Lie algebra, is given in the second section. This also contains some methods how to solve the system.

The third section contains nilpotent Lie algebras of maximal rank and of type \overline{F}_4 .

The corresponding systems of partial differential equations to these Lie algebras are included in the last section. This also has the general solutions of these systems.

2.

Let g be a Lie algebra over \mathbb{R} of dimension n , that is $\dim g = n$. If $\{x_1, \dots, x_n\}$ be a base of g , then we have:

$$[x_i, x_j] = \sum_{k=1}^n C_{ij}^k x_k \tag{2.1}$$

where C_{ij}^k $i, j, k = 1, \dots, n$ are the structure constants of g which satisfies the relations

$$C_{ij}^k = -C_{ji}^k \quad i, j, k = 1, \dots, n$$

$$\sum_{l=1}^n [C_{ij}^l C_{lk}^m + C_{jk}^l C_{li}^m + C_{ki}^l C_{lj}^m] = 0 \tag{2.2}$$

The relations (2.1) can be used in order to obtain the following system of partial differential equations.

$$\sum_{j,k=1}^{1,n} C_{ij}^k x_k \frac{\partial f}{\partial x_j} = 0 \quad i = 1, 2, \dots, n \tag{2.3}$$

A system of the form (2.3) or another one of the same form, which is obtained by (2.3) by means of a proper linear transformation, can be solved.

We consider the matrix

$$A = \begin{pmatrix} 0 & C_{12}^k x_k & \dots & C_{1n}^k x_n \\ C_{21}^k x_k & 0 & \dots & C_{2n}^k x_n \\ \dots & \dots & \dots & \dots \\ C_{n1}^k x_k & C_{n2}^k x_k & \dots & 0 \end{pmatrix} \tag{2.4}$$

If the $rank(A) = r$, then the system (2.3) has $n - r$ functionally independent solutions, which can be found with different methods. Now if $u_1(x_1, \dots, x_n), u_2(x_1, \dots, x_n), \dots, u_k(x_1, \dots, x_n)$, are linearly independent solution of this system, then the general solution of this system is:

$$f = f(u_1, \dots, u_k)$$

3.

Let \bar{F}_4 be the Cartan matrix of the exceptional Lie algebra F_4 , which is given by:

$$\bar{F}_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{3.1}$$

It is known that the nilpotent Lie algebras of maximal rank and of type \bar{F}_4 whose dimension is less or equal to eleven are the following:

$$F_4^1 = \bigoplus_{m=1}^7 KX_m \oplus KX_9, \dim F_4^1 = 8, ([1], \dots, [3], [6]) \quad (3.2)$$

$$F_4^2 = \bigoplus_{m=1}^7 KX_m \oplus \bigoplus_{m=9}^{10} KX_m, \dim F_4^2 = 9, ([1], \dots, [3], [6], \dots, [8]) \quad (3.3)$$

$$F_4^3 = \bigoplus_{m=1}^9 KX_m, \dim F_4^3 = 9, ([1], \dots, [6]) \quad (3.4)$$

$$F_4^4 = \bigoplus_{m=1}^8 KX_m \oplus KX_{10} \oplus KX_{13}, \dim F_4^4 = 8, ([1], \dots, [5], [7], [8], [14], [16]) \quad (3.5)$$

$$F_4^5 = \bigoplus_{m=1}^9 KX_m \oplus KX_{11}, \dim F_4^5 = 10, ([1], \dots, [6], [7], [9], [10]) \quad (3.6)$$

$$F_4^6 = \bigoplus_{m=1}^{10} KX_m, \dim F_4^6 = 10, ([1], \dots, [8]) \quad (3.7)$$

$$F_4^7 = \bigoplus_{m=1}^{10} KX_m \oplus KX_{13}, \dim F_4^7 = 11, ([1], \dots, [8], [14], \dots, [16]) \quad (3.8)$$

$$F_4^8 = \bigoplus_{m=1}^{10} KX_m \oplus KX_{12}, \dim F_4^8 = 11, ([1], \dots, [8], [11], \dots, [13]) \quad (3.9)$$

$$F_4^9 = \bigoplus_{m=1}^9 KX_m \oplus KX_{11} \oplus KX_{14}, \dim F_4^9 = 11, ([1], \dots, [6], [9], [10], [17], [18], [19]) \quad (3.10)$$

$$F_4^{10} = \bigoplus_{m=1}^7 KX_m \oplus \bigoplus_{m=9}^{10} KX_m \oplus KX_{13} \oplus KX_{16}, \dim F_4^{10} = 11, ([1], \dots, [3], [6], \dots, [8], [14], \dots, [16], [24], [25]) \quad (3.11)$$

$$F_4^{11} = \bigoplus_{m=1}^{11} KX_m, \dim F_4^{11} = 11, ([1], \dots, [10]) \quad (3.12)$$

where the brackets $[v]$, $v = 1 \dots 25$ in each parenthesis are the following Lie brackets:

$$\begin{aligned} [1] &= [x_1, x_2] = -x_5, [2] = [x_2, x_3] = -x_6, [3] = [x_3, x_4] = -x_7, \\ [4] &= [x_4, x_6] = -x_8, [5] = [x_3, x_5] = -x_8, [6] = [x_3, x_6] = -2x_9, \\ [7] &= [x_2, x_7] = -x_{10}, [8] = [x_4, x_6] = x_{10}, [9] = [x_1, x_9] = -x_{11}, \\ [10] &= [x_3, x_8] = -2x_{11}, [11] = [x_1, x_{10}] = -x_{12}, [12] = [x_4, x_8] = x_{12}, \\ [13] &= [x_5, x_7] = -x_{12}, [14] = [x_3, x_{10}] = -x_{13}, [15] = [x_4, x_9] = x_{13}, \\ [16] &= [x_6, x_7] = x_{13}, [17] = [x_2, x_{11}] = -x_{14}, [18] = [x_5, x_9] = x_{14}, \\ [19] &= [x_6, x_8] = -2x_{14}, [20] = [x_1, x_{13}] = -x_{15}, [21] = [x_3, x_{12}] = -x_{15}, \\ [22] &= [x_4, x_{11}] = x_{15}, [23] = [x_7, x_8] = -x_{15}, [24] = [x_4, x_{13}] = -2x_{16}, \\ [25] &= [x_7, x_{10}] = 2x_{16}. \end{aligned}$$

The whole number of nilpotent algebras of maximal rank and of type \overline{F}_4 is 43. We consider only those until to dimension 11 because some of them can be used as a model in the Neuclear Physics. In order to relate these two subjects we need to determine the invariants of the Lie algebras F_4^λ $\lambda = 1, \dots, 11$ and therefore to solve some systems of partial differential equation. The above relations will be given in the last section.

4.

For the Nilpotent Lie algebra F_4^1 we associated the following system of partial differential equations:

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0 \\ -x_7 \frac{\partial f}{\partial x_4} - 2x_9 \frac{\partial f}{\partial x_5} &= 0 \end{aligned} \quad (4.1)$$

From this system we obtain the matrix:

$$A = \begin{pmatrix} 0 & -x_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_5 & 0 & -x_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_6 & 0 & -x_7 & -2x_9 & 0 & 0 & 0 \\ 0 & 0 & x_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2x_9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The rank of A is 4. Therefore the system of partial differential equations has the following independent solutions:

$$u_1 = x_6 \quad u_2 = x_7 \quad u_3 = x_9 \quad u_4 = 2x_4x_9 - x_7x_5$$

Therefore the general solution of the system is given:

$$f_1 = f_1(x_6, x_7, 2x_4x_9 - x_7x_5) \quad (4.2)$$

The systems of partial differential equations associated to nilpotent Lie algebras F_4^λ $\lambda = 2, \dots, 11$ are the following:

$$\begin{aligned} &F_4^2 \\ \frac{\partial f}{\partial x_2} &= 0 \quad x_5 \frac{\partial f}{\partial x_1} - x_6 \frac{\partial f}{\partial x_3} - x_{10} \frac{\partial f}{\partial x_4} = 0 \quad -x_7 \frac{\partial f}{\partial x_4} - 2x_9 \frac{\partial f}{\partial x_6} = 0 \\ &\quad -x_7 \frac{\partial f}{\partial x_3} + x_{10} \frac{\partial f}{\partial x_6} = 0 \quad 2x_9 \frac{\partial f}{\partial x_3} - x_{10} \frac{\partial f}{\partial x_4} = 0 \\ &F_4^3 \\ \frac{\partial f}{\partial x_1} &= \frac{\partial f}{\partial x_3} = 0 \quad -x_5 \frac{\partial f}{\partial x_2} - x_8 \frac{\partial f}{\partial x_6} = 0 \\ x_5 \frac{\partial f}{\partial x_1} - x_6 \frac{\partial f}{\partial x_3} &= 0 \quad x_6 \frac{\partial f}{\partial x_2} - x_7 \frac{\partial f}{\partial x_4} + x_8 \frac{\partial f}{\partial x_5} - 2x_9 \frac{\partial f}{\partial x_6} = 0 \\ &F_4^4 \\ -x_5 \frac{\partial f}{\partial x_2} - x_8 \frac{\partial f}{\partial x_6} &= 0 \quad x_5 \frac{\partial f}{\partial x_1} - x_{10} \frac{\partial f}{\partial x_7} = 0 \\ x_6 \frac{\partial f}{\partial x_2} - x_7 \frac{\partial f}{\partial x_4} + x_8 \frac{\partial f}{\partial x_5} + x_{13} \frac{\partial f}{\partial x_{10}} &= 0 \quad x_7 \frac{\partial f}{\partial x_2} + x_{10} \frac{\partial f}{\partial x_6} = 0 \\ x_8 \frac{\partial f}{\partial x_3} = 0 \quad x_8 \frac{\partial f}{\partial x_1} - x_{10} \frac{\partial f}{\partial x_4} + x_{13} \frac{\partial f}{\partial x_7} &= 0 \quad x_{10} \frac{\partial f}{\partial x_2} - x_3 \frac{\partial f}{\partial x_6} = 0 \end{aligned}$$

$$\begin{aligned}
& F_4^5 \\
& -x_8 \frac{\partial f}{\partial x_6} - x_{11} \frac{\partial f}{\partial x_9} = 0 \\
& -x_7 \frac{\partial f}{\partial x_4} + x_8 \frac{\partial f}{\partial x_5} - 2x_9 \frac{\partial f}{\partial x_6} - 2x_{11} \frac{\partial f}{\partial x_8} = 0 \quad \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_3} = 0 \\
& F_4^6 \\
& \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_6} = 0 \quad x_5 \frac{\partial f}{\partial x_1} - x_{10} \frac{\partial f}{\partial x_7} = 0 \quad -x_7 \frac{\partial f}{\partial x_4} + x_8 \frac{\partial f}{\partial x_5} = 0 \\
& \quad x_8 \frac{\partial f}{\partial x_1} - x_{10} \frac{\partial f}{\partial x_4} = 0 \\
& F_4^7 \\
& \frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_4} = 0 \quad -x_5 \frac{\partial f}{\partial x_2} - x_8 \frac{\partial f}{\partial x_6} = 0 \quad x_5 \frac{\partial f}{\partial x_1} - x_{10} \frac{\partial f}{\partial x_7} = 0 \\
& \quad x_6 \frac{\partial f}{\partial x_2} + x_8 \frac{\partial f}{\partial x_5} - 2x_9 \frac{\partial f}{\partial x_6} = 0 \quad x_{10} \frac{\partial f}{\partial x_6} + x_{13} \frac{\partial f}{\partial x_9} = 0 \\
& \quad x_8 \frac{\partial f}{\partial x_1} + x_{13} \frac{\partial f}{\partial x_7} = 0 \quad x_{10} \frac{\partial f}{\partial x_2} - x_{13} \frac{\partial f}{\partial x_3} = 0 \\
& F_4^8 \\
& \frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_4} = 0 \quad -x_5 \frac{\partial f}{\partial x_2} - x_8 \frac{\partial f}{\partial x_6} = 0 \quad x_5 \frac{\partial f}{\partial x_1} - x_{10} \frac{\partial f}{\partial x_7} = 0 \\
& \quad x_6 \frac{\partial f}{\partial x_2} + x_8 \frac{\partial f}{\partial x_5} - 2x_9 \frac{\partial f}{\partial x_6} - x_{13} \frac{\partial f}{\partial x_{10}} = 0 \quad x_{10} \frac{\partial f}{\partial x_6} + x_{13} \frac{\partial f}{\partial x_9} = 0 \\
& \quad x_8 \frac{\partial f}{\partial x_1} + x_{13} \frac{\partial f}{\partial x_7} = 0 \quad x_{10} \frac{\partial f}{\partial x_2} - x_{13} \frac{\partial f}{\partial x_6} = 0 \\
& F_4^9 \\
& \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_5} = \frac{\partial f}{\partial x_6} = \frac{\partial f}{\partial x_8} = \frac{\partial f}{\partial x_9} = \frac{\partial f}{\partial x_{11}} = 0 \quad -x_7 \frac{\partial f}{\partial x_4} - 2x_{11} \frac{\partial f}{\partial x_7} = 0 \\
& F_4^{10} \\
& \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0 \quad x_5 \frac{\partial f}{\partial x_1} - x_{10} \frac{\partial f}{\partial x_7} = 0 \quad -x_7 \frac{\partial f}{\partial x_4} - 2x_9 \frac{\partial f}{\partial x_6} - x_{13} \frac{\partial f}{\partial x_{10}} = 0 \\
& \quad x_{10} \frac{\partial f}{\partial x_6} + x_{13} \frac{\partial f}{\partial x_9} - 2x_{16} \frac{\partial f}{\partial x_{13}} = 0 \quad x_8 \frac{\partial f}{\partial x_1} + x_{10} \frac{\partial f}{\partial x_4} + x_{13} \frac{\partial f}{\partial x_9} - 2x_{16} \frac{\partial f}{\partial x_{13}} = 0 \\
& \quad x_{13} \frac{\partial f}{\partial x_6} + 2x_{16} \frac{\partial f}{\partial x_{10}} = 0 \quad x_{11} \frac{\partial f}{\partial x_1} - x_{13} \frac{\partial f}{\partial x_4} = 0 \quad x_{12} \frac{\partial f}{\partial x_1} - 2x_{16} \frac{\partial f}{\partial x_7} = 0 \\
& F_4^{11} \\
& \left. \begin{aligned}
& \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = \frac{\partial f}{\partial x_6} = \frac{\partial f}{\partial x_7} = 0 \quad -x_{11} \frac{\partial f}{\partial x_9} - x_{12} \frac{\partial f}{\partial x_{10}} = 0 \\
& \quad x_8 \frac{\partial f}{\partial x_5} - 2x_{11} \frac{\partial f}{\partial x_8} = 0
\end{aligned} \right]
\end{aligned}$$

The general solutions of these system are:

$$\left. \begin{aligned} f_2 &= f_2(x_7, x_9, x_{10}) & f_7 &= f_7(x_8, x_{10}, x_{13}) \\ f_3 &= f_3(x_7, x_8, x_9) & f_8 &= f_8(x_8, x_{13}, x_{10}x_8 + x_5x_{13}) \\ f_4 &= f_4(x_8, x_9, x_{13}) & f_9 &= f_9(x_{10}, x_{14}, x_7^2 - 4x_4x_{11}) \\ f_5 &= f_5(x_7, x_{11}) & f_{10} &= f_{10}(x_5, x_{16}, x_{13}^2 - 4x_9x_{10}) \\ f_6 &= f_6(x_8, x_9, x_{10}) & f_{11} &= f_{11}(x_{11}, x_{11}x_{10} - x_9x_{12}, x_8^2 - 4x_5x_{11}) \end{aligned} \right] \quad (4.4)$$

Now we have the theorem:

Theorem 1 4.1 *The general solutions of the systems of partial differential equations (4.1) and (4.3) are given in (4.2) and (4.4) respectively.*

5.

It is known that the semisimple Lie groups, which are:

$$A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8 \quad (5.1)$$

play an important role in modern physics. From the Lie groups we obtain the corresponding Lie algebras denoted by:

$$A'_n, B'_n, C'_n, D'_n, G'_2, F'_4, E'_6, E'_7, E'_8 \quad (5.2)$$

From the (5.2) using the Kac-Moody theory we can construct nilpotent Lie algebras of type K (where K one of (5.2)) and of maximal rank.

For the physical applications it would be of interest to classify all real nilpotent Lie algebras of dimensions less or equal 11 into equivalence classes and to find their invariants. We must notice that such Lie algebras occur as subalgebras of the conformal group of:

$$(\mathbb{R}^4, d(+, +, +, \cdot))$$

This is the reason that we have estimate the invariants of the nilpotent Lie algebras of maximal rank and of type F_4 whose dimensions is at most eleven.

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